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BOOK REVIEW

# The remarkable effectiveness… A report on the book Manfred Einsiedler and Thomas Ward: "Ergodic theory, with a view towards number theory"

**Springer-Verlag, 2011, 467 pp.**

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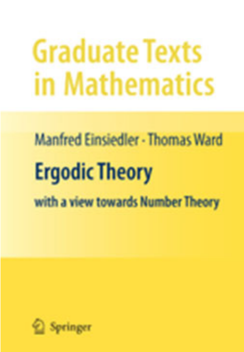


The book under review is an introductory textbook on ergodic theory, written with applications to number theory in mind. Among its many merits is its timeliness. It is inspired by dramatic recent developments in which progress on longstanding open questions in number theory is made using ergodic-theoretic tools. Examples of such developments are given in Chapter 1, among these are Margulis' proof of the Oppenheim conjecture, Furstenberg's ergodic proof of Szemerédi's theorem, and the progress of Einsiedler, Lindenstrauss and Katok toward Littlewood's conjecture. I will begin by briefly discussing the fruitful relationship of ergodic theory and number theory, which may seem surprising to some readers, and which motivated the authors in many of their choices. Although this theme is discussed in many interesting survey papers (among them I recommend [1], from whom the title of this review is taken), I hope some readers will find my remarks useful. Then I will focus on the book at hand in more detail.

Ergodic theory has its roots in the study of physical systems, and is classically concerned with the long-term statistical behavior of typical trajectories. To give a naive example, one might consider the motion of the earth around the sun, and ask: *Will the earth ever collide with the sun*? *Will it drift further and further from the sun*? An answer was famously given by Newton in his *Principia*. Modelling the system as two point particles in Euclidean space, and formulating his laws of motion and

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gravitation, Newton obtained a differential equation which he explicitly solved, recovering Kepler's laws of motion: the earth's orbit describes an ellipse with the sun at a focal point. In particular the long-term behavior is periodic, giving a negative answer to the above questions. Newton's tremendous achievement gave rise to the hope that long-term behavior of deterministic physical systems could always be analyzed successfully by explicitly solving differential equations. However it was gradually realized that many differential equations could not be explicitly solved, and in many cases, solutions depend sensitively on initial conditions. For instance this would be the case if one added a third body (e.g. the moon) to the above model: tiny changes in initial conditions, e.g. adding a milligram to the mass of one of the bodies, can lead to a different answer to questions about long-term behavior. Awareness of such phenomena led many researchers (famously, Poincaré in his memoir of 1890) to the study of *typical* trajectories. Naturally this study developed hand in hand with the emerging theories of probability and measure theory.

One of the resulting theories is ergodic theory, which can be loosely defined as the measure-theoretic analysis of group-actions on measure spaces. The classical setup is an action of the group of integers $\mathbb{Z}$, which is nothing but a measurable invertible transformation $T : X \to X$. In this setup $X$ is a Borel space and one often assumes that $T$ preserves a finite Borel measure on $X$, and inquires about the long-term statistical behavior of typical trajectories. More generally, one may replace the action of $\mathbb{Z}$ by the action of a topological group, and/or relax the assumption on the preservation of the measure. As the theory acquired a life of its own, two features emerged. It quickly became apparent that difficult questions can be asked concerning systems which are governed by much simpler laws than Newton's laws of motion and gravitation. Moreover it was realized that despite the desirability of a general abstract theory which would apply to all examples, many examples have special features which can be exploited to obtain a much more detailed understanding. An example of such a seemingly innocuous system is when $X$ is the unit circle, and $T : X \to X$ is rotation by an irrational angle. In this example, as shown by Kronecker and Weyl, one could completely understand the behavior of *all* orbits (they are all equidistributed) and classify *all* invariant measures (Lebesgue measure is the only one).

Note that the above example could be rewritten in fancier terms as $X = \mathbb{R}/\mathbb{Z}$, which is equipped with the transitive action of $\mathbb{R}$ (addition mod 1 or rotation by all possible angles) and $T$ is the $\mathbb{Z}$-action obtained by restricting the $\mathbb{R}$-action to a cyclic subgroup of $\mathbb{R}$. This fancier setup leads to a class of algebraic examples (the so-called *homogeneous spaces*) in which the space $X$ is equipped with the transitive action of a topological group $G$ and the action is obtained by restricting to subgroups of $G$ or using suitable endomophisms of $G$. A complete dynamical analysis of these richly structured actions is a very active field of research, and requires an array of tools from different mathematical fields (Lie theory, representation theory, number theory, Fourier analysis, to name a few). A feature which sets these examples apart from many other dynamical systems is that in certain cases, as in the case of the irrational rotation, one can obtain complete classification results describing the behavior of *all* orbits, invariant measures, etc. It is this feature which is often responsible for the remarkable effectiveness alluded to above.

The book of Einsiedler and Ward aims to equip the reader with the tools with which to study these remarkable applications of ergodic theory. There are several more specialized books in the same vein, focusing on a particular result (see [2–5]). The book at hand is different as it aims both to provide the reader with a solid comprehensive background in the main results of ergodic theory, and of reaching nontrivial applications to number theory. Loosely speaking, the foundations are laid in Chapters 1–6 (with some number-theoretic asides on continued fractions) and the fruits, in the form of substantial applications, are harvested in Chapters 7–10. This is very ambitious for a book of about 400 pages.

As the reader has presumably surmised, ergodic theory is not a linear theory with a universally accepted axiomatic framework, or to quote the authors, "is a rather diffuse subject with ill-defined boundaries". There are many choices which the authors had to make regarding the first part of the book, namely choosing the scope of their definitions, the generality of their results, and the assumed prerequisites. This is reflected in the flowchart on page (xi) describing the interdependencies of chapters—there is certainly not a unique way to read this book. For most of the first part of the book, Einsiedler and Ward chose to focus on a measure preserving transformation (i.e. a $\mathbb{Z}$-action), postponing the discussion of more general group actions and stationary measures to Chapter 8. Their discussion in the first few chapters is thorough and more comprehensive than in most other textbooks in ergodic theory. For example, in Chapter 5 they characterize the limit functions appearing in ergodic theorems in terms of both Hilbert-space projections onto the space of invariant functions, and conditional expectations with respect to the algebra of invariant sets. Other examples are detailed treatments of martingale theorems and ergodic decomposition. They sometimes give more than one proof of a result, and sometimes sketch alternative routes. By and large, they prefer long proofs giving detailed information over slick proofs giving only the best-known results. The text is interspersed with many footnotes and asides to the literature for possible extensions and refinements.

In writing the second part of the book, the authors were faced with the problem of choosing representative applications of ergodic theory to number theory. The reader should note that all of the celebrated applications listed above require detailed information about specific systems and prerequisites from various fields. It is not an easy task to choose applications which a reader will find sufficiently interesting and at the same time approachable. The authors have chosen two landmark results: Furstenberg's proof of Szemerédi's theorem (Chapter 7), and the results of Dani and Smillie on measure classification and equidistribution of horocycles on finite-volume hyperbolic surfaces (Chapter 10). These are excellent choices and the authors manage to reach important results without sacrificing detail. Along the way the authors present many other important results from the ergodic-theoretic literature which had previously only appeared in the research literature, e.g. Ledrappier's example of a $\mathbb{Z}^2$-action which is mixing but not mixing of all orders, Mozes' result on mixing of all orders for actions of $\mathrm{SL}_2(\mathbb{R})$, or results on translations on nilmanifolds. In all of these choices the authors display their fine taste. One may worry that the authors have tried to include too much material, but the treatment remains consistently solid and thorough. As in the preceding chapters, the authors often include several proofs of the same result (for instance they prove ergodicity of the geodesic flow both by following Hopf's original argument and via the Mautner property).

The authors were also forced to make difficult decisions concerning the mathematical prerequisites for reading the book. They have chosen to rely on a strong background in measure theory, functional analysis, and harmonic analysis. Much of the relevant background is recalled in the appendices. On the other hand the authors have included efficient crash courses in hyperbolic geometry, elementary Lie theory (via linear groups), and basics of continued fractions and diophantine approximation. The substantial prerequisites should not deter researchers from basing a graduate course on ergodic theory on this book, as the extra effort of the students will be amply rewarded in terms of both depth of coverage and exciting applications. The book should also be very appealing to more advanced readers already conducting research in representation theory or number theory, who are interested in understanding the basis of the recent interaction with ergodic theory.

The authors plan a sequel to the book, focusing on entropy theory and its relations to the recent work on Littlewood's conjecture. Based on the high standard set by this volume, the second volume will be eagerly anticipated by many readers.

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