Appendix A: Topological Groups and Homogeneous Spaces

A.1 Metrics on Topological Groups

Definition A.1. A collection $\mathcal{U}$ of neighbourhoods of a point x in a topological space is a *neighbourhood basis* if for any open neighbourhood $V \ni x$ there is some neighbourhood $U \in \mathcal{U}$ with $U \subseteq V$.

A topological space is *first countable* if every point has a countable neighbourhood basis.

A metric space (X, d) is automatically first countable, since the open neighbourhoods $B_{1/n}(x)$ for $n \in \mathbb{N}$ form a countable neighbourhood basis at x.(39)

Recall that a topological group is a group G together with a Hausdorff topology $\mathcal{T}$ with respect to which the maps $g \mapsto g^{-1}$ and $(g,h) \mapsto gh$ are continuous. This means that

- if U is a neighbourhood of a product $gh \in G$, then there are neighbourhoods $U_1 \ni g$ and $U_2 \ni h$ with $U_1U_2 \subseteq U$;
- if U is an open neighbourhood of $g \in G$, then there is an open neighbourhood V of g^{-1} with $V^{-1} \subseteq U$.

Lemma A.2 (Birkhoff–Kakutani [5], [76]). *The following properties of a topological group G are equivalent.*(40)

(1) *G has a left-invariant metric, that is a metric d giving the topology which additionally has $\mathsf{d}(gh_1, gh_2) = \mathsf{d}(h_1, h_2)$ for all $g, h_1, h_2 \in G$.*
(2) *Each $g \in G$ has a countable basis of open neighbourhoods.*
(3) *The identity $1 \in G$ has a countable basis of open neighbourhoods.*

Proof. It is clear that (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) since the rotation $g \mapsto gh$ is a homeomorphism of G for any $h \in G$. So we will assume (2). Let $\mathcal{U} = \{V_1, V_2, \dots\}$ be a countable neighbourhood basis at the identity I consisting of open sets. Without loss of generality we may assume that

$$V_1 \supseteq V_2 \supseteq \cdots \supseteq \{I\},$$

and since G is Hausdorff we have

$$\bigcap_{n\geqslant 1} V_n = \{I\}.$$

We wish to construct sets that mimic the behaviour of nested metric open sets. To that end we use the continuity properties of the two group operations to construct from the sequence of sets (V_n) another nested sequence of open neighbourhoods of the identity

$$U_1 = G \supseteq U_{1/2} \supseteq U_{1/2^2} \supseteq \cdots \supseteq \{I\}$$

with the property that $U_{1/2^n}^{-1} = U_{1/2^n}$ (each set is symmetric), $U_{1/2^n} \subseteq V_n$, and $U_{1/2^{n+1}}U_{1/2^{n+1}} \subseteq U_{1/2^n}$ for each $n \geqslant 1$. It follows that

$$\bigcap_{n\geqslant 0} U_{1/2^n} \subseteq \bigcap_{n\geqslant 1} V_n = \{I\}. \tag{A.1}$$

For any rational of the form $\frac{a}{2^n}$ with $a \in \{1,\dots,2^n-1\}$ and $n \in \mathbb{N}$ we define

$$U_{a/2^n} = U_{1/2^{n_1}} \cdots U_{1/2^{n_r}}$$

where

$$\frac{a}{2^n} = 2^{-n_1} + \cdots + 2^{-n_r}$$

is the binary expansion of $\frac{a}{2^n}$ arranged in the natural order with

$$1 \leqslant n_1 < \cdots < n_r.$$

By construction[†]

$$U_{a/2^n}U_{1/2^n} \subseteq U_{(a+1)/2^n} \tag{A.2}$$

for $n \geqslant 1$ and $1 \leqslant a \leqslant 2^n - 1$. Hence the sets $U_{a/2^n}$ are nested in the sense that[‡]

$$0 < a < b \leqslant 2^n \Rightarrow U_{a/2^n} \subseteq U_{b/2^n}.$$

Using this neighbourhood basis we can define a function f on G by

$$f(x) = \inf\{\tfrac{a}{2^n} \mid x \in U_{a/2^n}\}.$$

[†] If there is no carry in the binary addition of $a/2^n$ and $1/2^n$ this is just the definition, if there is a carry one uses the defining properties of $U_{1/2^n}$.

[‡] Suppose $a = 2^{-n_1} + \cdots + 2^{-n_r}$. If $b = a + \frac{c}{2^{n_r+1}}$ for an integer $c \geqslant 1$ this follows simply from the definition of U_b. If $b = a + \frac{1}{2^{n_r}}$ this follows from (A.2). If $b > a + \frac{1}{2^{n_r}}$ the conclusion follows from the latter case and induction on n_r.

We claim that f has the following properties[†]:

(a) $f(g) > 0$ for $g \in G \smallsetminus \{I\}$ and $f(I) = 0$;
(b) the collection $\bigl\{\{g \in G \bigm| f(g) < \frac{1}{n}\} \mid n \in \mathbb{N}\bigr\}$ form a neighbourhood base at the identity $I \in G$; and
(c) for any $\varepsilon > 0$ there is an open neighbourhood $U \ni e$ such that
$$|f(hg) - f(h)| \leqslant \varepsilon$$
for all $g \in U$ and $h \in G$.

Property (b) holds since the collection of sets $\{U_{a/2^n}\}$ form a neighbourhood basis at the identity, and (a) follows from (A.1). The uniform continuity property (c) is a consequence of (A.2). Indeed if $\varepsilon > 0$ is arbitrary we may choose some n with $\frac{1}{2^n} < \varepsilon$ and set $U = U_{1/2^{n+1}}$. We let $h \in G$ and $g \in U$ be arbitrary. If $f(h) \in [\frac{a}{2^{n+1}}, \frac{a+1}{2^{n+1}})$ then $h \in U_{(a+1)/2^{n+1}}$, and so $hg \in U_{(a+2)/2^{n+1}}$ which implies $f(hg) < \frac{a+2}{2^{n+1}} \leqslant f(h) + \frac{2}{2^{n+1}} < f(h) + \varepsilon$. Using $U^{-1} = U$ the inequality $f(h) < f(hg) + \varepsilon$ follows from the former.

We now define a metric-like function $\mathsf{d}^f \colon G \times G \to [0, \infty)$ by
$$\mathsf{d}^f(g_1, g_2) = \sup_{h \in G} |f(hg_1) - f(hg_2)|.$$

Clearly
$$\mathsf{d}^f(g_1, g_2) = \mathsf{d}^f(g_2, g_1)$$
and
$$\mathsf{d}^f(hg_1, hg_2) = \mathsf{d}^f(g_1, g_2)$$
for all $g_1, g_2, h \in G$, $\mathsf{d}^f(g, g) = 0$, and d^f obeys the triangle inequality. That is, d^f is a left-invariant *pseudometric* on G. Now assume that $\mathsf{d}^f(g_1, g_2) = 0$. Then $f(g_1^{-1}g_2) \leqslant \mathsf{d}^f(I, g_1^{-1}g_2) = 0$ implies that $g_1 = g_2$ by (a). Hence we see that d^f is a metric on G.

It remains to show that the metric topology induced from d^f is the original group topology. As both topologies make G into a topological group it is sufficient to study the neighbourhoods of I with respect to both topologies. Any $h \in G$ with $\mathsf{d}^f(h, I) < \frac{1}{2^n}$ satisfies $h \in U_{1/2^n}$, which shows that a neighbourhood in the original topology is also a neighbourhood in the the metric topology. Now let $\varepsilon > 0$ and let U be a neighbourhood as in the uniform continuity property (c). Then $\mathsf{d}^f(h, I) \leqslant \varepsilon$ for all $h \in U$, which shows that a metric neighbourhood is also a neighbourhood in the original topology. □

Notice in particular that this means the groups encountered in this volume, like $\mathrm{GL}_d(\mathbb{R})$ and $\mathrm{SL}_d(\mathbb{R})$, have left-invariant metrics that give the group topology.

[†] Notice that the existence of such a function would follow easily from the conclusion we seek. If G has a left-invariant metric d defining its topology, then the function f defined by $f(g) = \mathsf{d}(e, g)$ has the three properties claimed.

Exercise A.1. Let $d \geqslant 2$. Show that $G = \mathrm{SL}_d(\mathbb{R})$ cannot be equipped with a metric that gives the standard topology and is bi-invariant, that is, satisfies $\mathsf{d}(gh_1, gh_2) = \mathsf{d}(h_1, h_2) = \mathsf{d}(h_1 g, h_2 g)$ for all $g, h_1, h_2 \in G$.

A.2 Invariant Measures on Quotients of Groups

Proposition A.3. *Let G be a locally compact, σ-compact, metrizable group and let $H < G$ be a closed subgroup. Suppose that both G and H are unimodular. Then there exists a non-trivial locally finite G-invariant measure $m_{G/H}$ on G/H and this measure is unique up to scalar multiples.*

We refer to the work of Knapp [87] or Raghunathan [121] for a more general treatment of the existence of invariant measure on quotients. Concerning the necessity of the assumptions in Proposition A.3 we simply note that $\mathbb{P}^1(\mathbb{R})$ carries a transitive action of $G = \mathrm{SL}_2(\mathbb{R})$ and so is of the form G/H but possesses no invariant measure.

A.2.1 Existence of Cross-Sections

An important tool for us will be the existence of a cross-section for G/H. Set-theoretically a cross-section is simply a subset of G containing exactly one element from each coset of H, which is readily obtained using the axiom of choice. Where the construction becomes more difficult—and far more useful—is if additional measure-theoretic or topological requirements are sought in the cross-section.

Definition A.4 (Cross-section). We say that $S \subseteq G$ is a *cross-section* for $H < G$ if

(i) S is measurable,
(ii) $|S \cap gH| = 1$ for all $g \in G$, and
(iii) $S \ni s \mapsto sH \in G/H$ is a Borel isomorphism.

Lemma A.5 (Cross-sections exist). *Let $H < G$ be a closed subgroup of the group G as in Proposition* A.3*. Then there exists a cross-section $S \subseteq G$ for H. Moreover, there exists a subset $S_{\mathrm{local}} \subseteq S$ with compact closure such that $S_{\mathrm{local}} H \subseteq G/H$ has a non-trivial interior.*

We refer to [40, Sec. 5.5.4] for a general proof of Lemma A.5, but explain that for a Lie group (or for a closed linear p-adic group) it is quite straightforward to construct such a cross-section $S \subseteq G$ for $H < G$. Indeed, let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h} \subseteq \mathfrak{g}$ be the Lie algebra of H. Moreover, let $V \subseteq \mathfrak{g}$ be a subspace so that $\mathfrak{g} = V \oplus \mathfrak{h}$. We define a map

$$\begin{aligned}\phi\colon V\times\mathfrak{h}&\longrightarrow G\\ (v,w)&\longmapsto \exp(v)\exp(w).\end{aligned}$$

As the derivative of ϕ is the linear isomorphism

$$\mathfrak{h}\times V\ni(v,w)\longmapsto v+w\in\mathfrak{g}$$

it follows that there exists $\delta_0>0$ such that the restriction of ϕ to $B^V_{\delta_0}\times B^{\mathfrak{h}}_{\delta_0}$ is a diffeomorphism onto its image. Let $S_{\mathrm{local}}=\exp(B^V_\delta)$ for $\delta\in(0,\delta_0)$ determined as follows. Suppose $s_j=\exp(v_j)$ for $v_j\in B^V_\delta$ satisfy $s_1H=s_2H$. Then $\exp(v_1)=\exp(v_2)h$ for some $h\in H$ implies for sufficiently small $\delta>0$ that

$$h=\exp(v_2)^{-1}\exp(v_1)=\exp(w)$$

for some $w\in B^{\mathfrak{h}}_{\delta_0}$, which in turn implies $w=0$ and $v_1=v_2$.

To summarize, (for $\delta>0$ as above) the map

$$S_{\mathrm{local}}\ni s\longmapsto sH\in O=S_{\mathrm{local}}H\subseteq G/H$$

is continuous, injective, and has an open image O. Moreover, S_{local} is σ-compact which also implies that the map is a Borel isomorphism from S_{local} to O. For any $g\in G$ we may translate the above and obtain the fact that

$$gS_{\mathrm{local}}\ni gs\longmapsto gsH\in gO\subseteq G/H$$

is a Borel isomorphism with open image gO. As G is σ-compact we can cover G/H by countably many translates g_nO for $n\in\mathbb{N}$. We assume $g_1=e$ and define

$$S=\bigsqcup_{n=1}^{\infty}\Bigl(g_nS_{\mathrm{local}}\smallsetminus\bigcup_{k=1}^{n-1}g_kS_{\mathrm{local}}H\Bigr)\supseteq S_{\mathrm{local}}.$$

It then follows from the construction that $S\subseteq G$ is a cross-section for G (see also the proof of Lemma 1.20 or [40, Lem. 5.52]).

A.2.2 Two Actions on $S\times H$

Let $S\subseteq G$ be a cross-section for $H<G$ as in Lemma A.5. The properties of S imply that the map

$$\begin{aligned}\phi\colon S\times H&\longrightarrow G=SH\\ (s,h)&\longmapsto sh\end{aligned}$$

is a Borel isomorphism. Indeed, ϕ is continuous and the inverse can be obtained as $\phi^{-1}(g)=\bigl(p(gH),p(gH)^{-1}g\bigr)$, where $p\colon G/H\to S$ is the measurable selection map satisfying $gH\cap S=\{p(gH)\}$ for all $gH\in G/H$.

We now consider two actions on G and use ϕ to transport these to $S\times H$. Firstly we let H act on G on the right and this simply corresponds to the right action of H on the second factor in $S\times H$ (without changing the first coordinate).

Moreover, G acts on itself on the left which descends to the natural action of G on $G/H\cong S$. Within S we have the action $g\bullet s=p(gsH)$ for $g\in G$ and $s\in S$. The action of $g\in G$ on $S\times H$ therefore has the form

$$g\colon (s,h)\overset{\phi}{\longmapsto} sh \overset{g\bullet}{\longmapsto} gsh \overset{\phi^{-1}}{\longmapsto} \bigl(\underbrace{p(gsH)}_{=g\bullet s},\underbrace{p(gsH)^{-1}gs}_{=c(g,s)}\,h\bigr).$$

That is, g acts on the first coordinate $s\in S$ corresponding to the action of g on $G/H\cong S$ and simultaneously multiplies the second coordinate in H on the left by an element $c(g,s)\in H$ that depends on g and the first coordinate $s\in S$. We note that the function $c(\cdot,\cdot)$ is called an (exact) cocycle.

A.2.3 Existence and Uniqueness

Lemma A.6. *Let $H<G$ be as in Proposition* A.3 *and let $S\subseteq G$ be a cross-section for H. Using the Borel isomorphism $\phi\colon S\times H\to G$ the Haar measure m_G considered on $S\times H$ has the form $\mu\times m_H$ for a measure μ on S with $\mu(B_S)<\infty$ for any $B_S\subseteq S$ with compact closure.*

Proof. For a measurable set $B_S\subseteq S$ with compact closure we define

$$\nu(B)=m_G\bigl(\phi(B_S\times B)\bigr)$$

for measurable $B\subseteq H$. Using our discussions in Section A.2.2 and in particular the right action of H on G it follows that ν defines a (right) invariant measure on H. Moreover, if $B\subseteq H$ has compact closure then $\phi(B_S\times B)=B_SB$ has compact closure in G and $\nu(B)<\infty$ follows. If ν vanishes for a set $B\subseteq H$ with non-empty interior, then the invariance implies that $\nu(K)=0$ for any compact $K\subseteq H$ and hence $\nu(H)=0$ follows from σ-compactness.

The uniqueness of the Haar measure on H up to scalar multiples now implies in any case that $\nu=\mu(B_S)m_H$ for a scalar $\mu(B_S)\in[0,\infty)$. By varying B_S this defines a measure μ on S so that

$$m_G\bigl(\phi(B_S\times B)\bigr)=\mu(B_S)m_H(B)$$

for $B_S\subseteq S$ and $B\subseteq H$. Initially this holds for $B_S\subseteq S$ with compact closure, but σ-compactness of G means that this condition can be dropped. □

Proof of existence in Proposition A.3. Let μ be the measure on the quotient $S\cong G/H$ from Lemma A.6, let $f\geqslant 0$ be measurable on S and $g\in G$. Fix some compact $K\subseteq H$ with non-empty interior. Then

$$\begin{aligned}\int_S f(g\bullet s)\,\mathrm{d}\mu(s) &= \frac{1}{m_H(K)}\int_{S\times K} f(g\bullet s)\mathbb{1}_K(h)\,\mathrm{d}\mu\times m_H(s,h)\\ &= \frac{1}{m_H(K)}\int_{S\times K} f(g\bullet s)\mathbb{1}_K\bigl(c(g,s)h\bigr)\,\mathrm{d}\mu\times m_H(s,h)\end{aligned}$$

by Fubini's theorem and the invariance of m_H. Moreover $\mu\times m_H$ corresponds to m_G and the simultaneous action of g on $s\in S$ and multiplication of h on the left by $c(g,s)$ corresponds to left multiplication by g on G. Therefore we obtain

$$\int_S f(g\bullet s)\,\mathrm{d}\mu = \frac{1}{m_H(K)}\int_{S\times K} f(s)\mathbb{1}_K(h)\,\mathrm{d}\mu\times m_H(s,h) = \int_S f\,\mathrm{d}\mu.$$

Using the isomorphism $G/H\cong S$ we obtain a G-invariant measure μ on G/H.

Moreover, by construction $\mu(S_{\mathrm{local}}H)<\infty$. As $S_{\mathrm{local}}H$ has non-empty interior, invariance implies that $\mu(K)<\infty$ for any compact $K\subseteq G/H$. □

Proof of uniqueness in Proposition A.3. Suppose μ' is a non-trivial locally finite G-invariant measure on G/H. Since $S\cong G/H$ we can move μ' to S. Replacing μ as in Lemma A.6 by μ' we obtain a σ-finite G-invariant measure m' on $S\times H\cong G$—this follows from the same calculation as was used in the existence proof above. We claim that there exists a subset $B_H\subseteq H$ with compact closure such that $S_{\mathrm{local}}B_H$ has non-empty interior in G. Indeed by Lemma A.5 there exists an open subset $O\subseteq G$ with compact closure such that

$$OH\subseteq S_{\mathrm{local}}H.$$

Let $B_H=(S_{\mathrm{local}}^{-1}O)\cap H$. For $g\in O$ with $g=sh$, $s\in S_{\mathrm{local}}$, and $h\in H$ we then have $h\in B_H$ as claimed. As $\overline{S_{\mathrm{local}}H}$ is compact, it follows that $m'(S_{\mathrm{local}}B_H)$ is finite. Using invariance it follows that $m'(K)<\infty$ for all compact $K\subseteq G$. By uniqueness of Haar measure on G the measure m' is therefore a multiple of m_G, which shows that μ' is a multiple of the measure μ constructed above. □

Notes to Appendix A

(39)(Page 323) First countable topological spaces are not automatically metrizable. An example to see this is the *Sorgenfrey line* [147], the space $\mathbb{R}$ with the topology formed by using the half-open intervals $[a,b)$ with $a<b$ as basis. It is clear that this is first countable, since the sets $[a,a+\frac{1}{n})$ for $n\in\mathbb{N}$ form a countable neighbourhood basis at a. Much less clear is the fact that it is not metrizable, and we refer to Kelley [80] for the details.
(40)(Page 323) If d_ℓ is a left-invariant metric then $\mathsf{d}_r(x,y)=\mathsf{d}_\ell(x^{-1},y^{-1})$ is a right-invariant metric defining the same topology. A bi-invariant metric with

$$\mathsf{d}(xgy,xhy)=\mathsf{d}(g,h)$$

for all $x,y,g,h\in G$ only exists in special cases: We refer to [45, Lem. C.2] for the simple case that a compact metrizable group has a bi-invariant metric, and [45, Ex. C.3] for an explanation of why $\mathrm{GL}_2(\mathbb{C})$ has no bi-invariant metric. A striking result of Milnor [110] is that a connected

Lie group admits a bi-invariant metric if and only if it is isomorphic to $K \times \mathbb{R}^n$ for some compact Lie group K. The proof of Lemma A.2 given here is taken from the monograph of Montgomery and Zippin [112, Sec. 1.22] and Tao's blog [150].