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Integer sequences and dynamics

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Outline

Dynamics

Number theory

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Number theory

(but they interact fruitfully in Diophantine analysis, combinatorics, geometry,...)

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Algebraic example: $X = \mathbb{T}^d$, $T = T_A$ given by an integer matrix A in $SL_d(\mathbb{Z})$. Then

$$h(T_A) = \sum_{\text{eigenvalues } \lambda} \log^+ |\lambda|,$$

the 'Mahler measure' of the characteristic polynomial of A.

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It means we cannot entirely describe the measurable structure of group automorphisms.

A map $T\colon X \to X$ has two associated integer sequences:

- its 'count of periodic points' $\big(F_T(n)\big)$, where $F_T(n) = |\{x \in X \mid T^n x = x\}|$;
- its 'count of closed orbits', $\big(O_T(n)\big)$.

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$$(1, 1, 2, 3, 5, 8, \ldots)✗$$
$$(5, 5, 10, 15, 25, 40, \ldots)✓$$

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Sample results:

- $(1, a, 1+a, 1+2a, 2+3a, 3+5a, \dots)$ counts periodic points $\iff a = 3$;
- binary recurrences understood;
- some 'dominant root' cases understood;
- much deeper results known for T of a specific type (a 'shift of finite type').

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Modest knowledge:

- 'local' (at each prime) conditions are necessary but not sufficient;
- toral examples are deceptive;
- linear recurrence + divisibility are not sufficient: no group automorphism has $(1, 1, 1, 1, 6, 1, 1, 1, 1, 6, \dots)$ as periodic point count.

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$$\zeta_T(z) := \exp \sum_{n\geqslant 1} \tfrac{z^n}{n} F_T(n) = \prod_{n\geqslant 1} (1 - z^n)^{-O_T(n)}$$

$$d_T(z) := \sum_{n\geqslant 1} \frac{O_T(n)}{n^z} = \frac{1}{\zeta(z+1)} \sum_{n\geqslant 1} \frac{F_T(n)/n}{n^z}$$

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A baby case

If $T\colon X \to X$ has exactly one orbit of each length, then $d_T(z) = \zeta(z)$.

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Hardy / Ramanujan formula for $\sum_{n \geqslant 1} \frac{\sigma_a(n)\sigma_b(n)}{n^z}$

$$\implies d_{T\times T}(z) = \frac{\zeta^2(z)\zeta(z-1)}{\zeta(2z)},$$

$$\implies \sum_{n \leqslant N} O_{T\times T}(n) \sim \frac{\pi^2}{12} N^2.$$

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The expression $\left((2^n - 1) \times |2^n - 1|_3\right)$ gives a dynamical zeta function with a natural boundary at $|z| = \frac{1}{2}$.

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Cautionary example: there is a smooth map T with $F_T(n) = \binom{2n}{n}$ for all $n \geqslant 1$, and hence with an irrational algebraic zeta function.

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Nonetheless...

Theorem: M is uncountable.