# Chapter 9
# Unitary Representations of $\mathbf{SL}_2(\mathbb{R})$

In this chapter we will again focus our attention on $\mathrm{SL}_2(\mathbb{R})$ and its unitary representations. For this, we recall some of our previous discussions:

- We already classified all finite-dimensional representations of $\mathrm{SL}_2(\mathbb{R})$ in Section 6.1. This result, and even more so the method of proof, will be of importance here.
- In Section 8.3 we recalled the geometric significance of $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SU}_{1,1}(\mathbb{R})$ by connecting it to the hyperbolic plane.
- In Section 8.4 we already found our first two types of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$, namely the discrete series and the mock discrete series representations.
- In Section 8.5 we studied the regular representation of $\mathrm{SL}_2(\mathbb{R})$.
- This allowed us to characterize tempered unitary representations of $\mathrm{SL}_2(\mathbb{R})$ in terms of integrability and decay of matrix coefficients in Section 8.6.

We will extend these results here, to obtain a complete description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$. Moreover, we will decompose natural unitary representations into irreducible representations. In particular, we will study the Koopman representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{H})$, and see that the hyperbolic Fourier transform is intimately related to the principal series representations of $\mathrm{SL}_2(\mathbb{R})$.

## 9.1 The Universal Enveloping Algebra and Smooth Vectors

We recall that the Casimir operator for $\mathrm{SU}_2(\mathbb{R})$ in Proposition 7.19 and Corollary 7.20 commutes with $\mathrm{SU}_2(\mathbb{R})$. Because of the connection between $\mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ developed in Section 6.1.2, it stands to reason that $\mathrm{SL}_2(\mathbb{R})$ should also possess a Casimir operator. However, the Casimir operator of $\mathrm{SL}_2(\mathbb{R})$ will have 'mixed signature', and will not arise from an application of Proposition 7.19. After developing the necessary abstract machinery in this section, it will also be relatively straightforward to define the raising and lowering operators for any unitary representation of $\mathrm{SL}_2(\mathbb{R})$, which will lead to the description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ in Section 9.2.

### 9.1.1 The Universal Enveloping Algebra

We briefly introduce the algebra $\mathfrak{E}$ containing the Lie algebra $\mathfrak{g}$ of a Lie group $G$ as well as higher-order terms like the Casimir elements. We refer to Knapp [39, Ch. 3] for a more complete introduction to this concept.

**Definition 9.1 (The algebra $\mathfrak{E}$).** The *universal enveloping algebra* $\mathfrak{E}$ of a Lie algebra $\mathfrak{g}$ is the linear hull of all formal multi-linear associative non-commuting products $\mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n$ for $n \in \mathbb{N}_0$ and $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathfrak{g}$ modulo the ideal generated by the expressions $\mathbf{a}\circ\mathbf{b} - \mathbf{b}\circ\mathbf{a} - [\mathbf{a},\mathbf{b}]$ for $\mathbf{a},\mathbf{b} \in \mathfrak{g}$. By also allowing the empty product $\mathbb{1}_{\mathfrak{E}}$ (corresponding to $n = 0$) the algebra $\mathfrak{E}$ is also unital. We will write $\mathbf{b}^{\circ n} = \mathbf{b}\circ\mathbf{b}\circ\cdots\circ\mathbf{b}$ for powers of $\mathbf{b} \in \mathfrak{E}$ in the algebra $\mathfrak{E}$ for all $n \in \mathbb{N}$. We define the complexification of $\mathfrak{E}$ by $\mathfrak{E}_{\mathbb{C}} = \mathfrak{E} \otimes_{\mathbb{R}} \mathbb{C}$, which is also the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{g}$. For $\mathbf{a} + \mathrm{i}\mathbf{b} \in \mathfrak{E}_{\mathbb{C}}$ with $\mathbf{a}, \mathbf{b} \in \mathfrak{E}$, we also define $\overline{\mathbf{a} + \mathrm{i}\mathbf{b}} = \mathbf{a} - \mathrm{i}\mathbf{b}$.

Because of its definition, $\mathfrak{E}$ is not a graded algebra (that is, there is no good definition of homogeneous degree in $\mathfrak{E}$), but it can be written as an increasing union

$$\mathfrak{E} = \bigcup_{n=0}^{\infty} \mathfrak{E}_{\leqslant n},$$

where $\mathfrak{E}_{\leqslant 0} = \mathbb{C}\mathbb{1}_{\mathfrak{E}}$, $\mathfrak{E}_{\leqslant 1} = \mathfrak{E}_{\leqslant 0} + \mathfrak{g}$, and $\mathfrak{E}_{\leqslant n}$ is the subspace of $\mathfrak{E}$ generated by all products $\mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_k$ with $k \leqslant n$ and $\mathbf{b}_1, \dots, \mathbf{b}_k \in \mathfrak{g}$. We may think of $B \in \mathfrak{E}$ as having *degree less than or equal to* $k \in \mathbb{N}_0$ if $B \in \mathfrak{E}_{\leqslant k}$.

We also extend the adjoint representation to $\mathfrak{E}$, by defining

$$\mathrm{Ad}_g(\mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n) = (\mathrm{Ad}_g \mathbf{b}_1) \circ \cdots \circ (\mathrm{Ad}_g \mathbf{b}_n)$$

for all $g \in G$, $n \in \mathbb{N}_0$, and $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathfrak{g}$. We note that

$$\begin{aligned}\mathrm{Ad}_g(\mathbf{a}\circ\mathbf{b} - \mathbf{b}\circ\mathbf{a} - [\mathbf{a},\mathbf{b}]) &= (\mathrm{Ad}_g \mathbf{a})\circ(\mathrm{Ad}_g \mathbf{b}) - (\mathrm{Ad}_g \mathbf{b})\circ(\mathrm{Ad}_g \mathbf{a}) \\ &\qquad - \mathrm{Ad}_g([\mathbf{a},\mathbf{b}]) \\ &= (\mathrm{Ad}_g \mathbf{a})\circ(\mathrm{Ad}_g \mathbf{b}) - (\mathrm{Ad}_g \mathbf{b})\circ(\mathrm{Ad}_g \mathbf{a}) \\ &\qquad - [\mathrm{Ad}_g(\mathbf{a}), \mathrm{Ad}_g(\mathbf{b})]\end{aligned}$$

for $g \in G$ and $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$ (see (6.9)). Hence the adjoint action of $G$ on formal products sends the ideal appearing in Definition 9.1 to itself, and we obtain a well-defined representation of $G$ on $\mathfrak{E}$, and by restriction also on $\mathfrak{E}_{\leqslant n}$ for all $n \in \mathbb{N}_0$.

By the discussion in Section 6.1.3, we may also take the derivative of the adjoint representation of $G$ on $\mathfrak{E}$ (or, equivalently, on $\mathfrak{E}_{\leqslant n}$ for all $n \in \mathbb{N}_0$) to obtain a representation of $\mathfrak{g}$ on $\mathfrak{E}$. We will again call this representation the adjoint representation, and denote it by

$$\begin{aligned} \mathrm{ad}\colon \mathfrak{g} &\longrightarrow \mathrm{End}(\mathfrak{E}) \\ \mathbf{c} &\longmapsto \mathrm{ad}_{\mathbf{c}}\,. \end{aligned}$$

For $n \in \mathbb{N}$, $\mathbf{b}_1, \ldots, \mathbf{b}_n, \mathbf{c} \in \mathfrak{g}$ the adjoint representation is defined by

$$\begin{aligned} \mathrm{ad}_{\mathbf{c}}(\mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \bigl(\mathrm{Ad}_{\exp(t\mathbf{c})}\,\mathbf{b}_1\bigr) \circ \cdots \circ \bigl(\mathrm{Ad}_{\exp(t\mathbf{c})}\,\mathbf{b}_n\bigr) \\ &= (\mathrm{ad}_{\mathbf{c}}\,\mathbf{b}_1) \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n + \cdots + \mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ (\mathrm{ad}_{\mathbf{c}}\,\mathbf{b}_n). \end{aligned} \tag{9.1}$$

This generalized product rule follows, for example, from the approximation formula

$$\mathrm{Ad}_{\exp(t\mathbf{c})}(\mathbf{b}_j) = \mathbf{b}_j + t\,\mathrm{ad}_{\mathbf{c}}\,\mathbf{b}_j + \mathrm{o}(t)$$

for $t \to 0$ and $j \in \{1, \ldots, n\}$, by expanding the product above, and letting $t$ go to zero.

The reader who wants to see an abstract argument for the existence of the Casimir element in $\mathfrak{E}_{\leqslant 2}$ for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ belonging to the centre of $\mathfrak{E}$, or who hopes to find more central elements in $\mathfrak{E}_{\leqslant n}$ for bigger values of $n \in \mathbb{N}$, is invited to solve the following exercise.

**Exercise 9.2.** Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$.
(a) For every $n \in \mathbb{N}$, calculate the weight of $\mathbf{e}^{\circ n} \in \mathfrak{E}_{\leqslant n}$, where

$$\mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$$

has weight 2. Show that all other eigenvectors of $\mathfrak{E}_{\leqslant n}$ have smaller weight. Conclude that $\mathfrak{E}_{\leqslant n}$ contains a $(2n+1)$-dimensional irreducible invariant subspace that is not contained in $\mathfrak{E}_{\leqslant n-1}$.
(b) Using the notation from (6.3), show that $\mathfrak{E}_{\leqslant 2}$ has $\mathbb{1}_{\mathfrak{E}}, \mathbf{a}, \mathbf{e}, \mathbf{f}, \mathbf{a}^{\circ 2}, \mathbf{a}\circ\mathbf{e}, \mathbf{a}\circ\mathbf{f}, \mathbf{e}^{\circ 2}, \mathbf{e}\circ\mathbf{f}, \mathbf{f}^{\circ 2}$ as a basis, and conclude that $\mathfrak{E}_{\leqslant 2}$ contains an element $\Omega$ that is not contained in $\mathfrak{E}_{\leqslant 1}$ and is fixed by $\mathfrak{sl}_2(\mathbb{R})$.
(c) Calculate the dimension of $\mathfrak{E}_{\leqslant 3}$ and $\mathfrak{E}_{\leqslant 4}$, analyze the representation appearing and show that the centre of $\mathfrak{E}$ intersected with $\mathfrak{E}_{\leqslant 4}$ is the linear hull of $\mathbb{1}, \Omega, \Omega^{\circ 2}$.
(d) Repeat (c) for all $\mathfrak{E}_{\leqslant 2n}$ with $n \in \mathbb{N}$ to see that the centre of $\mathfrak{E}$ is the linear hull of $\mathbb{1}_{\mathfrak{E}}$ and $\Omega^{\circ n}$ for $n \in \mathbb{N}$.

### 9.1.2 Higher-order Differential Operators

**Proposition 9.3 (Differential operators coming from $\mathfrak{E}$).** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\pi$ be a unitary representation of $G$. Then the representation of $\mathfrak{g}$ via $\pi_\partial$ on smooth vectors extends to a representation* *of the universal enveloping algebra* $\mathfrak{E}$ *of* $\mathfrak{g}$ *on smooth vectors in such a way that*

$$\pi_\partial(\mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n) = \pi_\partial(\mathbf{b}_1) \cdots \pi_\partial(\mathbf{b}_n)$$

*for all* $n \in \mathbb{N}$ *and* $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathfrak{g}$*. Moreover,* $\pi_\partial$ *also extends to a representation of the complexification* $\mathfrak{E}_{\mathbb{C}}$*.*

Proof. Let $v \in \mathcal{H}_\pi$ be smooth. Recall that by Lemma 7.3 the vector $\pi_\partial(\mathbf{b})v$ depends linearly on $\mathbf{b} \in \mathfrak{g}$. This extends to multilinear dependence of $\pi_\partial(\mathbf{b}_1) \cdots \pi_\partial(\mathbf{b}_n)v$ on $\pi_\partial(\mathbf{b}_1), \dots, \pi_\partial(\mathbf{b}_n) \in \mathfrak{g}$. Therefore $\pi_\partial$ extends from $\mathfrak{g}$ to the algebra of formal non-commuting products of elements of $\mathfrak{g}$ as appearing in Definition 9.1.

However, to see that $\pi_\partial$ extends to $\mathfrak{E}$ we have to show that $\pi_\partial$ sends the ideal appearing in Definition 9.1 to zero, or equivalently that

$$\pi_\partial\bigl([\mathbf{a}, \mathbf{b}]\bigr) = \bigl(\pi_\partial(\mathbf{a})\pi_\partial(\mathbf{b}) - \pi_\partial(\mathbf{b})\pi_\partial(\mathbf{a})\bigr) \tag{9.2}$$

for all $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$. To see this, we fix a vector $w \in \mathcal{H}_\pi$ and look at the matrix coefficient $\varphi = \varphi^\pi_{w,v}$. Let us use right-translation by elements of $G$ to define a vector field $\lambda_\partial(\mathbf{m})$ on $G$ for every $\mathbf{m} \in \mathfrak{g}$, as in Proposition 7.5. Smoothness of $v$ and Lemma 7.16 show that

$$\begin{aligned}\lambda_\partial(\mathbf{m})\varphi(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi^\pi_{w,v}\bigl(\exp(-t\mathbf{m})g\bigr) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \bigl\langle \pi_{\exp(-t\mathbf{m})}\pi_g w, v\bigr\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \bigl\langle \pi_g w, \pi_{\exp(t\mathbf{m})} v\bigr\rangle \\ &= \bigl\langle \pi_g w, \pi_\partial(\mathbf{m})v\bigr\rangle = \varphi^\pi_{w,\pi_\partial(\mathbf{m})v}(g)\end{aligned}$$

exists for all $\mathbf{m} \in \mathfrak{g}$ and $g \in G$, which, when iterated, shows that $\varphi \in C^\infty(G)$. However, for smooth functions on $G$, the formula

$$\lambda_\partial([\mathbf{a}, \mathbf{b}])\varphi = \lambda_\partial(\mathbf{a})\lambda_\partial(\mathbf{b})\varphi - \lambda_\partial(\mathbf{b})\lambda_\partial(\mathbf{a})\varphi \tag{9.3}$$

is the definition of the Lie bracket for general Lie groups (see Exercise 9.4), and at $g = e$, together with the above, this becomes

$$\begin{aligned}\bigl\langle w, \pi_\partial\bigl([\mathbf{a}, \mathbf{b}]\bigr)v\bigr\rangle &= \lambda_\partial([\mathbf{a}, \mathbf{b}])\varphi^\pi_{w,v}(e) \\ &= \bigl(\lambda_\partial(\mathbf{a})\lambda_\partial(\mathbf{b})\varphi^\pi_{w,v} - \lambda_\partial(\mathbf{b})\lambda_\partial(\mathbf{a})\varphi^\pi_{w,v}\bigr)(e) \\ &= \bigl(\lambda_\partial(\mathbf{a})\varphi^\pi_{w,\pi_\partial(\mathbf{b})v} - \lambda_\partial(\mathbf{b})\varphi^\pi_{w,\pi_\partial(\mathbf{a})v}\bigr)(e) \\ &= \langle w, \pi_\partial(\mathbf{a})\pi_\partial(\mathbf{b})v\rangle - \langle w, \pi_\partial(\mathbf{b})\pi_\partial(\mathbf{a})v\rangle\end{aligned}$$

As this holds for all $w \in \mathcal{H}_\pi$ we obtain (9.2).

For $\mathbf{a}, \mathbf{b} \in \mathfrak{E}$, we define $\pi_\partial(\mathbf{a}+\mathrm{i}\mathbf{b}) = \pi_\partial(\mathbf{a})+\mathrm{i}\pi_\partial(\mathbf{b})$. This gives an extension of the above representation to $\mathfrak{E}_{\mathbb{C}}$ satisfying linearity over $\mathbb{C}$ in addition to the already established properties: if $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2 \in \mathfrak{E}$, then

$$(\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1) \circ (\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2) = \mathbf{a}_1 \circ \mathbf{a}_2 - \mathbf{b}_1 \circ \mathbf{b}_2 + \mathrm{i}(\mathbf{a}_1 \circ \mathbf{b}_2 + \mathbf{b}_1 \circ \mathbf{a}_2),$$

and hence

$$\begin{aligned}\pi_\partial\big((\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1) \circ (\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2)\big) &= \pi_\partial\big(\mathbf{a}_1 \circ \mathbf{a}_2 - \mathbf{b}_1 \circ \mathbf{b}_2\big) + \mathrm{i}\pi_\partial\big(\mathbf{a}_1 \circ \mathbf{b}_2 + \mathbf{b}_1 \circ \mathbf{a}_2\big)\\ &= \pi_\partial(\mathbf{a}_1)\pi_\partial(\mathbf{a}_2) - \pi_\partial(\mathbf{b}_1)\pi_\partial(\mathbf{b}_2)\\ &\qquad + \mathrm{i}\big(\pi_\partial(\mathbf{a}_1)\pi_\partial(\mathbf{b}_2) - \pi_\partial(\mathbf{b}_1)\pi_\partial(\mathbf{a}_2)\big)\\ &= \pi_\partial(\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1)\pi_\partial(\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2).\end{aligned}$$

□

**Exercise 9.4.** Prove (9.3) for closed linear groups, where the Lie bracket is defined by (6.1) using matrix products.

**Corollary 9.5 (Adjoint representation and formal adjoint of differential operators).** *Let $G$ be a closed linear group with Lie algebra $\mathfrak{g}$ and let $\pi$ be a unitary representation of $G$. Then*

$$\pi_g \pi_\partial(\mathbf{e})\pi_{g^{-1}} = \pi_\partial\big(\mathrm{Ad}_g(\mathbf{e})\big) \tag{9.4}$$

*for all elements $\mathbf{e}$ of the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{g}$ and $g \in G$. Moreover, there exists a conjugate linear anti-homomorphism ${}^*\colon \mathfrak{E}_\mathbb{C} \to \mathfrak{E}_\mathbb{C}$ with*

$$\begin{cases}\mathbf{a}^* = -\mathbf{a},\\ (\alpha\mathbf{e})^* = \overline{\alpha}\mathbf{e}^*,\\ (\mathbf{e}\circ\mathbf{f})^* = \mathbf{f}^* \circ \mathbf{e}^*,\\ \langle \pi_\partial(\mathbf{e})u, v\rangle = \big\langle u, \pi_\partial(\mathbf{e}^*)v\big\rangle\end{cases}$$

*for all $\mathbf{a} \in \mathfrak{g}$, $\mathbf{e},\mathbf{f} \in \mathfrak{E}_\mathbb{C}$, $\alpha \in \mathbb{C}$, and smooth $u, v \in \mathcal{H}_\pi$.*

Proof. For $g \in G$ and $\mathbf{e} \in \mathfrak{g}$, the identity in (9.4) is simply the chain rule in Lemma 7.16. Moreover, if $\mathbf{e} = \mathbf{a}_1 \circ \cdots \circ \mathbf{a}_n \in \mathfrak{E}$ then we also have

$$\begin{aligned}\pi_g \pi_\partial(\mathbf{e})\pi_{g^{-1}} &= \pi_g \pi_\partial(\mathbf{a}_1)\pi_{g^{-1}}\pi_g\pi_\partial(\mathbf{a}_2)\pi_{g^{-1}} \cdots \pi_g\pi_\partial(\mathbf{a}_n)\pi_{g^{-1}}\\ &= \pi_\partial\big(\mathrm{Ad}_g(\mathbf{a}_1)\big)\pi_\partial\big(\mathrm{Ad}_g(\mathbf{a}_2)\big)\cdots\pi_\partial\big(\mathrm{Ad}_g(\mathbf{a}_n)\big)\\ &= \pi_\partial\big(\mathrm{Ad}_g(\mathbf{a}_1)\circ\cdots\circ\mathrm{Ad}_g(\mathbf{a}_n)\big) = \pi_\partial\big(\mathrm{Ad}_g(\mathbf{e})\big).\end{aligned}$$

This proves (9.4). We define ${}^*\colon \mathfrak{g} \ni \mathbf{a} \mapsto -\mathbf{a} \in \mathfrak{g}$ and extend ${}^*$ to a map from $\mathfrak{E}$ to $\mathfrak{E}$ by requiring

$$\big(\mathbf{e}_1 \circ \mathbf{e}_2\big)^* = \mathbf{e}_2^* \circ \mathbf{e}_1^* \tag{9.5}$$

for all $\mathbf{e}_1, \mathbf{e}_2 \in \mathfrak{E}$.

To see that ${}^*$ is well-defined, we verify that the relations appearing in Definition 9.4 are preserved. For $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$ we have $\mathbf{a}^* = -\mathbf{a}$, $\mathbf{b}^* = -\mathbf{b}$, and that $[\mathbf{a},\mathbf{b}] \in \mathfrak{g}$ satisfies $[\mathbf{a},\mathbf{b}]^* = -[\mathbf{a},\mathbf{b}]$. Together with (9.5), we obtain that

$$\begin{aligned}\big(\mathbf{a}\circ\mathbf{b}-\mathbf{b}\circ\mathbf{a}-[\mathbf{a},\mathbf{b}]\big)^* &= (-\mathbf{b})\circ(-\mathbf{a})-(-\mathbf{a})\circ(-\mathbf{b})+[\mathbf{a},\mathbf{b}]\\ &= \mathbf{b}\circ\mathbf{a}-\mathbf{a}\circ\mathbf{b}+[\mathbf{a},\mathbf{b}]\end{aligned}$$

indeed is once more in the ideal appearing in the definition of $\mathfrak{E}$. Finally, we may extend $^*$ conjugate-linearly from $\mathfrak{E}$ to $\mathfrak{E}_{\mathbb{C}}=\mathfrak{E}+\mathrm{i}\,\mathfrak{E}$, so that

$$(\mathbf{e}+\mathrm{i}\mathbf{f})^*=\mathbf{e}^*-\mathrm{i}\mathbf{f}^*$$

for all real $\mathbf{e},\mathbf{f}\in\mathfrak{E}$.

Now let $u,v\in\mathcal{H}_\pi$ be smooth and $\mathbf{a}\in\mathfrak{g}$. Then

$$\begin{aligned}\langle\pi_\partial(\mathbf{a})u,v\rangle &= \lim_{t\to 0}\big\langle \tfrac{1}{t}\big(\pi(\exp(t\mathbf{a}))u-u\big),v\big\rangle\\ &= \lim_{t\to 0}\tfrac{1}{t}\big(\langle u,\pi(\exp(-t\mathbf{a}))v\rangle-\langle u,v\rangle\big)\\ &= \lim_{t\to 0}\big\langle u,\tfrac{1}{t}\big(\pi(\exp(-t\mathbf{a}))v-v\big)\big\rangle\\ &= \langle u,-\pi_\partial(\mathbf{a})v\rangle=\big\langle u,\pi_\partial(\mathbf{a}^*)v\big\rangle\,.\end{aligned}$$

Iterating this, and combining it with sesqui-linearity of the inner product, the corollary follows. □

We will show in Section 9.2.1 that the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{sl}_2(\mathbb{R})$ has a central element, to which we will apply the following general result.

**Proposition 9.6 (Operators coming from the centre of $\mathfrak{E}$).** *Let $G$ be a closed linear group with Lie algebra $\mathfrak{g}$, and let $\Omega$ be a central element of the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{g}$. Then, for any unitary representation $\pi$ of $G$, the closure of $\pi_\partial(\Omega)$ (defined on smooth vectors) is a well-defined closed equivariant operator. If $\Omega^*=\Omega$, then the closure is a self-adjoint operator. If $\pi$ is irreducible, then the closure is multiplication by a scalar $\alpha_{\Omega,\pi}\in\mathbb{C}$ (respectively $\alpha_{\Omega,\pi}\in\mathbb{R}$ if $\Omega^*=\Omega$ also).*

Proof. Let $\Omega\in\mathfrak{E}$ be central and let $\pi$ be a unitary representation as in the proposition. We define $T_\pi$ as the closure of $\pi_\partial(\Omega)$ acting on smooth vectors. More formally, we have that $v\in D_{T_\pi}$ belongs to the domain of $T_\pi$, and $T_\pi v=w$ is the image, if there exists a sequence $(v_n)$ in $\mathcal{H}_\pi$ of smooth vectors with

$$\big(v_n,\pi_\partial(\Omega)v_n\big)\in\mathrm{Graph}\big(\pi_\partial(\Omega)\big)$$

converging to $(v,w)$ as $n\to\infty$. To see that this defines a well-defined operator, we need to show that

$$(v,w)=\lim_{n\to\infty}\big(v_n,\pi_\partial(\Omega)v_n\big)\in\overline{\mathrm{Graph}\big(\pi_\partial(\Omega)\big)}$$

with $v=0$ implies that $w=0$. To see this, suppose $u\in\mathcal{H}_\pi$ is smooth, then

$$\langle w,u\rangle=\lim_{n\to\infty}\langle\pi_\partial(\Omega)v_n,u\rangle=\lim_{n\to\infty}\big\langle v_n,\pi_\partial(\Omega^*)u\big\rangle\,,$$

where we applied Corollary 9.5. Since $\lim_{n\to\infty} v_n = 0$ by assumption, we obtain $\langle w, u\rangle = 0$ for all smooth $u \in \mathcal{H}_\pi$. Since smooth vectors are dense by Proposition 7.5, it follows that $w = 0$.

Suppose now that $v \in D_{T_\pi}$ so that, by definition,

$$(v, T_\pi v) = \lim_{n\to\infty} (v_n, \pi_\partial(\Omega) v_n) \in \mathrm{Graph}(T_\pi),$$

and fix some $g \in G$. Then

$$\pi_g \pi_\partial(\Omega) v_n = \pi_\partial \underbrace{\big(\mathrm{Ad}_g(\Omega)\big)}_{=\Omega} \pi_g v = \pi_\partial(\Omega) \pi_g v_n$$

by Corollary 9.5 and our assumptions on $\Omega$.

Since $v_n \to v$ and $\pi_\partial(\Omega) v_n \to T_\pi v$ as $n \to \infty$, we see that $\pi_g v_n \to \pi_g v$ and

$$\pi_\partial(\Omega) \pi_g v_n = \pi_g \pi_\partial(\Omega) v_n \longrightarrow \pi_g T_\pi v$$

as $n \to \infty$. Therefore

$$\big(\pi_g v, \pi_g T_\pi v\big) = \lim_{n\to\infty} \big(\pi_g v_n, \pi_\partial(\Omega) \pi_g v_n\big),$$

which shows that $T_\pi \pi_g v = \pi_g T_\pi v$ for all $g \in G$ and $v \in D_{T_\pi}$. In other words, $T_\pi$ is a well-defined closed equivariant operator.

We note that if $\pi$ is irreducible then Schur's lemma in the form of Corollary 1.36 implies that $T_\pi$ is simply multiplication by some $\lambda_\pi \in \mathbb{C}$. If, in addition, $\Omega^* = \Omega$ and $v \in \mathcal{H}_\pi$ is a smooth unit vector, then

$$\lambda_\pi = \lambda_\pi \langle v, v\rangle = \langle \pi_\partial(\Omega) v, v\rangle = \big\langle v, \pi_\partial(\Omega^*) v\big\rangle = \overline{\lambda_\pi} \langle v, v\rangle = \overline{\lambda_\pi}$$

shows that $\lambda_\pi \in \mathbb{R}$. We note that this case suffices (for instance) for the classification of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$ in the next section.

For completeness we now drop the assumption of irreducibility but still suppose that $\Omega^* = \Omega$ , and will show that $T_\pi$ is in this case a closed self-adjoint operator. For this we define the operator

$$B = P_1 P_{\mathrm{Graph}(T_\pi)} \imath_1,$$

where

$$\imath_1 \colon \mathcal{H}_\pi \ni v \longmapsto (v, 0) \in \mathcal{H}_\pi^2$$

is the embedding of $\mathcal{H}_\pi$ into the first factor, $P_{\mathrm{Graph}(T_\pi)}$ is the orthogonal projection onto the closed subspace $\mathrm{Graph}(T_\pi) \subseteq \mathcal{H}_\pi^2$ and

$$P_1 \colon \mathcal{H}_\pi^2 \ni (v_1, v_2) \longmapsto v_1 \in \mathcal{H}_\pi$$

is the projection onto the first factor.

Since the adjoint of $\imath_1$ is $P_1$, it follows that $B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ is self-adjoint. Note that if $P_{\mathrm{Graph}(T_\pi)}(v,0) = (\widetilde{v}, T_\pi\widetilde{v})$ for some $v \in \mathcal{H}_\pi$, then $Bv = \widetilde{v} \in D_{T_\pi}$. From this and $(\widetilde{v}, T_\pi\widetilde{v}) - (v,0) \in \mathrm{Graph}(T_\pi)^\perp$, it follows that

$$\langle Bv, v\rangle = \langle \widetilde{v}, v\rangle = \langle (\widetilde{v}, T_\pi\widetilde{v}), (v,0)\rangle = \langle (\widetilde{v}, T_\pi\widetilde{v}), (\widetilde{v}, T_\pi\widetilde{v})\rangle = \|(\widetilde{v}, T_\pi\widetilde{v})\|^2_{\mathcal{H}^2_\pi}$$

belongs to $[0, \|v\|^2]$. Since $B$ is self-adjoint, this proves that the spectrum of $B$ is contained in $[0,1]$. Moreover, suppose now that $Bv = \widetilde{v} = 0$. Recalling that by definition $(\widetilde{v}, T_\pi\widetilde{v}) = (0,0)$ is the orthogonal projection of $(v,0)$ onto $\mathrm{Graph}(T_\pi)$, density of the domain of $D_{T_\pi} \supseteq D_{\pi_\partial(\Omega)}$ now forces $v = 0$. This implies that $B$ is injective.

Recall that $T_\pi$ is equivariant by the first part of the proof. Hence $\mathrm{Graph}(T_\pi)$ is invariant, and $B$ is also equivariant. We use the measurable functional calculus for $B$ to define

$$\mathcal{V}_n = \bigl(\mathbb{1}_{(\frac{1}{n+1}, \frac{1}{n}]}\bigr)_{\mathrm{FC}}\mathcal{H}_\pi.$$

Hence

$$\mathcal{H}_\pi = \bigl(\mathbb{1}_{(0,1]}\bigr)_{\mathrm{FC}}\mathcal{H}_\pi = \bigoplus_{n\in\mathbb{N}} \mathcal{V}_n \tag{9.6}$$

is a decomposition into closed invariant subspaces. Indeed, the measurable functional calculus also shows that all operators that commute with $B$ (for example, $\pi_g$ for any $g \in G$) also commute with any operators constructed by the measurable functional calculus. Since the spectrum of $B$ is contained in $[0,1]$, but the kernel of $B$ is trivial, this gives (9.6).

Let $\pi_n = \pi|_{\mathcal{V}_n}$. Then $B|_{\mathcal{V}_n}\colon \mathcal{V}_n \to \mathcal{V}_n$ has a bounded inverse defined on all of $\mathcal{V}_n$, which shows that for any $\widetilde{v} \in \mathcal{V}_n$ we have that

$$(\widetilde{v}, T_\pi\widetilde{v}) = P_{\mathrm{Graph}(T_\pi)}(B^{-1}\widetilde{v}, 0) \in \mathrm{Graph}(T_\pi)$$

and

$$\|(\widetilde{v}, T_\pi\widetilde{v})\|_{\mathcal{H}^2_\pi} \leqslant \|B^{-1}\widetilde{v}\| \leqslant (n+1)\widetilde{v}.$$

Therefore, $\mathcal{V}_n \subseteq D_{T_\pi}$ for all $n \in \mathbb{N}$.

Let $v = \sum_{n\in\mathbb{N}} v_n \in \mathcal{H}_\pi$ be smooth with $v_n = P_{v_n} v \in \mathcal{V}_n$ for all $n \in \mathbb{N}$, where $P_{v_n}\colon \mathcal{H}_\pi \to \mathcal{V}_n$ is the (equivariant) orthogonal projection. Then $v_n$ is smooth for $\pi_n$ for all $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}$ we have that $v_n \in \mathcal{V}_n$ is smooth for $\pi_n$ if and only if $v_n$ is smooth for $\pi$. Using the definition

$$\mathrm{Graph}(T_\pi) = \overline{\mathrm{Graph}\bigl(\pi_\partial(\Omega)\bigr)},$$

this implies that $T_\pi|_{\mathcal{V}_n} = T_{\pi_n}$. Using $\Omega^* = \Omega$, Corollary 9.5 implies that $T_{\pi_n}$ is a bounded self-adjoint operator.

We now define the subspace

$$D = \left\{ \sum_{n=1}^\infty v_n \,|\, v_n \in \mathcal{V}_n \text{ for } n \in \mathbb{N}, \sum_{n=1}^\infty \|v_n\|^2 < \infty, \text{ and} \sum_{n=1}^\infty \|T_{\pi_n} v_n\|^2 < \infty \right\}.$$

For $\sum_{n=1}^\infty v_n \in D$ and any $N \in \mathbb{N}$ we have $\sum_{n=1}^N v_n \in D_{T_\pi}$ and

$$\left(\sum_{n=1}^N v_n, \sum_{n=1}^N T_{\pi_n} v_n\right) \in \mathrm{Graph}(T_\pi).$$

Letting $N \to \infty$, and using the fact that $\mathrm{Graph}(T_\pi)$ is closed, we obtain that $D \subseteq D_{T_\pi}$.

Assume now that, on the other hand, $v \in \mathrm{Graph}(T_\pi)$. We may write

$$v = \sum_{n=1}^\infty v_n$$

for some $v_n \in \mathcal{V}_n$ for $n \in \mathbb{N}$, with $\sum_{n=1}^\infty \|v_n\|^2 < \infty$. Projecting $(v, T_\pi v)$ onto $\mathcal{V}_n \oplus \mathcal{V}_n \subseteq \mathcal{H}_\pi \oplus \mathcal{H}_\pi$ and noting again that the projection of a smooth vector in $\mathcal{H}_\pi$ to $\mathcal{V}_n$ is a smooth vector in $\mathcal{V}_n$, it follows that

$$T_\pi v = \sum_{n=1}^\infty T_{\pi_n} v_n$$

and hence $\sum_{n=1}^\infty \|T_{\pi_n} v_n\|^2 < \infty$. This shows that $D = D_{T_\pi}$. Since the operator $T_{\pi_n} \colon \mathcal{V}_n \to \mathcal{V}_n$ are bounded and self-adjoint, it is now a standard exercise (see [21, Ch. 3] and Exercise 9.7) to show that $T_\pi$ is a closed self-adjoint operator. □

**Exercise 9.7.** Let $\mathcal{H}_n$ be a Hilbert space and $T_n \colon \mathcal{H}_n \to \mathcal{H}_n$ a self-adjoint bounded operator for all $n \in \mathbb{N}$. Then

$$T\left(\sum_{n=1}^\infty v_n\right) = \sum_{n=1}^\infty T_n v_n$$

for all $\sum_{n=1}^\infty v_n \in \bigoplus_{n \geqslant 1} \mathcal{H}_n$ with $\sum_{n=1}^\infty \|T_n v_n\|^2 < \infty$ defines a self-adjoint operator.

## 9.2 Casimir, Raising, Lowering, and the Dual of $\mathbf{SL_2}(\mathbb{R})$

We now specialize and extend the material from the previous section in the case of $\mathrm{SL}_2(\mathbb{R})$. Our final goal of the section is a description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$.

### 9.2.1 The Casimir Element and Operator

We let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and will use the elements $\mathbf{a}, \mathbf{e}, \mathbf{f}$ as in (6.3) and, in addition,

$$\mathbf{d} = \mathbf{e} + \mathbf{f} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$$

and

$$\mathbf{k} = -\mathbf{e} + \mathbf{f} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}_2(\mathbb{R}).$$

With these abbreviations we define the *Casimir element* for $\mathfrak{sl}_2(\mathbb{R})$ as

$$\begin{aligned} \Omega &= \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{d}^{\circ 2} - \mathbf{k}^{\circ 2} \qquad (9.7) \\ &= \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + (\mathbf{e}+\mathbf{f}) \circ (\mathbf{e}+\mathbf{f}) - (-\mathbf{e}+\mathbf{f}) \circ (-\mathbf{e}+\mathbf{f}) \\ &= \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{e}^{\circ 2} + \mathbf{e} \circ \mathbf{f} + \mathbf{f} \circ \mathbf{e} + \mathbf{f}^{\circ 2} - \big(\mathbf{e}^{\circ 2} - \mathbf{e} \circ \mathbf{f} - \mathbf{f} \circ \mathbf{e} + \mathbf{f}^{\circ 2}\big) \\ &= \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + 2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}. \qquad (9.8) \end{aligned}$$

We note that $\mathrm{Ad}_g(\mathbb{1}_{\mathfrak{E}}) = \mathbb{1}_{\mathfrak{E}}$ for all $g \in G$, and $\mathrm{ad}_{\mathbf{m}}(\mathbb{1}_{\mathfrak{E}}) = 0$ for all $\mathbf{m} \in \mathfrak{g}$. Hence we obtain by the product rule in (9.1) that

$$\begin{aligned} \mathrm{ad}_{\mathbf{a}}(\Omega) &= \mathrm{ad}_{\mathbf{a}}\big(\mathbf{a}^{\circ 2} + 2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}\big) \\ &= 2(\mathrm{ad}_{\mathbf{a}}\,\mathbf{e}) \circ \mathbf{f} + 2\mathbf{e} \circ (\mathrm{ad}_{\mathbf{a}}\,\mathbf{f}) + 2(\mathrm{ad}_{\mathbf{a}}\,\mathbf{f}) \circ \mathbf{e} + 2\mathbf{f} \circ \mathrm{ad}_{\mathbf{a}}(\mathbf{e}) \\ &= 2(2\mathbf{e}) \circ \mathbf{f} + 2\mathbf{e} \circ (-2\mathbf{f}) + 2(-2\mathbf{f}) \circ \mathbf{e} + 2\mathbf{f} \circ (2\mathbf{e}) = 0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{ad}_{\mathbf{e}}(\Omega) &= \mathrm{ad}_{\mathbf{e}}\big(\mathbf{a}^{\circ 2} + 2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}\big) \\ &= \mathrm{ad}_{\mathbf{e}}(\mathbf{a}) \circ \mathbf{a} + \mathbf{a} \circ \mathrm{ad}_{\mathbf{e}}(\mathbf{a}) + 2\mathbf{e} \circ (\mathrm{ad}_{\mathbf{e}}\,\mathbf{f}) + 2(\mathrm{ad}_{\mathbf{e}}\,\mathbf{f}) \circ \mathbf{e} \\ &= -2\mathbf{e} \circ \mathbf{a} - 2\mathbf{a} \circ \mathbf{e} + 2\mathbf{e} \circ \mathbf{a} + 2\mathbf{a} \circ \mathbf{e} = 0. \end{aligned}$$

The calculation $\mathrm{ad}_{\mathbf{f}}(\Omega) = 0$ is similar, but also follows from the properties of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$ in Section 6.1.4 applied to ad on $\mathfrak{E}_{\leqslant 2}$ (since $\mathrm{ad}_{\mathbf{a}}(\Omega) = \mathrm{ad}_{\mathbf{e}}(\Omega) = 0$ shows that $\Omega \in \mathfrak{E}_{\leqslant 2}$ is a highest weight vector with weight zero, and so generates the trivial representation of $\mathfrak{sl}_2(\mathbb{R})$).

It follows from this that Proposition 9.6 applies to unitary representations of $\mathrm{SL}_2(\mathbb{R})$ as in Corollary 9.8 below. Indeed, the Casimir element of $\mathrm{SL}_2(\mathbb{R})$ satisfies $\Omega^* = \Omega$ by its definition in (9.7).

**Corollary 9.8 (Casimir operator).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ and let $\Omega \in \mathfrak{E}$ be the Casimir element in the centre of the enveloping algebra $\mathfrak{E}$ of $\mathfrak{sl}_2(\mathbb{R})$ defined in (9.7). Then the closure $T_\pi$ of $\pi_\partial(\Omega)$ is a self-adjoint equivariant operator. If $\pi$ is in addition irreducible, then $T_\pi = \alpha_\pi I$ for some $\alpha_\pi \in \mathbb{R}$.*

We also note that our definition of $\pi_\partial(\Omega)$ with $\Omega$ as in (9.7) formally matches the definition of the Casimir operator in Corollary 7.20. Indeed, there we used the basis vectors

$$\mathbf{b}_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \mathrm{i}\mathbf{a}, \mathbf{b}_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\mathbf{d}, \mathbf{b}_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{k}$$

of $\mathrm{SU}_2(\mathbb{R}) \subseteq \mathrm{SL}_2(\mathbb{C})$, and hence

$$\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{d}^{\circ 2} - \mathbf{k}^{\circ 2} = \mathbb{1}_{\mathfrak{E}} - \mathbf{b}_1^{\circ 2} - \mathbf{b}_2^{\circ 2} - \mathbf{b}_3^{\circ 2}$$

within the complex universal enveloping algebra $\mathfrak{E}_{\mathbb{C}}$ of $\mathfrak{sl}_2(\mathbb{C})$.

### 9.2.2 The Raising and Lowering Operators

We define[†] the elements $\mathbf{r}^+ = \frac{1}{2}(\mathbf{a} - \mathrm{i}\mathbf{d})$ and $\mathbf{r}^- = \frac{1}{2}(\mathbf{a} + \mathrm{i}\mathbf{d})$ of $\mathfrak{sl}_2(\mathbb{C})$, and note that $\overline{\mathbf{r}^+} = \mathbf{r}^-$. We calculate

$$\mathrm{ad}_{\mathbf{k}}(\mathbf{a}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = 2\mathbf{d}$$

and

$$\mathrm{ad}_{\mathbf{k}}(\mathbf{d}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -2\mathbf{a},$$

which gives

$$\mathrm{ad}_{\mathbf{k}}(\mathbf{r}^+) = \mathbf{d} - \mathrm{i}(-\mathbf{a}) = \mathrm{i}(\mathbf{a} - \mathrm{i}\mathbf{d}) = 2\mathrm{i}\mathbf{r}^+$$

and, by conjugation,

$$\mathrm{ad}_{\mathbf{k}}(\mathbf{r}^-) = -2\mathrm{i}\mathbf{r}^-$$

also. With $\mathrm{Ad}_{\exp(\theta\mathbf{k})} = \exp(\mathrm{ad}_{\theta\mathbf{k}})$ for $\theta \in \mathbb{R}$, this also implies

$$\begin{cases} \mathrm{Ad}_{k_\theta}(\mathbf{r}^+) = \mathrm{e}^{2\mathrm{i}\theta}\mathbf{r}^+ \\ \mathrm{Ad}_{k_\theta}(\mathbf{r}^-) = \mathrm{e}^{-2\mathrm{i}\theta}\mathbf{r}^- \end{cases} \tag{9.9}$$

for all $k_\theta \in K$.

For a unitary representation $\pi$ of $\mathrm{SL}_2(\mathbb{R})$, we will call $\pi_\partial(\mathbf{r}^+)$ the *raising operator* and $\pi_\partial(\mathbf{r}^-)$ the *lowering operator*.

**Proposition 9.9 (Raising and lowering operators).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. For any smooth $K$-eigenvector $v_n \in \mathcal{H}_\pi$ of weight $n \in \mathbb{Z}$ the vector $\pi_\partial(\mathbf{r}^+)v_n$ has weight $n+2$ and $\pi_\partial(\mathbf{r}^-)v_n$ has weight $n-2$.*

Proof. Let $k_\theta \in K$ and let $v_n \in \mathcal{H}_\pi$ be a smooth vector of $K$-weight $n$. Then the chain rule in Lemma 7.16 and (9.9) implies that

[†] It is an unfortunate coincidence that $\mathrm{i} \in \mathbb{C}$ multiplied by $\mathbf{d} \in \mathfrak{sl}_2(\mathbb{R})$, giving $\mathrm{i}\mathbf{d}$, is notationally close to a familiar 'identity' notation, id.

$$\begin{aligned}\pi_{k_\theta}\pi_\partial(\mathbf{r}^+)v_n &= \pi_\partial\big(\mathrm{Ad}_{k_\theta}(\mathbf{r}^+)\big)\pi_{k_\theta}v_n\\ &= \pi_\partial\big(\mathrm{e}^{2\mathrm{i}\theta}\mathbf{r}^+\big)\mathrm{e}^{\mathrm{i}n\theta}v_n\\ &= \mathrm{e}^{\mathrm{i}(n+2)\theta}\pi_\partial(\mathbf{r}^+)v_n,\end{aligned}$$

and the calculation for $\pi_\partial(\mathbf{r}^-)v_n$ is identical. □

We also wish to relate the elements $\mathbf{r}^\pm$ to the Casimir operator in (9.7). By definition of $\mathbf{r}^+$ and $\mathbf{r}^-$, we have

$$\begin{aligned}\mathbf{a} &= \mathbf{r}^+ + \mathbf{r}^-,\\ \mathbf{d} &= \mathrm{i}(\mathbf{r}^+ - \mathbf{r}^-),\end{aligned}$$

and

$$\begin{aligned}[\mathbf{r}^+,\mathbf{r}^-] &= \tfrac{1}{4}[\mathbf{a}-\mathrm{i}\mathbf{d},\mathbf{a}+\mathrm{i}\mathbf{d}]\\ &= \tfrac{1}{4}\left(\mathrm{i}[\mathbf{a},\mathbf{d}]-\mathrm{i}[\mathbf{d},\mathbf{a}]\right)\\ &= \tfrac{\mathrm{i}}{2}[\mathbf{a},\mathbf{d}] = -\mathrm{i}\mathbf{k} \qquad (9.10)\end{aligned}$$

since

$$\begin{aligned}[\mathbf{a},\mathbf{d}] &= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} - \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\\ &= \begin{pmatrix}0&1\\-1&0\end{pmatrix} - \begin{pmatrix}0&-1\\1&0\end{pmatrix} = -2\mathbf{k}.\end{aligned}$$

Using (9.7), we now obtain

$$\begin{aligned}\varOmega &= \mathbb{1}_{\mathfrak{E}} + (\mathbf{r}^+ + \mathbf{r}^-)\circ(\mathbf{r}^+ + \mathbf{r}^-) - (\mathbf{r}^+ - \mathbf{r}^-)\circ(\mathbf{r}^+ - \mathbf{r}^-) - \mathbf{k}^{\circ 2}\\ &= \mathbb{1}_{\mathfrak{E}} + (\mathbf{r}^+\circ\mathbf{r}^+ + \mathbf{r}^+\circ\mathbf{r}^- + \mathbf{r}^-\circ\mathbf{r}^+ + \mathbf{r}^-\circ\mathbf{r}^-)\\ &\qquad - (\mathbf{r}^+\circ\mathbf{r}^+ - \mathbf{r}^+\circ\mathbf{r}^- - \mathbf{r}^-\circ\mathbf{r}^+ + \mathbf{r}^-\circ\mathbf{r}^-) - \mathbf{k}^{\circ 2}\\ &= \mathbb{1}_{\mathfrak{E}} + 2\mathbf{r}^+\circ\mathbf{r}^- + 2\mathbf{r}^-\circ\mathbf{r}^+ - \mathbf{k}^{\circ 2}.\end{aligned}$$

We can also write this in the form

$$\varOmega = \mathbb{1}_{\mathfrak{E}} + 4\mathbf{r}^+\circ\mathbf{r}^- - 2[\mathbf{r}^+,\mathbf{r}^-] - \mathbf{k}^{\circ 2}.$$

Using (9.10), this gives

$$\varOmega = 4\mathbf{r}^+\circ\mathbf{r}^- + \underbrace{\mathbb{1}_{\mathfrak{E}} + 2\mathrm{i}\mathbf{k} - \mathbf{k}^{\circ 2}}_{(\mathbb{1}_{\mathfrak{E}}+\mathrm{i}\mathbf{k})^{\circ 2}}.$$

Therefore

$$\varOmega = 4\mathbf{r}^+\circ\mathbf{r}^- + (\mathbb{1}_{\mathfrak{E}}+\mathrm{i}\mathbf{k})^{\circ 2} = 4\mathbf{r}^-\circ\mathbf{r}^+ + (\mathbb{1}_{\mathfrak{E}}-\mathrm{i}\mathbf{k})^{\circ 2}, \qquad (9.11)$$

where the second formula follows from the first by conjugation.

**Corollary 9.10 (Norm of raised and lowered vectors).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ so that the closure of $\pi_\partial(\Omega)$ is multiplication by $\alpha_\pi \in \mathbb{R}$. Then for any smooth vector $v_n \in \mathcal{H}_\pi$ with $K$-weight $n \in \mathbb{Z}$, we have*

$$\begin{cases} \|\pi_\partial(\mathbf{r}^+)v_n\|^2 & = \frac{1}{4}\big((n+1)^2 - \alpha_\pi\big)\|v_n\|^2 \\ \|\pi_\partial(\mathbf{r}^-)v_n\|^2 & = \frac{1}{4}\big((n-1)^2 - \alpha_\pi\big)\|v_n\|^2. \end{cases}$$

To make these two formulas more memorable, we note that in both cases $n \pm 1$ is precisely the weight that lies in between the weight $n$ of $v_n$ and the weight $n \pm 2$ of the vector $\pi_\partial(\mathbf{r}^\pm)v_n$.

Proof of Corollary 9.10. We apply the formal adjoint in Corollary 9.5 and (9.11) in the form

$$\mathbf{r}^- \circ \mathbf{r}^+ = \frac{1}{4}\left(\Omega - (\mathbb{1}_{\mathfrak{E}} - \mathrm{i}\mathbf{k})^{\circ 2}\right),$$

which gives

$$\begin{aligned} \|\pi_\partial(\mathbf{r}^+)v_n\|^2 &= \left\langle \pi_\partial(\mathbf{r}^+)v_n, \pi_\partial(\mathbf{r}^+)v_n \right\rangle \\ &= -\left\langle \pi_\partial(\mathbf{r}^- \circ \mathbf{r}^+)v_v, v_n \right\rangle \\ &= -\frac{1}{4}\left\langle \pi_\partial(\Omega - (\mathbb{1}_{\mathfrak{E}} - \mathrm{i}\mathbf{k})^{\circ 2})v_n, v_n \right\rangle. \end{aligned}$$

Using

$$\pi_\partial(\mathbb{1}_{\mathfrak{E}} - \mathrm{i}\mathbf{k})v_n = v_n - \mathrm{i}\pi_\partial(\mathbf{k})v_n = v_n + nv_n = (n+1)v_n,$$

this gives

$$\|\pi_\partial(\mathbf{r}^+)v_n\|^2 = -\frac{1}{4}\left(\alpha_\pi - (n+1)^2\right)\|v_n\|^2.$$

Similarly,

$$\begin{aligned} \|\pi_\partial(\mathbf{r}^-)v_n\|^2 &= \left\langle \pi_\partial(\mathbf{r}^-)v_n, \pi_\partial(\mathbf{r}^-)v_n \right\rangle \\ &= -\left\langle \pi_\partial(\mathbf{r}^+ \circ \mathbf{r}^-)v_v, v_n \right\rangle \\ &= -\frac{1}{4}\left\langle \pi_\partial(\Omega - (\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})^{\circ 2})v_n, v_n \right\rangle \\ &= -\frac{1}{4}\left(\alpha_\pi - (n-1)^2\right)\|v_n\|^2. \end{aligned}$$

□

We note that the raising and lowering operators will be very useful for proving irreducibility of the principal and complementary series representations in the next sections. In fact we already used them implicitly in the proofs of irreducibility of the discrete and mock discrete series representations of $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SU}_{1,1}(\mathbb{R})$ in Section 8.4. More importantly, we will use the more abstract framework above involving the Casimir operator to classify all irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$. In fact Corollary 9.10 already implies some restrictions on the Casimir eigenvalue $\alpha_\pi$: If $v_n$ as in the corollary is non-zero, then we must have

$$\alpha_\pi \leqslant (n \pm 1)^2. \tag{9.12}$$

Applying our above discussion to the irreducible unitary representation of Section 8.4, we obtain their Casimir eigenvalue.

**Corollary 9.11 (Casimir for discrete and mock discrete series).** *For every integer $n \geqslant 2$, we have*

$$\alpha_{\delta^{n,+}} = \alpha_{\delta^{n,-}} = (n-1)^2$$

*for the discrete series representations $\delta^{n,\pm}$. Similarly, we have*

$$\alpha_{\delta^{1,+}} = \alpha_{\delta^{1,-}} = 0$$

*for the two mock discrete series representations $\delta^{1,\pm}$.*

PROOF. Let $n \geqslant 2$ and write $e_0$ for the constant function in $A_n(\mathbb{D})$ which has $K$-weight $n$ by Lemma 8.22. The proof of irreducibility in Theorem 8.23 shows that $e_0$ is smooth (see also Lemma 9.13). By Proposition 9.9 we know that $\delta_\partial^{n,+}(\mathbf{r}^-)e_0$ has weight $n-2$, and by Corollary 9.10 we have

$$\|\delta_\partial^{n,+}(\mathbf{r}^-)e_0\| = \tfrac{1}{4}\left((n-1)^2 - \alpha_{\delta^{n,+}}\right)\|e_0\|.$$

However, by Theorem 8.23, there is no vector of weight $n-2$. Hence $\delta_\partial^{n,+}(\mathbf{r}^-)e_0$ must be trivial, and so

$$\alpha_{\delta^{n,+}} = (n-1)^2,$$

as claimed. The argument for the mock discrete series representation $\delta^{1,+}$ uses Theorem 8.30, but is otherwise identical.

This implies the same formulas for the contragredient representations $\delta^{n,-}$ for $n \geqslant 2$ and for $\delta^{1,-}$. □

**Exercise 9.12.** Explain the last step in the proof of Corollary 9.11 more carefully.

### 9.2.3 Smooth $K$-finite Vectors

Because of the argument above, the following lemma is useful for the study of general unitary representations of $\mathrm{SL}_2(\mathbb{R})$.

**Lemma 9.13 (Smooth $K$-finite vectors).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. Then the subspace of smooth $K$-finite vectors is dense in $\mathcal{H}_\pi$. Moreover, if for some $n \in \mathbb{Z}$ the space of $K$-eigenvectors in $\mathcal{H}_\pi$ of weight $n$ is finite-dimensional, then every $K$-eigenvector of weight $n$ is smooth.*

PROOF. Let $v \in \mathcal{H}_\pi$ and let $\psi \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$. By Proposition 7.5 we know that $\pi_*(\psi)v$ is a smooth vector. Now let $n \in \mathbb{Z}$ and let $\chi_n(k_\theta) = \mathrm{e}^{\mathrm{i}n\theta}$ for $k_\theta$ in $K$ be the $n$th character on $K$. Recall that $\bigl(\pi|_K\bigr)_*\bigl(\overline{\chi_n}\bigr)w$ is a $K$-eigenvector of weight $n$ for all $w \in \mathcal{H}_\pi$. Combining these two with $w = \pi_*(\psi)v$, we obtain using Fubini's theorem and the substitution $h = k_\theta g$ that

$$\begin{aligned}\bigl(\pi|_K\bigr)_*\bigl(\overline{\chi_n}\bigr)\pi_*(\psi)v &= \int_K \overline{\chi_n}(k_\theta)\pi_{k_\theta}\int_G \psi(g)\pi_g v\,\mathrm{d}m(g)\,\mathrm{d}m_K(k_\theta)\\ &= \int_K\int_G \overline{\chi_n}(k_\theta)\psi(g)\underbrace{\pi_{k_\theta g}v}_{=\pi_h v}\,\mathrm{d}m(g)\,\mathrm{d}m_K(k_\theta)\\ &= \int_G \underbrace{\int_K \overline{\chi_n}(k_\theta)\psi(k_\theta^{-1}h)\,\mathrm{d}m_K(k_\theta))}_{\psi_n(h)}\,\pi_h(v)\,\mathrm{d}m(h)\end{aligned}$$

where

$$\psi_n(h) = \int_K \overline{\chi_n}(k_\theta)\psi(k_\theta^{-1}h)\,\mathrm{d}m_K(k_\theta)$$

for $h \in \mathrm{SL}_2(\mathbb{R})$. We note that $\psi_n$ has compact support with

$$\operatorname{supp}\psi_n \subseteq K\operatorname{supp}\psi.$$

Moreover, $\psi_n$ is also smooth, which follows for instance by considering the derivatives

$$\rho_{\mathbf{m}}(\psi_n) = \lim_{s\to 0}\frac{1}{s}\bigl(\rho_{\exp(\mathbf{m})}\psi_n - \psi_n\bigr)$$

for the right-regular representation and proving that

$$\rho_{\mathbf{m}}(\psi_n) = \int_K \overline{\chi_n}(k_\theta)(\rho_{\mathbf{m}}\psi)(k_\theta^{-1}h)\,\mathrm{d}m_K(k_\theta)$$

for all $\mathbf{m} \in \mathfrak{sl}_2(\mathbb{R})$. Hence Proposition 7.5 shows that

$$\bigl(\pi|_K\bigr)_*\bigl(\overline{\chi_n}\bigr)\pi_*(\psi)v = \pi_*(\psi_n)v$$

is a smooth vector. The first claim in the lemma now follows from

$$\pi_*(\psi)v = \sum_{n\in\mathbb{Z}} \bigl(\pi|_K\bigr)_* \bigl(\overline{\chi_n}\bigr)\pi_*(\psi)v = \lim_{N\to\infty} \sum_{n=-N}^{N} \pi_*(\psi_n)v.$$

Fix some $n \in \mathbb{Z}$ and suppose now that

$$\mathcal{V}_n = \{v \in \mathcal{H}_\pi \mid v \text{ has } K\text{-weight } n\}$$

is finite-dimensional. Using Proposition 1.47 for a smooth approximate identity and each vector in a basis of $\mathcal{V}_n$, we can find some $\psi \in C_c^\infty(G)$ such that $\pi_*(\psi)v$ is close to $v$ for each of the basis vectors $v$ of $\mathcal{V}_n$. Since $(\pi|_K)_*(\overline{\chi_n})$ is the orthogonal projection onto $\mathcal{V}_n$, it follows that

$$\pi_*(\psi_n)|_{\mathcal{V}_n} \colon \mathcal{V}_n \longrightarrow \mathcal{V}_n$$

is as close to the identity on $\mathcal{V}_n$ as we desire. In particular, we may ensure that $\pi_*(\psi_n)(\mathcal{V}_n) = \mathcal{V}_n$, and the final claim of the lemma follows from the first part of the proof. □

### 9.2.4 A Differential Equation for Matrix Coefficients

We continue our journey towards the description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ by examining matrix coefficients of $K$-eigenvectors in irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$. In fact we now show how the Casimir eigenvalue $\alpha_\pi$ for some unitary representation $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$, and the mere existence of a smooth unit vector $v \in \mathcal{H}_\pi$ of $K$-weight $n$, determines its matrix coefficient $\varphi_v^\pi$.

**Lemma 9.14 (The differential equation for the matrix coefficient).** *Let $\pi$ be a unitary representation of the group $\mathrm{SL}_2(\mathbb{R})$. Suppose that $v \in \mathcal{H}_\pi$ is a smooth $K$-eigenvector with weight $n \in \mathbb{Z}$ and Casimir eigenvalue $\alpha \in \mathbb{R}$. Then*

$$\varphi_v^\pi\bigl(k_\theta a_t k_\psi\bigr) = \mathrm{e}^{\mathrm{i}n(\theta+\psi)}\varphi_v^\pi(a_t)$$

*for all $k_\theta a_t k_\psi \in \mathrm{SL}_2(\mathbb{R})$ and the smooth function $\phi\colon \mathbb{R} \to \mathbb{C}$ defined by*

$$\phi(t) = \varphi_v^\pi(a_t)$$

*for $t \in \mathbb{R}$ satisfies the second order linear differential equation*

$$\phi''(t) + 2\tfrac{\cosh(2t)}{\sinh(2t)}\phi'(t) + \Bigl(1 - \alpha + \tfrac{n^2}{\cosh^2 t}\Bigr)\phi(t) = 0$$

*of degree two for all $t \in \mathbb{R}\smallsetminus\{0\}$. Moreover, $\phi$ is real-valued and satisfies*

$$\phi(t) = \phi(-t)$$

*for all* $t \in \mathbb{R}$.

Proof. The first part of the lemma follows simply because $v$ is a $K$-eigenvector of weight $n$. Indeed, we have

$$\varphi_v^\pi\big(k_\theta a_t k_\psi\big) = \Big\langle \pi_{a_t}\pi_{k_\psi}v, \pi_{k_{-\theta}}v\Big\rangle = \Big\langle \pi_{a_t}(\mathrm{e}^{\mathrm{i}n\psi}v), \mathrm{e}^{-\mathrm{i}n\theta}v\Big\rangle = \mathrm{e}^{\mathrm{i}n(\theta+\psi)}\varphi_v^\pi(a_t)$$

for all $k_\theta a_t k_\psi \in \mathrm{SL}_2(\mathbb{R})$. We note that

$$k_{\pi/2}a_t k_{-\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = a_{-t},$$

for all $t \in \mathbb{R}$. For $\phi$ this implies

$$\phi(-t) = \varphi_v^\pi(a_{-t}) = \varphi_v^\pi\big(k_{\pi/2}a_t k_{-\pi/2}\big) = \varphi_v^\pi(a_t) = \phi(t)$$

for all $t \in \mathbb{R}$ by Lemma 9.14. In other words, $\phi$ is an even function on $\mathbb{R}$. Moreover,

$$\overline{\phi(t)} = \overline{\langle \pi_{a_t}v, v\rangle} = \langle v, \pi_{a_t}v\rangle = \langle \pi_{a_{-t}}v, v\rangle = \phi(-t) = \phi(t)$$

for all $t \in \mathbb{R}$, which also shows that $\phi$ only takes values in $\mathbb{R}$.

To obtain the differential equation for $\phi(t) = \varphi_v^\pi(a_t)$ for $t \in \mathbb{R}\smallsetminus\{0\}$, we will combine the consequence above of $v$ having $K$-weight $n \in \mathbb{Z}$, the assumption $\pi_\partial(\Omega)v = \alpha v$, and the formula

$$\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + 2\mathbf{e}\circ\mathbf{f} + 2\mathbf{f}\circ\mathbf{e}$$

in (9.8). Recall also that $v$ having $K$-weight $n$ implies that $\pi_\partial(\mathbf{k})v = \mathrm{i}nv$ and $\pi_\partial(\mathbf{k}^{\circ 2})v = -n^2 v$.

We start by calculating how to express $\phi'$ and $\phi''$ as matrix coefficients. Indeed,

$$\begin{aligned} \phi'(t) &= \lim_{h\to 0}\frac{1}{h}\big(\phi(t+h) - \phi(t)\big) \\ &= \lim_{h\to 0}\frac{1}{h}\left\langle \pi_{a_t}(\pi_{a_h}v - v), v\right\rangle = \left\langle \pi_{a_t}\pi_\partial(\mathbf{a})v, v\right\rangle \end{aligned}$$

and, similarly,

$$\phi''(t) = \lim_{h\to 0}\frac{1}{h}\left\langle \pi_{a_t}\big(\pi_{a_h}\pi_\partial(\mathbf{a})v - \pi_\partial(\mathbf{a})v\big), v\right\rangle = \Big\langle \pi_{a_t}\pi_\partial(\mathbf{a}^{\circ 2})v, v\Big\rangle$$

for all $t \in \mathbb{R}$.

We note that $\mathbf{a}^{\circ 2}$ is one of the terms in $\Omega$, but that we do not yet have an interpretation of the term $2\mathbf{e}\circ\mathbf{f} + 2\mathbf{f}\circ\mathbf{e}$ in terms of $\phi$. To obtain such an interpretation, we use the consequence of $v$ having $K$-weight $n$ mentioned above to express $-n^2\phi$ in three different ways. Indeed, we have

$$\begin{aligned}
-n^2\phi(t) &= -n^2\left\langle \pi_{a_t}v, v\right\rangle = \left\langle \pi_{a_t}(-n^2 v), v\right\rangle \\
&= \left\langle \pi_{a_t}\pi_\partial(\mathbf{k}^{\circ 2})v, v\right\rangle; \\
-n^2\phi(t) &= -\left\langle \pi_{a_t} \mathrm{i}nv, \mathrm{i}nv\right\rangle = -\left\langle \pi_{a_t}\pi_\partial(\mathbf{k})v, \pi_\partial(\mathbf{k})v\right\rangle = \left\langle \pi_\partial(\mathbf{k})\pi_{a_t}\pi_\partial(\mathbf{k})v, v\right\rangle \\
&= \left\langle \pi_{a_t}\pi_\partial\bigl(\mathrm{Ad}_{a_{-t}}(\mathbf{k})\circ\mathbf{k}\bigr)v, v\right\rangle; \\
-n^2\phi(t) &= \left\langle \pi_{a_t}v, -n^2 v\right\rangle = \left\langle \pi_{a_t}v, \pi_\partial(\mathbf{k}^{\circ 2})v\right\rangle = \left\langle \pi_\partial(\mathbf{k}^{\circ 2})\pi_{a_t}v, v\right\rangle \\
&= \left\langle \pi_{a_t}\pi_\partial\Bigl(\bigl(\mathrm{Ad}_{a_{-t}}(\mathbf{k})\bigr)^{\circ 2}\Bigr)v, v\right\rangle
\end{aligned}$$

for all $t\in\mathbb{R}$. We recall that $\mathbf{k} = -\mathbf{e}+\mathbf{f}$ and note that

$$\mathrm{Ad}_{a_{-t}}(\mathbf{k}) = -\mathrm{e}^{-2t}\mathbf{e} + \mathrm{e}^{2t}\mathbf{f}$$

for all $t\in\mathbb{R}$. We put these two formulas into the three expressions above, expand the resulting parentheses, and obtain from this that

$$\begin{aligned}
-n^2\phi(t) &= \left\langle \pi_{a_t}\pi_\partial\bigl(\mathbf{e}^{\circ 2} - \mathbf{e}\circ\mathbf{f} - \mathbf{f}\circ\mathbf{e} + \mathbf{f}^{\circ 2}\bigr)v, v\right\rangle \\
&= \left\langle \pi_{a_t}\pi_\partial\bigl(\mathrm{e}^{-2t}\mathbf{e}^{\circ 2} - \mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f} - \mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e} + \mathrm{e}^{2t}\mathbf{f}^{\circ 2}\bigr)v, v\right\rangle \\
&= \left\langle \pi_{a_t}\pi_\partial\bigl(\mathrm{e}^{-4t}\mathbf{e}^{\circ 2} - \mathbf{e}\circ\mathbf{f} - \mathbf{f}\circ\mathbf{e} + \mathrm{e}^{4t}\mathbf{f}^{\circ 2}\bigr)v, v\right\rangle
\end{aligned}$$

for all $t\in\mathbb{R}$.

As our aim is to find a formula involving $2\mathbf{e}\circ\mathbf{f} + 2\mathbf{f}\circ\mathbf{e}$, we wish to rid ourselves of all expressions involving the two terms $\mathbf{e}^{\circ 2}$ and $\mathbf{f}^{\circ 2}$. As we have three formulas for $-n^2\phi$, this is an exercise in linear algebra. In fact, multiplying the first line by 1, the second by $-\bigl(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\bigr)$, the third by 1, and taking the sum gives

$$-n^2\phi(t)\bigl(1-\bigl(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\bigr)+1\bigr) = n^2\phi(t)\bigl(\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}\bigr) = 4n^2\sinh^2 t\,\phi(t)$$

on the left-hand side.

On the right-hand side, we use the similarities between the three formulas and obtain the matrix coefficient

$$\left\langle \pi_{a_t}\pi_\partial(\mathbf{m}_t)v, v\right\rangle,$$

where $\mathbf{m}_t$ is the element of $\mathfrak{E}_{\leqslant 2}$ defined by

$$\begin{aligned}
\mathbf{m}_t = \bigl(&\mathbf{e}^{\circ 2} - \mathbf{e}\circ\mathbf{f} - \mathbf{f}\circ\mathbf{e} + \mathbf{f}^{\circ 2}\bigr) \\
&- \bigl(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\bigr)\bigl(\mathrm{e}^{-2t}\mathbf{e}^{\circ 2} - \mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f} - \mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e} + \mathrm{e}^{2t}\mathbf{f}^{\circ 2}\bigr) \\
&+ \bigl(\mathrm{e}^{-4t}\mathbf{e}^{\circ 2} - \mathbf{e}\circ\mathbf{f} - \mathbf{f}\circ\mathbf{e} + \mathrm{e}^{4t}\mathbf{f}^{\circ 2}\bigr).
\end{aligned}$$

Our choice of the three coefficients 1, $-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)$, and 1 was made so that the coefficient in front of $\mathbf{e}^{\circ 2}$ is

$$1-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)\mathrm{e}^{-2t}+\mathrm{e}^{-4t}=1-1-\mathrm{e}^{-4t}+\mathrm{e}^{-4t}=0$$

and the coefficient in front of $\mathbf{f}^{\circ 2}$ is

$$1-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)\mathrm{e}^{2t}+\mathrm{e}^{4t}=1-\mathrm{e}^{4t}-1+\mathrm{e}^{4t}=0$$

also. Therefore,

$$\begin{aligned}\mathbf{m}_t &= \big(-\mathbf{e}\circ\mathbf{f}-\mathbf{f}\circ\mathbf{e}\big)-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)\big(-\mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f}-\mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e}\big)+\big(-\mathbf{e}\circ\mathbf{f}-\mathbf{f}\circ\mathbf{e}\big)\\ &= \big(-1+1+\mathrm{e}^{-4t}-1\big)\mathbf{e}\circ\mathbf{f}+\big(-1+\mathrm{e}^{4t}+1-1\big)\mathbf{f}\circ\mathbf{e}\\ &= \big(\mathrm{e}^{-4t}-1\big)\mathbf{e}\circ\mathbf{f}+\big(\mathrm{e}^{4t}-1\big)\mathbf{f}\circ\mathbf{e}.\end{aligned}$$

Using

$$\begin{aligned}\mathrm{e}^{4t}-1&=\mathrm{e}^{2t}\big(\mathrm{e}^{2t}-\mathrm{e}^{-2t}\big)=2\mathrm{e}^{2t}\sinh(2t),\\ \mathrm{e}^{-4t}-1&=\mathrm{e}^{-2t}\big(\mathrm{e}^{2t}-\mathrm{e}^{-2t}\big)=-2\mathrm{e}^{-2t}\sinh(2t),\end{aligned}$$

we also have

$$\mathbf{m}_t=-2\mathrm{e}^{-2t}\sinh(2t)\mathbf{e}\circ\mathbf{f}+2\mathrm{e}^{2t}\sinh(2t)\mathbf{f}\circ\mathbf{e}.$$

To summarize, we have shown that

$$\begin{aligned}4n^2\sinh^2 t\,\phi(t)&=\big\langle\pi_{a_t}\pi_\partial(\mathbf{m}_t)v,v\big\rangle\\ &=2\sinh(2t)\Big\langle\pi_{a_t}\pi_\partial\big(-\mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f}+\mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e}\big)v,v\Big\rangle.\end{aligned}$$

Dividing by $2\sinh(2t)=4\sinh t\cosh t$, we obtain

$$n^2\frac{\sinh t}{\cosh t}\phi(t)=\Big\langle\pi_{a_t}\pi_\partial\big(-\mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f}+\mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e}\big)v,v\Big\rangle.$$

Next we note that for $s_1,s_2\in\mathbb{R}$, we have

$$s_1\mathbf{e}\circ\mathbf{f}+s_2\mathbf{f}\circ\mathbf{e}=\frac{s_1+s_2}{2}(\mathbf{e}\circ\mathbf{f}+\mathbf{f}\circ\mathbf{e})+\frac{s_1-s_2}{2}\underbrace{[\mathbf{e},\mathbf{f}]}_{=\mathbf{a}},$$

which, with the choice $s_1=-\mathrm{e}^{-2t}$ and $s_2=\mathrm{e}^{2t}$, gives

$$\begin{aligned}n^2\frac{\sinh t}{\cosh t}\phi(t)&=\left\langle\pi_{a_t}\pi_\partial\left(\frac{\mathrm{e}^{2t}-\mathrm{e}^{-2t}}{2}(\mathbf{e}\circ\mathbf{f}+\mathbf{f}\circ\mathbf{e})-\frac{\mathrm{e}^{2t}+\mathrm{e}^{-2t}}{2}\mathbf{a}\right)v,v\right\rangle\\ &=\frac{\sinh(2t)}{2}\big\langle\pi_{a_t}\pi_\partial(2\mathbf{e}\circ\mathbf{f}+2\mathbf{f}\circ\mathbf{e})v,v\big\rangle-\cosh(2t)\phi'(t).\end{aligned}$$

Dividing by $\frac{\sinh(2t)}{2} = \sinh t \cosh t$, we also obtain

$$\left\langle \pi_{a_t} \pi_{\partial}(2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e})v, v \right\rangle = \frac{n^2}{\cosh^2 t}\phi(t) + 2\frac{\cosh(2t)}{\sinh(2t)}\phi'(t).$$

Using the relations $\pi_{\partial}(\Omega)v = \alpha v$ and $\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + 2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}$, we now obtain

$$\begin{aligned} \alpha\phi(t) &= \left\langle \pi_{a_t}(\alpha v), v \right\rangle \\ &= \left\langle \pi_{a_t}\pi_{\partial}(\Omega)v, v \right\rangle \\ &= \phi(t) + \phi''(t) + \left\langle \pi_{a_t}\pi_{\partial}(2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e})v, v \right\rangle \\ &= \phi''(t) + \left(1 + \frac{n^2}{\cosh^2 t}\right)\phi(t) + 2\frac{\cosh(2t)}{\sinh(2t)}\phi'(t), \end{aligned}$$

and hence the lemma. □

**Proposition 9.15 (Determining the matrix coefficient).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. Suppose that $v \in \mathcal{H}_\pi$ is a smooth $K$-eigenvector with weight $n \in \mathbb{Z}$ and Casimir eigenvalue $\alpha \in \mathbb{R}$. Then the matrix coefficient $\varphi_v^\pi$ is uniquely determined by $\alpha$, $n$, and $\|v\|$.*

We would like to apply the uniqueness part of the theorem of Picard–Lindelöf(16) to the function $\phi(t) = \varphi_v^\pi(a_t)$ for $t \in \mathbb{R}$ introduced in Lemma 9.14. In fact $\phi(0) = \|v\|^2$ and $\phi'(0) = 0$ (see Exercise 9.16) give two initial conditions for the second-order differential equation satisfied by $\phi$. Unfortunately, applying the Picard–Lindelöf theorem is not straightforward, as the differential equation is really only defined on the domain $\mathbb{R}\smallsetminus\{0\}$.

**Exercise 9.16.** Show that the function $\phi$ defined in Lemma 9.14 satisfies $\phi'(0) = 0$.

Proof of Proposition 9.15. Let $\pi$, $v$, $\alpha$, $n$, and $\phi$ be as in Lemma 9.14. We briefly discuss the structure of all real-valued solutions to the second-order linear differential equation

$$y'' + 2\tfrac{\cosh(2t)}{\sinh(2t)}y' + \left(1 - \alpha + \tfrac{n^2}{\cosh^2 t}\right)y = 0 \tag{9.13}$$

for $t \in \mathbb{R}\smallsetminus\{0\}$. Restricting to the connected component $(-\infty, 0)$ of $\mathbb{R}\smallsetminus\{0\}$, a corollary of the theorem of Picard–Lindelöf shows that

$$\mathscr{F} = \{y\colon (-\infty,0) \to \mathbb{R} \mid y \text{ solves (9.13) for all } t \in (-\infty, 0)\}$$

is a two-dimensional real vector space. In fact this corresponds to the independence and sufficiency (to determine the unique solution) of the two initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$ for some $t_0 < 0$. By Lemma 9.14 we have

$$y_v = \phi|_{(-\infty,0)} \in \mathscr{F}.$$

We claim that $\mathscr{F}$ contains an element $y_\infty$ with

$$\lim_{t\nearrow 0} y'_\infty(t) = +\infty. \tag{9.14}$$

Assuming the claim for now, we see that $y_v, y_\infty \in \mathscr{F}$ are linearly independent, and so they form a basis for $\mathscr{F}$. Suppose now that $\tau$ and $w$ are another unitary representation and vector as in Lemma 9.14 with the same $K$-weight $n$, the same Casimir eigenvalue $\alpha$, and the same norm $\|v\| = \|w\|$. Then

$$y_w(t) = \varphi_w^\tau(a_{-t})$$

for $t \in (-\infty, 0)$ defines another element of $\mathscr{F}$. It follows that

$$y_w = s_v y_v + s_\infty y_\infty$$

for some $s_v, s_\infty \in \mathbb{R}$. Recall that $\varphi_v^\pi(a_t)$ and $\varphi_w^\tau(a_t)$ defining $y_v$ and $y_w$ are both even by Lemma 9.14. If $s_\infty \neq 0$ we may use (9.14) to obtain the contradiction

$$0 = \lim_{t\nearrow 0} y'_w(t) = \lim_{t\nearrow 0} |s_v y'_v + s_\infty y'_\infty| = \infty.$$

Hence $s_\infty = 0$, and so

$$\|w\|^2 = \lim_{t\nearrow 0} y_w(t) = \lim_{t\nearrow 0} s_v y_v(t) = s_v \|v\|^2$$

shows that $s_v = 1$, and so $y_w = y_v$.

Using Lemma 9.14, we see that the claim implies the proposition.

It remains to construct $y_\infty \in \mathscr{F}$ as in the claim. We note that since the coefficient $2\frac{\cosh 2t}{\sinh 2t}$ of $y'$ in the differential equation (9.13) goes to infinity in absolute value as $t \to \infty$, it is natural to expect the claim to hold. The following elementary proof of the claim precisely relies on this property of (9.13).

To bound the effect of the term involving $y$ in (9.13), we define

$$M = \max_{t\in[-1,0]} \left| 1 - \alpha + \frac{n^2}{\cosh^2 t} \right|$$

and choose $t_0 \in [-1, 0)$ so that

$$\left| \frac{\cosh(2t)}{\sinh(2t)} \right| \geqslant M + 1$$

for all $t \in [t_0, 0)$. Using the existence part of the theorem of Picard–Lindelöf, we define $y_\infty$ as the solution of (9.13) on $(-\infty, 0)$ with the initial value conditions

$$\begin{cases} y_\infty(t_0) = 0, \\ y'_\infty(t_0) = 1. \end{cases}$$

We note that this gives

$$y''_\infty(t_0) = -2\frac{\cosh(2t_0)}{\sinh(2t_0)} > 0$$

by (9.13). We will show that

$$\lim_{t \nearrow 0} y_\infty(t) = \infty.$$

For this we first define

$$B = \{t \in [t_0, 0) \mid y'_\infty \geqslant y_\infty \geqslant 0\},$$

so that $t_0 \in B$. Moreover, since $y'_\infty(t_0) > y_\infty(t_0) = 0$, there exists some $\delta_0$ with $[t_0, t_0 + \delta_0) \subseteq B$ and $y_\infty(t_0 + \delta_0) > 0$. We also define

$$s = \sup\{t \in [-t_0, 0) \mid [0, t) \subseteq B\}$$

and note that $[t_0, s) \subseteq B$. Now consider the derivative of $y'_\infty - y_\infty$, which is given by

$$\begin{aligned} y''_\infty(t) - y'_\infty(t) &= \left|2\frac{\cosh(2t)}{\sinh(2t)}\right| y'_\infty(t) - \left(1 - \alpha + \frac{n^2}{\cosh^2 t}\right) y_\infty(t) - y'_\infty(t) \\ &= \underbrace{\left(\left|2\frac{\cosh(2t)}{\sinh(2t)}\right| - 1\right)}_{\geqslant M} y'_\infty(t) - \underbrace{\left(1 - \alpha + \frac{n^2}{\cosh^2 t}\right)}_{\leqslant M} y_\infty(t), \end{aligned}$$

where the indicated estimates hold for all $t \in [t_0, 0)$. For $t \in [t_0, s)$ we obtain from our definitions of $t_0$ and of $B$ that

$$y''_\infty(t) - y'_\infty(t) \geqslant M y'_\infty(t) - M y_\infty(t) \geqslant 0.$$

However, this shows that $t \mapsto y'_\infty(t) - y_\infty(t)$ is monotone non-decreasing on $[t_0, s)$. Suppose for a moment that $s < 0$. Then monotonicity of $y'_\infty - y_\infty$ and of $y_\infty$ on $[t_0, s]$ imply

$$\begin{cases} y'_\infty(s) - y_\infty(s) \geqslant y'_\infty(t_0) - y_\infty(t_0) = 1 > 0, \\ y_\infty(s) \geqslant y_\infty(t_0 + \delta_0). \end{cases}$$

However, this also implies the existence of some $\delta > 0$ with $[s, s + \delta) \subseteq B$ and contradicts the definition of $s$. Therefore we have $s = 0$.

Equivalently, we have shown that

$$y'_\infty(t) \geqslant y_\infty(t) \geqslant 0$$

for all $t \in [t_0, 0)$. With this we now estimate the growth of $y'_\infty$ on $[t_0, 0)$. Indeed,

$$\begin{aligned} y''_\infty(t) &= \left|2\frac{\cosh(2t)}{\sinh(2t)}\right| y'_\infty(t) - \left(1 - \alpha + \frac{n^2}{\cosh^2 t}\right) y_\infty(t) \\ &\geqslant \left|\frac{\cosh(2t)}{\sinh(2t)}\right| y'_\infty(t) + \underbrace{M y'_\infty(t) - M y_\infty(t)}_{\geqslant 0} \geqslant 0 \end{aligned}$$

for all $t \in [t_0, 0)$ implies that $y'_\infty$ is monotone non-decreasing.

With $y'_\infty(t_0) = 1$, this gives $y'_\infty(t) \geqslant 1$ for all $t \in [t_0, 0)$, which leads to

$$y''_\infty(t) \geqslant \left|\frac{\cosh(2t)}{\sinh(2t)}\right| \geqslant \frac{c}{|t|}$$

for all $t \in [t_0, 0)$ and some absolute constant $c > 0$. Therefore

$$\lim_{t \nearrow 0} y''_\infty(t) = \lim_{t \nearrow 0} y'_\infty(t) = \infty.$$

This proves the claim, and hence the proposition. □

### 9.2.5 The Unitary Dual of $\mathbf{SL_2}(\mathbb{R})$

The following corollary to Proposition 9.15 will be our main tool for the classification of the elements of $\widehat{\mathrm{SL}_2(\mathbb{R})}$.

**Corollary 9.17 (Isomorphisms).** *Let $\pi$ and $\tau$ be irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$. Suppose that the Casimir eigenvalues $\alpha_\pi = \alpha_\tau$ agree, and that there exists some $n \in \mathbb{Z}$ such that both $\mathcal{H}_\pi$ and $\mathcal{H}_\tau$ contain a $K$-eigenvector of weight $n$. Then $\pi \cong \tau$.*

Proof. Let $v \in \mathcal{H}_\pi$ and $w \in \mathcal{H}_\tau$ be $K$-eigenvectors of weight $n$. Without loss of generality, we may assume that $\|v\| = \|w\| = 1$. By Proposition 9.15 this implies $\phi^\pi_v = \phi^\tau_w$. However, Proposition 1.60 now shows that

$$\mathcal{H}_\pi = \langle v \rangle_\pi \cong \langle w \rangle_\tau = \mathcal{H}_\tau$$

are isomorphic as unitary representations of $\mathrm{SL}_2(\mathbb{R})$. □

We are now in a position to completely describe the unitary dual of $\mathrm{SL}_2(\mathbb{R})$. Since $\mathrm{SL}_2(\mathbb{R})$ has the non-trivial centre $\{\pm I\}$, the first distinction we can make between irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$ is by use of the central character as in Corollary 1.30. Since the centre of $\mathrm{SL}_2(\mathbb{R})$ is given by $\{\pm I\}$, we say that $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$ is *even* if $\pi_{-I} = I$ and *odd* if $\pi_{-I} = -I$. The second distinction is in terms of the 'infinitesimal character' obtained by applying Proposition 9.6 to all central elements of the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{sl}_2(\mathbb{R})$. Actually this centre is generated† by $\mathbb{1}_{\mathfrak{E}}$ and the Casimir element $\Omega$ of Section 9.2.1. Hence the infinitesimal character is in our case simply the Casimir eigenvalue $\alpha_\pi \in \mathbb{R}$. The third and final distinction is in terms of which $K$-weights are present in the representation $\pi$.

The following result of Bargmann [1] contains all the possibilities of these three aspects, and introduces the final two types of irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$; these will be studied in detail in the next sections.

We depict $\widehat{\mathrm{SL}_2(\mathbb{R})}$ in Figure 9.1, where we draw even representations on the top half and odd ones on the bottom half.

**Theorem 9.18 (Unitary dual of $\mathrm{SL}_2(\mathbb{R})$).** *Suppose that $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$ is even. Then one of the following four possibilities holds:*

- $\alpha_\pi = 1$ *and* $\pi = \mathbb{1}$ *is the trivial representation.*
- $\alpha_\pi = (n-1)^2$ *for some* $n \in 2\mathbb{N}$, *and* $\pi = \delta^{n,\pm}$ *is either the holomorphic or the anti-holomorphic discrete series representation with terminal weight* $\pm n$ *(see Section 8.4).*
- $\alpha_\pi = -\xi^2 \leqslant 0$ *for some* $\xi \in [0,\infty)$ *and* $\pi = \pi^{\xi,\mathrm{e}}$ *is the even principal series representation for the parameter* $\xi$ *(see Section 9.3).*
- $\alpha_\pi = \zeta^2$ *for some* $\zeta \in (0,1)$, *and* $\pi = \gamma^\zeta$ *is the complementary series representation for the parameter* $\zeta$ *(see Section 9.5).*

*Suppose that $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$ is odd. Then one of the following three possibilities holds:*

- $\alpha_\pi = 0$ *and* $\pi = \delta^{1,\pm}$ *is either the holomorphic or the anti-holomorphic mock discrete series representations (see Section 8.4).*
- $\alpha_\pi = (n-1)^2$ *for some* $n \in (2\mathbb{N}+1)$, *and* $\pi = \delta^{n,\pm}$ *is either the holomorphic or the anti-holomorphic discrete series representation with terminal weight* $\pm n$ *(see Section 8.4).*
- $\alpha_\pi = -\xi^2$ *for some* $\xi \in (0,\infty)$ *and* $\pi = \pi^{\xi,\mathrm{o}}$ *is the odd principal series representation for the parameter* $\xi$ *(see Section 9.3).*

Proof of Theorem 9.18. In the following we let $\pi$ be an irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$. By Corollary 9.8, the closure of $\pi_\partial(\Omega)$ is multiplication by $\alpha_\pi$ for some $\alpha_\pi \in \mathbb{R}$. By Lemma 9.13, there also exists a smooth $K$-eigenvector $v \in \mathcal{H}_\pi$ with weight $n \in \mathbb{Z}$ and unit length $\|v\| = 1$. By Corollary 9.17, we have that $(n, \alpha_\pi)$ uniquely determines $\pi$ up to isomorphism. Hence the question is really which $(n, \alpha_\pi) \in \mathbb{Z} \times \mathbb{R}$ are possible. As already explained, Corollary 9.10 gives the constraint

† We do not have to know this (see Exercise 9.2) if we simply define the infinitesimal character as the Casimir eigenvalue.

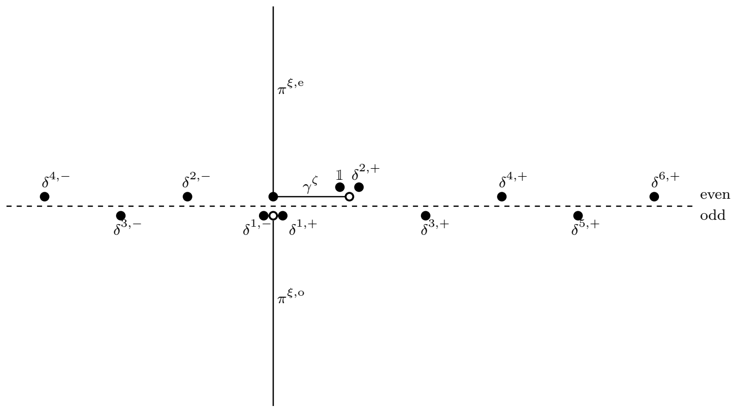


**Fig. 9.1:** The graphical representation of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ as a subset of $\mathbb{C}$ (with the representations $\pi^{0,\mathrm{e}}, \delta^{1,-}, \delta^{1,+}$ at the origin and both $\mathbb{1}$ and $\delta^{2,+}$ at $1 \in \mathbb{C}$) also has the property that $\alpha_\pi$ is the square of the position of $\pi$ when drawn in $\mathbb{C}$ (except for the artificial small gap between the even and odd representations, and problems arising from $\{\pi^{0,\mathrm{e}}, \delta^{1,+}, \delta^{1,-}\}$ and $\{\delta^{2,+}, \mathbb{1}\}$ which should be drawn at the same point).

$$\alpha_\pi \leqslant (n \pm 1)^2;$$

(see the discussion leading to (9.12), and Figure 9.2).

We first go through the list of representations that we have already encountered.

- If $\pi = \mathbb{1}_G$ is the trivial representation on $\mathbb{C}$, then $v = 1 \in \mathbb{C}$ has $K$-weight $n = 0$ and Casimir eigenvalue 1, since $\pi_\partial(\Omega) = \pi_\partial(\mathbb{1}_{\mathfrak{E}}) = 1$. By the above, it follows that $\mathbb{1}_G$ is characterized by the pair $(n = 0, \alpha = 1)$.
- For the holomorphic discrete series representation $\delta^{n,+}$ or the anti-holomorphic discrete series representation $\delta^{n,-}$ with $n \geqslant 2$ we have, by Theorem 8.23, that $\delta^{n,\pm}$ contains vectors with $K$-weights $\pm(n+2\mathbb{N}_0)$ and only these. Moreover, Corollary 9.11 gives $\alpha_{\delta^{n,\pm}} = (n-1)^2$.
- This also holds similarly for the mock discrete series representations $\delta^{1,\pm}$ with $K$-weights $\pm(1 + 2\mathbb{N}_0)$ and $\alpha_{\delta^{1,\pm}} = 0$ (see Theorem 8.30 and Corollary 9.11).

We next describe the remaining irreducible unitary representations listed in the theorem.

- Suppose $\pi$ is an irreducible unitary representation with $\alpha_\pi < 0$. Now let $v_n \in \mathcal{H}_\pi$ be a smooth $K$-eigenvector of weight $n \in \mathbb{Z}$. Using the raising and lowering operators in Proposition 9.9, we can define two other $K$-eigenvectors $\pi_\partial(\mathbf{r}^\pm)v_n$ of $K$-weights $n \pm 2$. By Corollary 9.10, we have

$$\|\pi_\partial(\mathbf{r}^\pm)v_n\|^2 = \tfrac{1}{4}\bigl((n \pm 1)^2 - \alpha_\pi\bigr)\|v_n\|^2. \tag{9.15}$$

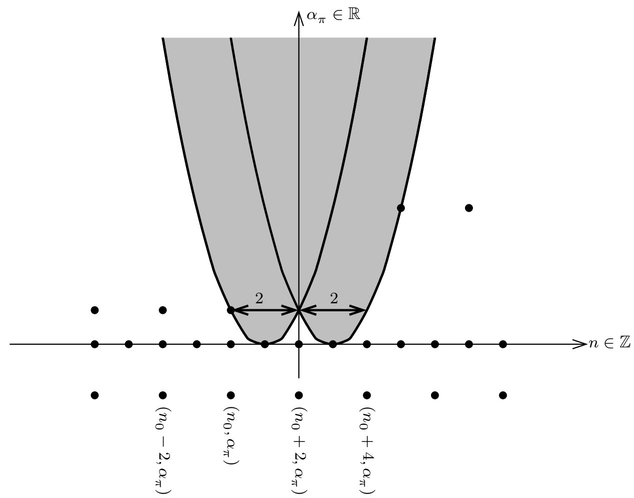


**Fig. 9.2:** The shaded region is the union of the region defined by the two inequalities $\alpha_\pi > (n+1)^2$ and $\alpha_\pi > (n-1)^2$. By (9.12), we know that this region cannot contain a pair $(n, \alpha_\pi)$ corresponding to a non-zero $K$-eigenvector for some representation $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$. In addition, we see how Proposition 9.9 creates additional pairs from one pair. We also note that the two parabolas have width 2 precisely at height $\alpha_\pi = 1$.

Since we assume $\alpha_\pi < 0$, it follows that $(n \pm 1)^2 - \alpha_\pi > 0$ and so that $\pi$ contains non-zero $K$-eigenvalues of $K$-weights $n \pm 2$. Iterating this shows that $\pi$ contains $K$-eigenvectors with $K$-weights $m$ for all $m \in n + 2\mathbb{Z}$. Suppose next that $\tau$ is another irreducible unitary representation with $\alpha_\tau = \alpha_\pi$. If $\pi$ and $\tau$ are both even (or both odd) irreducible representations, this, together with the first argument of the proof, implies that $\pi \cong \tau$. In other words, if there is an even (or, similarly, an odd) irreducible unitary representation $\pi$ with a given $\alpha_\pi < 0$, then $\alpha_\pi$ uniquely determines $\pi$ up to isomorphism. We will show in Section 9.3 that for any $\xi > 0$ there exists an irreducible unitary representation $\pi^{\xi,\mathrm{e}}$ with $\alpha_{\pi^{\xi,\mathrm{e}}} = -\xi^2$ and $K$-weights in $2\mathbb{Z}$, and an irreducible unitary representation $\pi^{\xi,\mathrm{o}}$ with $\alpha_{\xi,\mathrm{o}} = -\xi^2$ and $K$-weights in $1 + 2\mathbb{Z}$. These are the even and odd principal series representations.

- The discussion above almost applies to the case of Casimir eigenvalue $\alpha_\pi = 0$. Indeed, if $\pi$ is an even irreducible unitary representation with $\alpha_\pi = 0$, then the $K$-weight $n$ is even, $n \pm 1$ is odd, and hence

$$(n \pm 1)^2 - \alpha_\pi = (n \pm 1)^2 > 0.$$

Applying the argument above, it follows that $\pi$ must contain $K$-eigenvectors for all even $K$-weights, and that $\pi$ is uniquely determined up to isomorphism. We will show in Section 9.3 that the even principal series representation $\pi^{0,\mathrm{e}}$ is this irreducible unitary representation with $\alpha_{\pi^{0,\mathrm{e}}} = 0$. We note that the argument above fails in the odd case precisely when $n = \pm 1$. Moreover, we already found the two odd irreducible unitary representations $\delta^{1,\pm}$ with vanishing Casimir eigenvalue and with $K$-weights in $\pm(1 + 2\mathbb{N}_0)$ respectively.

- Suppose now that $\pi$ is an even irreducible unitary representation with

$$\alpha_\pi \in (0, 1).$$

We note that

$$(n \pm 1)^2 - \alpha_\pi \geqslant 1 - \alpha_\pi > 0$$

for all even $n \in \mathbb{Z}$. Hence the argument above applies once more, $\pi$ contains $K$-eigenvectors for all even $K$-weights, and is uniquely determined up to isomorphism by $\alpha_\pi$. We will show in Section 9.5 that this so-called complementary series representation exists.

It remains to show that the cases above give all possible irreducible unitary representations. So let $\pi$ be an irreducible unitary representation with Casimir eigenvalue $\alpha_\pi$ and let $v_n \in \mathcal{H}_\pi$ be a smooth $K$-eigenvector of $K$-weight $n \in \mathbb{Z}$. If $\alpha_\pi < 0$, then $\pi$ is a principal series representation. If $\alpha_\pi = 0$, then $\pi$ is either the even principal series representation $\pi^{0,\mathrm{e}}$ (when $n$ is even), the holomorphic mock discrete series representation $\delta^{1,+}$ (when $n > 0$ is odd), or the anti-holomorphic mock discrete series representation $\delta^{1,-}$ (when $n < 0$ is odd). If $\alpha_\pi \in (0, 1)$ and $n$ is even, then $\pi$ is a complementary series representation.

For the remaining cases we use Figure 9.2 to extend the argument that we used above for the principal series representation. By Proposition 9.9 and Corollary 9.10, the vectors $\pi_\partial(\mathbf{r}^\pm)v_n$ have $K$-weight $n \pm 2$ and satisfy (9.15). In particular, this implies that $\alpha_\pi \leqslant (n \pm 1)^2$, or equivalently that $(n, \alpha_\pi)$ does not belong to the 'forbidden' shaded region in Figure 9.2. Moreover, if $(n, \alpha_\pi)$ does not belong to either of the two parabolas defined by $\alpha = (n \pm 1)^2$, then we may replace $n$ by $n \pm 2$ and iterate this argument to obtain further eigenvectors with different $K$-weights. If, however, $\alpha_\pi = (n \pm 1)^2$ then (9.15) shows that $\pi_\partial(\mathbf{r}^\pm)v_n = 0$.

This argument creates a chain of points $(n, \alpha_\pi)$ avoiding the forbidden region in Figure 9.2 with end points belonging to either of the parabolas defined by $\alpha_\pi = (n \pm 1)^2$. There are a few possibilities for this chain of points, as follows.

- $\alpha_\pi < 0$, and the chain is bi-infinite.
- $\alpha_\pi = 0$, $n$ is even, and the chain is bi-infinite.
- $\alpha_\pi = 0$, $n$ is odd, and the chain is one of $\pm(1 + 2\mathbb{N}_0) \times \{0\}$.

- $\alpha_\pi \in (0,1)$, $n$ is even, and the chain is bi-infinite and jumps over the two shaded regions that have width less than two at the height $\alpha_\pi$.
- $\alpha_\pi \in (0,1]$ and $n$ is odd is impossible, since the chain starting on either side would lead to the creation of one of the points $(\pm 1, \alpha_\pi)$ inside the forbidden region.
- $\alpha_\pi = 1$ and $n$ even has three such chains, one starting at $(2,1)$ going to the right, one starting at $(-2,1)$ going to the left, and one consisting of $(0,1)$ only.
- $\alpha_\pi > 1$ and $n > 0$ creates a half-infinite chain going to the right. It cannot go infinitely far to the left, as the forbidden region has width larger than 4 and the gaps in the chain have size 2. Hence the chain has to stop with a point on the right parabola. Replacing $n$ by this minimal $K$-weight, we see that $\alpha = (n-1)^2$. However, this shows that $\pi$ has the same Casimir eigenvalue and the same $K$-weights as the holomorphic discrete series representation $\delta^{n,+}$.
- $\alpha_\pi > 1$ and $n < 0$ gives rise to a half-infinite chain going to the left, and corresponds by the same argument to an anti-holomorphic discrete series representation.

We leave it to the reader to match the cases above to the irreducible unitary representations discussed earlier, which concludes the proof. □

## 9.3 The Principal Series Representations

We will now modify the representation $\pi^0$ from Section 8.5.2, which will give rise to the even and odd principal series representations appearing in Theorem 9.18. Along the way we will also explain the connection to Example 1.6.

**Definition 9.19 (Principal series representation).** For a given $\xi \in \mathbb{R}$, we define the character $\chi_\xi$ on

$$B = \{a_t u_x \mid t, x \in \mathbb{R}\}$$

by

$$\chi_\xi(a_t u_x) = \mathrm{e}^{\mathrm{i}\xi t}$$

for all $a_t u_x \in B$. The representation $\pi^\xi$ of $G = \mathrm{SL}_2(\mathbb{R})$ is defined by

$$(\mathcal{H}_\xi, \pi^\xi) = \mathrm{Ind}_B^G(\mathbb{C}, \chi_\xi),$$

or, more concretely, by the left-regular representation on the space $\mathcal{H}_\xi$ of those functions $f\colon G \to \mathbb{C}$ with the following properties:

(1) $f$ is measurable,
(2) $f(gb) = \overline{\chi_\xi(b)}\Delta_B(b)^{\frac{1}{2}} f(g)$ for all $g \in G$ and $b \in B$, and

(3) $\|f|_K\|_{L^2(K)} < \infty$.

The *even principal series representation* $\pi^{\xi,\mathrm{e}} = \pi^{\xi,\mathrm{even}}$ (for frequency parameter $\xi$) is defined as the restriction of $\pi^\xi$ to the subspace

$$\mathcal{H}_\xi^{\mathrm{even}} = \{f \in \mathcal{H}_\xi \mid f(-g) = f(g) \text{ for all } g \in G\}.$$

Similarly, the *odd principal series representation* $\pi^{\xi,\mathrm{o}} = \pi^{\xi,\mathrm{odd}}$ (for frequency parameter $\xi$) is defined as the restriction of $\pi^\xi$ to the subspace

$$\mathcal{H}_\xi^{\mathrm{odd}} = \{f \in \mathcal{H}_\xi \mid f(-g) = -f(g) \text{ for all } g \in G\}.$$

Let us summarize the properties of the even and odd principal series representations that we will prove in this section.

**Theorem 9.20 (Even and odd principal series representations).** *The representations $\pi^\xi$, $\pi^{\xi,\mathrm{e}}$, and $\pi^{\xi,\mathrm{o}}$ are tempered unitary representations with Casimir eigenvalue $-\xi^2$ for any $\xi \in \mathbb{R}$. The representation $\pi^{\xi,\mathrm{e}}$ is irreducible for any $\xi \in \mathbb{R}$ and $\pi^{\xi,\mathrm{o}}$ is irreducible for all $\xi \in \mathbb{R}\smallsetminus\{0\}$. Moreover, $\pi^{-\xi,\mathrm{e}}$ is isomorphic to $\pi^{\xi,\mathrm{e}}$, and $\pi^{-\xi,\mathrm{o}}$ is isomorphic to $\pi^{\xi,\mathrm{o}}$ for all $\xi \in \mathbb{R}$. Finally, $\pi^{0,\mathrm{o}}$ is isomorphic to the sum $\delta^{1,+} \oplus \delta^{1,-}$ of the holomorphic and anti-holomorphic mock discrete series representations.*

Proof of unitarity in Theorem 9.20. Recall from Example 1.6 that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{S}^1 \subseteq \mathbb{R}^2$ via

$$\mathbb{S}^1 \ni v \longmapsto g\boldsymbol{\cdot}v = \frac{1}{\|gv\|} gv$$

for $g \in \mathrm{SL}_2(\mathbb{R})$ and $v \in \mathbb{S}^1$, and let $m$ again denote the normalized length measure so that the Radon–Nikodym derivative is given by

$$\frac{\mathrm{d}g_* m}{\mathrm{d}m}(v) = \|g^{-1}v\|^{-2}.$$

Let $\xi \in \mathbb{R}$ and note that the map

$$\mathrm{SL}_2(\mathbb{R}) \times \mathbb{S}^1 \ni (g,v) \longmapsto c(g,v) = \|g^{-1}v\|^{-1+\mathrm{i}\xi}$$

satisfies the equation

$$\begin{aligned} c(g_1,v)c(g_2,g_1^{-1}\boldsymbol{\cdot}v) &= \|g_1^{-1}v\|^{-1-\mathrm{i}\xi} \left\| g_2^{-1} \frac{g_1^{-1}v}{\|g_1^{-1}v\|} \right\|^{-1-\mathrm{i}\xi} \\ &= \|(g_1g_2)^{-1}v\|^{-1-\mathrm{i}\xi} = c(g_1g_2,v) \end{aligned}$$

for $g_1, g_2 \in \mathrm{SL}_2(\mathbb{R})$ and $v \in \mathbb{S}^1$. By Proposition 1.5 the formula

$$\pi_g^{\mathbb{S}^1,\xi}(f)(v) = \|g^{-1}v\|^{-1-\mathrm{i}\xi} f(g^{-1}\boldsymbol{\cdot}v)$$

for $g \in \mathrm{SL}_2(\mathbb{R})$, $f \in L^2_m(\mathbb{S}^1)$, and $v \in \mathbb{S}^1$ defines a unitary representation $\pi^{\mathbb{S}^1,\xi}$ of $\mathrm{SL}_2(\mathbb{R})$ on $L^2_m(\mathbb{S}^1)$.

We now show that $\pi^\xi$ is $\pi^{\mathbb{S}^1,\xi}$ in disguise. In fact, we define for $f \in L^2_m(\mathbb{S}^1)$ the function

$$U(f)\colon \mathrm{SL}_2(\mathbb{R}) \ni g \longmapsto U(f)(g) = \|ge_1\|^{-1-\mathrm{i}\xi} f(g\bullet e_1).$$

Then $\|U(f)\|_{L^2(K)} = \|f\|_{L^2_m(\mathbb{S}^1)}$ since the normalized Haar measure $m_K$ is mapped under the action to the normalized length measure $m$ on $\mathbb{S}^1$. Moreover, $g \in \mathrm{SL}_2(\mathbb{R})$ and $b = a_t u_x \in B$ imply $be_1 = \mathrm{e}^t e_1$ and so

$$\begin{aligned} U(f)(gb) &= \|gbe_1\|^{-1-\mathrm{i}\xi} f(gb\bullet e_1) \\ &= \mathrm{e}^{-t-\mathrm{i}\xi t}\|ge_1\|^{-1-\mathrm{i}\xi} f(g\bullet v) \\ &= \overline{\chi_\xi(b)}\Delta_B(b)^{\frac{1}{2}} U(f)(g) \end{aligned}$$

by (8.23), which shows that $U(f) \in \mathcal{H}_\xi$. We note that since $f \in L^2_m(\mathbb{S}^1)$ was arbitrary, this shows in particular that every $F \in L^2(K)$ has an extension to an element of $\mathcal{H}_\xi$. Moreover, the Iwasawa decomposition and Definition 9.19(2) this extension is also uniquely determined.

Finally, we let $g_0 \in \mathrm{SL}_2(\mathbb{R})$ and calculate

$$U(f)(g_0^{-1}g) = \|g_0^{-1}ge_1\|^{-1-\mathrm{i}\xi} f(g_0^{-1}g\bullet e_1)$$

and

$$\begin{aligned} U\big(\pi^{\mathbb{S}^1,\xi}_{g_0} f\big)(g) &= \|ge_1\|^{-1-\mathrm{i}\xi}\big(\pi^{\mathbb{S}^1,\xi}_{g_0} f\big)(g\bullet e_1) \\ &= \underbrace{\|ge_1\|^{-1-\mathrm{i}\xi}\|g_0^{-1}(g\bullet e_1)\|^{-1-\mathrm{i}\xi}}_{\|g_0^{-1}ge_1\|^{-1-\mathrm{i}\xi}} f(g_0^{-1}g\bullet e_1). \end{aligned}$$

Together these show that $U\colon L^2_m(\mathbb{S}^1) \to \mathcal{H}_\xi$ is an equivariant isomorphism, and hence that $\pi^\xi$ is a unitary representation.

Since $-I$ belongs to the centre of $\mathrm{SL}_2(\mathbb{R})$, the subspaces $\mathcal{H}_\xi^{\mathrm{even}}$, $\mathcal{H}_\xi^{\mathrm{odd}}$ of $\mathcal{H}_\xi$ are closed invariant subspaces. It follows that $\pi^\xi$, $\pi^{\xi,\mathrm{e}}$, $\pi^{\xi,\mathrm{o}}$ are well-defined unitary representations of $\mathrm{SL}_2(\mathbb{R})$. □

**Exercise 9.21.** As an alternative, use Corollary 8.36 to show that $\pi^\xi_g$ is unitary for all $g \in \mathrm{SL}_2(\mathbb{R})$.

For the proof of irreducibility, we will use the following lemma.

**Lemma 9.22 (Casimir eigenvalue for $\pi^\xi$).** *Let $\xi \in \mathbb{R}$. Then the closure of $\pi^\xi_\partial(\Omega)$ is equal to multiplication by $\alpha_\xi = -\xi^2$. Moreover, for every $n \in \mathbb{Z}$ the extension of $\chi_{-n} \in L^2(K)$ to an element $F_{\xi,n} \in \mathcal{H}_\xi$ has $K$-weight $n$, is given by*

$$F_{\xi,n}(k_\psi a_t u_x) = \mathrm{e}^{-\mathrm{i}n\psi - \mathrm{i}\xi t - t} \tag{9.16}$$

*for all* $k_\psi a_t u_x \in KAU = G$, *and satisfies*

$$\pi_\partial^\xi(\mathbf{r}^+)F_{\xi,n} = \frac{n+1+\mathrm{i}\xi}{2}F_{\xi,n+2},$$
$$\pi_\partial^\xi(\mathbf{r}^-)F_{\xi,n} = \frac{-n+1+\mathrm{i}\xi}{2}F_{\xi,n-2},$$

*and*

$$\pi_\partial^\xi(\mathbf{a})F_{\xi,n} = \frac{n+1+\mathrm{i}\xi}{2}F_{\xi,n+2} + \frac{-n+1+\mathrm{i}\xi}{2}F_{\xi,n-2}. \tag{9.17}$$

Proof. For any $n \in \mathbb{Z}$, we define $F_{\xi,n} \in \mathcal{H}_\xi$ by setting

$$F_{\xi,n}|_K = \chi_{-n} \in L^2(K)$$

to be the character defined by $-n$ and extending it by the defining properties of its elements to an element of $\mathcal{H}_\xi$ (see Exercise 9.23). Using the formula $\Delta_B(a_t u_x) = \mathrm{e}^{-2t}$ for all $a_t u_x \in AN = B$ and the definition of $\mathcal{H}_\xi$, this gives (9.16).

To see that $F_{\xi,n}$ has $K$-weight $n$, we calculate

$$\pi_{k_\psi}^\xi(F_{\xi,n})(k_\theta) = F_{\xi,n}(k_\psi^{-1}k_\theta) = \mathrm{e}^{\mathrm{i}n\psi - \mathrm{i}n\theta} = \mathrm{e}^{\mathrm{i}n\psi}F_{\xi,n}(k_\theta)$$

for all $k_\psi, k_\theta \in K$. Since the characters $\chi_{-n}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $L^2(K)$, it follows that the functions $F_{\xi,n}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $\mathcal{H}_\xi$. We note that each $F_{\xi,n}$ is a smooth function on $G$, which implies, by dominated convergence, that it is also a smooth vector for $\pi^\xi$. Alternatively, the latter also follows from Lemma 9.13.

Next we wish to calculate $\pi_\partial^\xi(\mathbf{a})F_{\xi,n}$. For $t \in \mathbb{R}$ we have

$$\exp(t\mathbf{a}) = \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix},$$

and

$$\left(\pi_{\exp(t\mathbf{a})}^\xi F_{\xi,n}\right)(k_\theta) = F_{\xi,n}\left(\exp(-t\mathbf{a})k_\theta\right) = F_{\xi,n}\left(\begin{pmatrix} \mathrm{e}^{-t} & \\ & \mathrm{e}^t \end{pmatrix} k_\theta\right).$$

In order to apply the definition of $F_{\xi,n}$ in (9.16), we need to write the argument in the form

$$\begin{pmatrix} \mathrm{e}^{-t} & \\ & \mathrm{e}^t \end{pmatrix} k_\theta = k_\psi a_{t_0} u_{x_0},$$

where in fact we are only interested in the angle parameter $\psi = \psi(t,\theta)$ and the diagonal parameter $t_0 = t_0(t,\theta)$ considered as functions in $t$ and $\theta$. As in the proof of the estimate for the Harish-Chandra spherical function in Proposition 8.39, we obtain $\psi$ and $t_0$ by using polar co-ordinates in $\mathbb{R}^2$. Indeed,

$$k_\psi a_{t_0} u_{x_0} e_1 = \mathrm{e}^{t_0} \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \tag{9.18}$$

must equal

$$\begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{t} \end{pmatrix} k_\theta e_1 = \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{t} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-t}\cos\theta \\ \mathrm{e}^{t}\sin\theta \end{pmatrix}. \tag{9.19}$$

Therefore

$$\mathrm{e}^{2t_0} = \mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta.$$

Since we will later take the partial derivative with respect to $t$ at $t = 0$, we calculate from this that

$$\begin{aligned} 2\mathrm{e}^{2t_0}\frac{\partial}{\partial t}t_0 = \frac{\partial}{\partial t}(\mathrm{e}^{2t_0}) &= \frac{\partial}{\partial t}\big(\mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta\big) \\ &= -2\mathrm{e}^{-2t}\cos^2\theta + 2\mathrm{e}^{2t}\sin^2\theta. \end{aligned}$$

For $t = 0$, this gives, with $t_0(0,\theta) = 0$ for all $\theta \in \mathbb{R}$,

$$\frac{\partial}{\partial t}\bigg|_{t=0}(t_0) = -\cos^2\theta + \sin^2\theta = -\cos(2\theta) = -\tfrac{1}{2}\big(\mathrm{e}^{2\theta\mathrm{i}} + \mathrm{e}^{-2\theta\mathrm{i}}\big). \tag{9.20}$$

For the angle $\psi = \psi(\theta, t)$, we obtain from (9.18) and (9.19) that

$$\tan\psi = \frac{\mathrm{e}^{t}\sin\theta}{\mathrm{e}^{-t}\cos\theta} = \mathrm{e}^{2t}\tan\theta,$$

$$(1+\tan^2\psi)\frac{\partial}{\partial t}\psi = \frac{\partial}{\partial t}(\tan\psi) = \frac{\partial}{\partial t}\big(\mathrm{e}^{2t}\tan\theta\big) = 2\mathrm{e}^{2t}\tan\theta.$$

Setting $t = 0$ and using in addition $\psi(0,\theta) = \theta$ for all $\theta \in \mathbb{R}$, we obtain

$$\frac{\partial}{\partial t}\bigg|_{t=0}(\psi) = \frac{2\tan\theta}{1+\tan^2\theta} = 2\sin\theta\cos\theta = \sin 2\theta = \tfrac{1}{2\mathrm{i}}\big(\mathrm{e}^{2\theta\mathrm{i}} - \mathrm{e}^{-2\theta\mathrm{i}}\big). \tag{9.21}$$

Combining (9.20), (9.21), and using again $t_0(0,\theta) = 0$ and $\psi(0,\theta) = \theta$ for all $\theta \in \mathbb{R}$, this gives

$$\begin{aligned}
\pi_\partial^\xi(\mathbf{a})F_{\xi,n}(k_\theta) &= \frac{\partial}{\partial t}\Big|_{t=0}\big(F_{\xi,n}(k_\psi a_{t_0} u_{x_0})\big) = \frac{\partial}{\partial t}\Big|_{t=0}\Big(\mathrm{e}^{-\mathrm{i}n\psi - \mathrm{i}\xi t_0 - t_0}\Big) \\
&= \mathrm{e}^{-\mathrm{i}n\theta}\left(-\mathrm{i}n\Big(\frac{\partial}{\partial t}\Big|_{t=0}\psi\Big) - (\mathrm{i}\xi+1)\Big(\frac{\partial}{\partial t}\Big|_{t=0} t_0\Big)\right) \\
&= \mathrm{e}^{-\mathrm{i}n\theta}\left(-\mathrm{i}n\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{2\theta\mathrm{i}} - \mathrm{e}^{-2\theta\mathrm{i}}\right) + (\mathrm{i}\xi+1)\frac{1}{2}\left(\mathrm{e}^{2\theta\mathrm{i}} + \mathrm{e}^{-2\theta\mathrm{i}}\right)\right) \\
&= \left(\frac{n+1+\mathrm{i}\xi}{2}\right)\mathrm{e}^{-\mathrm{i}(n+2)\theta} + \left(\frac{-n+1+\mathrm{i}\xi}{2}\right)\mathrm{e}^{-\mathrm{i}(n-2)\theta} \\
&= \left(\frac{n+1+\mathrm{i}\xi}{2}\right)F_{\xi,n+2}(k_\theta) + \left(\frac{-n+1+\mathrm{i}\xi}{2}\right)F_{\xi,n-2}(k_\theta)
\end{aligned}$$

for all $k_\theta \in K$. To summarize, we have shown (9.17).

Recalling that $\mathbf{a} = \mathbf{r}^+ + \mathbf{r}^-$ and that, by Proposition 9.9, $\pi_\partial(\mathbf{r}^\pm)F_{\xi,n}$ has weight $n \pm 2$, we obtain

$$\pi_\partial^\xi(\mathbf{r}^+)F_{\xi,n} = \frac{n+1+\mathrm{i}\xi}{2}F_{\xi,n+2}$$

and

$$\pi_\partial^\xi(\mathbf{r}^-)F_{\xi,n} = \frac{-n+1+\mathrm{i}\xi}{2}F_{\xi,n-2},$$

as claimed in the lemma.

Using the formula for $\Omega$ in (9.11) in terms of $\mathbf{r}^+$, $\mathbf{r}^-$, and $\mathbf{k}$, we obtain with $\pi_\partial^\xi(\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})F_{\xi,n} = (1-n)F_{\xi,n}$ that

$$\begin{aligned}
\pi_\partial^\xi(\Omega)F_{\xi,n} &= 4\pi_\partial^\xi(\mathbf{r}^+\circ\mathbf{r}^-)F_{\xi,n} + \pi_\partial^\xi\left((\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})^{\circ 2}\right)F_{\xi,n} \\
&= 2\pi_\partial^\xi(\mathbf{r}^+)(-n+1+\mathrm{i}\xi)F_{\xi,n-2} + (1-n)^2F_{\xi,n} \\
&= (n-1+\mathrm{i}\xi)(-n+1+\mathrm{i}\xi)F_{\xi,n} + (1-n)^2F_{\xi,n} \\
&= \big(-(n-1)^2 - \xi^2 + (1-n)^2\big)F_{\xi,n} \\
&= -\xi^2F_{\xi,n}
\end{aligned}$$

for all $n \in \mathbb{Z}$. Since the functions $F_{\xi,n}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $\mathcal{H}^\xi$, the lemma follows. □

Proof of irreducibility claims in Theorem 9.20. For $\xi \in \mathbb{R}$ let $\rho = \pi^{\xi,\mathrm{e}}$ or $\rho = \pi^{\xi,\mathrm{o}}$ be the restriction of $\pi^\xi$ to

$$\mathcal{H}_\xi^{\mathrm{even}} = \big\langle F_{\xi,2n} \mid n \in \mathbb{Z}\big\rangle = \{f \in \mathcal{H}_\xi \mid \pi_{-I}^\xi f = f\}$$

or

$$\mathcal{H}_\xi^{\mathrm{odd}} = \big\langle F_{\xi,2n+1} \mid n \in \mathbb{Z}\big\rangle = \{f \in \mathcal{H}_\xi \mid \pi_{-I}^\xi f = -f\}$$

respectively. Suppose $\mathcal{V} < \mathcal{H}_\xi$ is a non-trivial closed $\rho$-invariant subspace. Since $K = \mathrm{SO}_2(\mathbb{R}) < \mathrm{SL}_2(\mathbb{R})$ is compact and abelian, $\mathcal{V}$ contains a $K$-eigenfunction.

As $\mathcal{H}_\rho$ is given as the linear hull of orthonormal $K$-eigenfunctions of different weights, it follows that $f_n \in \mathcal{V}$ for some $n \in \mathbb{Z}$. By Lemma 9.22, this implies

$$\pi_\partial^\xi(\mathbf{r}^+)F_{\xi,n} = \left(\frac{n+1+\mathrm{i}\xi}{2}\right)F_{\xi,n+2} \in \mathcal{V}$$

and

$$\pi_\partial^\xi(\mathbf{r}^-)F_{\xi,n} = \left(\frac{-n+1+\mathrm{i}\xi}{2}\right)F_{\xi,n-2} \in \mathcal{V}.$$

If $\xi \neq 0$, then certainly $\frac{\pm n+1+\mathrm{i}\xi}{2} \neq 0$, and we obtain $F_{\xi,n+2}, F_{\xi,n-2} \in \mathcal{V}$. Moreover, in this case we can iterate this argument and obtain

$$\mathcal{H}_\rho = \langle F_{\xi,n+2k} \mid k \in \mathbb{Z}\rangle \subseteq \mathcal{V} \leqslant \mathcal{H}_\rho,$$

which implies that $\rho$ is irreducible.

If $\xi = 0$ and $\rho = \pi^{0,\mathrm{e}}$ is the even principal series representation, then the $K$-weight $n$ of $F_{\xi,n}$ is even, $\frac{\pm n+1}{2}$ is non-zero, and we again obtain $F_{\xi,n-2}, F_{\xi,n+2} \in \mathcal{V}$. Once more we can iterate this and thus obtain $\mathcal{V} = \mathcal{H}_0^{\mathrm{even}}$, and so deduce that $\pi^{0,\mathrm{e}}$ is irreducible. □

While the above was independent of Section 9.2, for the following step we are going to use the Bargmann classification (Theorem 9.18).

Proof of isomorphism claims in Theorem 9.20. Let $\xi \in \mathbb{R}$. By Lemma 9.22 the representations $\pi^{\xi,\mathrm{e}}$ and $\pi^{\xi,\mathrm{o}}$ have Casimir eigenvalue $-\xi^2$. By the previous part of the proof, we know that these are irreducible with the exception of $\pi^{0,\mathrm{o}}$. By Theorem 9.18 there is however only one even irreducible representation with Casimir eigenvalue $-\xi^2$, which gives $\pi^{\xi,\mathrm{e}} \cong \pi^{-\xi,\mathrm{e}}$. Similarly, for $\xi \in \mathbb{R}\smallsetminus\{0\}$ we have $\pi^{\xi,\mathrm{o}} \cong \pi^{-\xi,\mathrm{o}}$.

Let us now discuss $\pi^{0,\mathrm{o}}$ with Casimir eigenvalue 0. By Corollary 9.11 we also have $\alpha_{\delta^{1,+}} = \alpha_{\delta^{1,-}} = 0$. By the construction of $\pi^{0,\mathrm{o}}$, it contains all odd $K$-weights. Let $v_1 \in \mathcal{H}_{\pi^{0,\mathrm{o}}}$ be a unit vector with $K$-weight 1, and let $e_0 \in \mathcal{H}_{\delta^{1,+}}$ be the unit vector with $K$-weight 1 as in Lemma 8.25 (see also the paragraph after Theorem 8.30). By Proposition 9.15, we conclude that $\varphi_{v_1}^{\pi^{0,\mathrm{o}}} = \varphi_{e_0}^{\delta^{1,+}}$. However, Proposition 1.60 now implies that the cyclic representations $\langle v_1\rangle_{\pi^{0,\mathrm{o}}}$ and $\langle e_0\rangle_{\delta^{1,+}} = \mathcal{H}_{\delta^{1,+}}$ are isomorphic. In other words, we have shown that $\delta^{1,+} < \pi^{0,\mathrm{o}}$ (up to isomorphisms). Using a unit vector $v_{-1} \in \mathcal{H}_{\pi^{0,\mathrm{o}}}$ of $K$-weight $-1$, we obtain, from the same argument, that $\delta^{1,-} < \pi^{0,\mathrm{o}}$. Together we have $\delta^{1,+} \oplus \delta^{1,-} < \pi^{0,\mathrm{o}}$. However, since in $\delta^{1,+} \oplus \delta^{1,-}$ and $\pi^{0,\mathrm{o}}$ each odd $K$-weight appears with multiplicity one, we must have equality. □

We leave the remaining claim in Theorem 9.20 as an exercise.

**Essential Exercise 9.23.** Let $\xi \in \mathbb{R}$. Show that $\pi^{\xi}$, $\pi^{\xi,\mathrm{e}}$, $\pi^{\xi,\mathrm{o}}$ are tempered.

**Corollary 9.24 (Odd representations are tempered).** *Every odd unitary representation of* $\mathrm{SL}_2(\mathbb{R})$ *is tempered, and so has decay exponent* 1.

Proof. By Theorems 8.23 and 8.30 the discrete series representations and mock discrete series representations are tempered. By Exercise 9.23 (see also the hint on p. 485), the principal series representations $\pi^{\xi,\mathrm{o}}$ are also tempered for $\xi \in \mathbb{R}\smallsetminus\{0\}$. By Theorem 9.18, it follows that every odd irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$ is tempered.

Suppose now that $\rho$ is an odd unitary representation of $\mathrm{SL}_2(\mathbb{R})$. We note that being odd is equivalent to $\varphi_v^{\rho}(-I) = -\|v\|^2$ for any $v \in \mathcal{H}_\rho$. Therefore every irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$ that is weakly contained in $\rho$ must be odd too. By the Bargmann classification theorem (Theorem 9.18) and our discussion of the principal series representation (Theorem 9.20, resp. Exercise 9.23; see also the hint on p. 485), this shows that every irreducible unitary representation weakly contained in $\rho$ is tempered. We now combine the definition of temperedness, the characterization of weak containment in Theorem 4.28 ($\prec_{\mathrm{diag}}$), and Proposition 4.34. By the latter, we know that for any unit vector $v \in \mathcal{H}_\rho$ the matrix coefficient $\varphi_v^{\rho}$ can be approximated in the compact-open topology by sums of the form $\sum_{j=1}^{n} \varphi_{v_j}^{\pi_j}$ for some irreducible unitary representations $\pi_j \prec \rho$ for $j = 1, \dots, n$. Since $\pi_j \prec \lambda_{\mathrm{SL}_2(\mathbb{R})}$, we can approximate $\varphi_{v_j}^{\pi_j}$ for $j = 1, \dots, n$ by some sum of principal matrix coefficients for the regular representation. Putting these together, we obtain the same property for $\varphi_v^{\rho}$, which gives $\rho \prec \lambda_{\mathrm{SL}_2(\mathbb{R})}$. □

## 9.4 Two Koopman Representations of $\mathrm{SL}_2(\mathbb{R})$*

We wish to show here how the principal series representations can naturally occur as components of other unitary representations. Since $\mathrm{SL}_2(\mathbb{R})$ acts both on the Euclidean plane $\mathbb{R}^2$ and on the hyperbolic plane $\mathbb{H}$, preserving area measure on the space in each case, this already gives rise to two natural unitary representations of $\mathrm{SL}_2(\mathbb{R})$. As we will see, the case of $\mathbb{R}^2$ will be relatively straightforward to analyze. On the other hand, understanding the case of $\mathbb{H}$ will require more work, but we will motivate the formulas arising.

### 9.4.1 The Koopman Representation on $\mathbb{R}^2$

Almost by definition, the group $\mathrm{SL}_2(\mathbb{R})$ acts continuously on $\mathbb{R}^2$, preserving the two-dimensional Lebesgue measure $m = m_{\mathbb{R}^2}$. Using Proposition 1.3, this gives rise to a Koopman representation $\pi^{\mathbb{R}^2}$ of $\mathrm{SL}_2(\mathbb{R})$ on $L^2_m(\mathbb{R}^2)$, where

$$\pi^{\mathbb{R}^2}_g(f)(x) = f(g^{-1}x)$$

for $g \in \mathrm{SL}_2(\mathbb{R})$, $f \in L^2_m(\mathbb{R}^2)$, and $x \in \mathbb{R}^2$.

Using polar coordinates

$$(r,\theta) \in (0,\infty) \times [0, 2\pi)$$

for $\mathbb{R}^2 \smallsetminus \{0\}$ with $\mathrm{d}m = r\,\mathrm{d}r\,\mathrm{d}\theta$, we make the following definition.

**Definition 9.25 (Radial Fourier transform).** For a function $f \in L^2_m(\mathbb{R}^2)$, an element $h \in \mathrm{SL}_2(\mathbb{R})$, and a frequency parameter $\xi \in \mathbb{R}$, we define the *radial Fourier transform* of $f$ at $(h, \xi)$ by

$$\widehat{f}^{\mathrm{rad}}(h,\xi) = \int_0^\infty f(rhe_1) r^{\mathrm{i}\xi}\,\mathrm{d}r, \tag{9.22}$$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

denotes the first basis vector of $\mathbb{R}^2$.

Just as in the case of the usual Fourier transform, the integral in (9.22) may not make sense as a Lebesgue integral. Thus we also need to discuss the meaning of this expression more carefully (which we will do in the proof of Proposition 9.29). The following lemma, together with the definition of the principal series representation in Definition 9.19 reveal why (9.22) is really the right definition.

**Lemma 9.26 (Equivariance properties).** *For $f \in C_c(\mathbb{R}^2)$ the radial Fourier transform $\widehat{f}^{\mathrm{rad}}(h,\xi)$ is well-defined and satisfies*

$$\begin{cases} \widehat{\pi^{\mathbb{R}^2}_g(f)}^{\mathrm{rad}}(h,\xi) = \widehat{f}^{\mathrm{rad}}(g^{-1}h,\xi) \\ \widehat{f}^{\mathrm{rad}}(hb,\xi) = \overline{\chi_\xi(b)}\,\Delta_B(b)^{\frac{1}{2}}\,\widehat{f}^{\mathrm{rad}}(h,\xi) \end{cases}$$

*for all $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$, $g \in \mathrm{SL}_2(\mathbb{R})$, and $b \in B = AU$.*

Proof. It is clear that for $f \in C_c(\mathbb{R}^2)$, the domain of integration in 9.22 can be chosen to be a compact interval, which gives the first claim in the lemma.

Now fix some $g, h \in \mathrm{SL}_2(\mathbb{R})$ and $\xi \in \mathbb{R}$. Then

$$
\begin{aligned}
\widehat{\pi_g^{\mathbb{R}^2}(f)}^{\mathrm{rad}}(h,\xi) &= \int_0^\infty \bigl(\pi_g^{\mathbb{R}^2}(f)\bigr)(rhe_1)r^{\mathrm{i}\xi}\,\mathrm{d}r\\
&= \int_0^\infty f(g^{-1}rhe_1)r^{\mathrm{i}\xi}\,\mathrm{d}r\\
&= \widehat{f}^{\mathrm{rad}}(g^{-1}h,\xi),
\end{aligned}
$$

as claimed.

For the second claim, we calculate for $b = a_t u_x \in B = AU$ that

$$
\begin{aligned}
\widehat{f}^{\mathrm{rad}}(ha_tu_x,\xi) &= \int_0^\infty f\bigl(rh\underbrace{a_tu_xe_1}_{=\mathrm{e}^t e_1}\bigr)r^{\mathrm{i}\xi}\,\mathrm{d}r\\
&= \int_0^\infty f\bigl(\widetilde{r}he_1\bigr)\bigl(\widetilde{r}\mathrm{e}^{-t}\bigr)^{\mathrm{i}\xi}\mathrm{e}^{-t}\,\mathrm{d}\widetilde{r}\\
&= \mathrm{e}^{-\mathrm{i}\xi t}\mathrm{e}^{-t}\widehat{f}^{\mathrm{rad}}(h,\xi),
\end{aligned}
$$

where we used the substitution $\widetilde{r} = \mathrm{e}^t r$ with $\mathrm{d}\widetilde{r} = \mathrm{e}^t\,\mathrm{d}r$. The lemma follows by recalling that $\Delta_B(a_tu_x) = \mathrm{e}^{-2t}$ and $\chi_\xi(a_tu_x) = \mathrm{e}^{\mathrm{i}\xi t}$ for all $a_tu_x \in B$. □

In addition to the correct equivariance properties as shown above, the radial Fourier transform is also isometric in the following sense.

**Lemma 9.27 (Isometry).** *For $f \in C_c(\mathbb{R}^2)$ we have*

$$
\|f\|_{L^2(\mathbb{R}^2)} = \Bigl\|\widehat{f}^{\mathrm{rad}}|_{K\times\mathbb{R}}\Bigr\|_{L^2(K\times\mathbb{R})},
$$

*where we equip $K \times \mathbb{R}$ with the Haar measure $\mathrm{d}m_K\,\mathrm{d}\xi = \frac{1}{2\pi}\,\mathrm{d}\theta\,\mathrm{d}\xi$.*

Proof. We first note that

$$
\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^2)} &= \int_0^{2\pi}\int_0^\infty \bigl|f\bigl(rk_\theta e_1\bigr)\bigr|^2\,r\,\mathrm{d}r\,\mathrm{d}\theta\\
&= \int_0^{2\pi}\int_0^\infty \bigl|rf\bigl(rk_\theta e_1\bigr)\bigr|^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\theta.
\end{aligned}
$$

We now define, for $\theta \in [0,2\pi)$, the function $F_\theta\colon \mathbb{R} \to \mathbb{C}$ by

$$
F_\theta(s) = \mathrm{e}^s f\bigl(\mathrm{e}^s k_\theta e_1\bigr)
$$

for all $s \in \mathbb{R}$, and note that $F_\theta$ corresponds, roughly speaking, to the restriction of $f$ to the ray from 0 at angle $\theta$ to the positive $x$-axis. Using the fact that $f$ has compact support in $\mathbb{R}^2$ and is bounded near 0, we see that $F_\theta \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for all $\theta \in \mathbb{R}$. Using the substitution $r = \mathrm{e}^s$ with $\frac{\mathrm{d}r}{r} = \mathrm{d}s$ we see that

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \int_0^{2\pi} \underbrace{\int_{-\infty}^{\infty} |F_\theta(s)|^2 \,\mathrm{d}s}_{=\|F_\theta\|^2_{L^2(\mathbb{R})}} \,\mathrm{d}\theta.$$

Next we use the fact that

$$\|F_\theta\|_{L^2(\mathbb{R})} = \|\widecheck{F_\theta}\|_{L^2(\mathbb{R})},$$

where $\widecheck{F_\theta}$ again denotes the Fourier back transform. Using the definitions and the substitution $r = \mathrm{e}^s$ with $\mathrm{d}r = \mathrm{e}^s\,\mathrm{d}s$ again, we obtain

$$\begin{aligned} \widecheck{F_\theta}(\zeta) &= \int_{-\infty}^{\infty} F_\theta(s)\mathrm{e}^{2\pi\mathrm{i}\zeta s}\,\mathrm{d}s \\ &= \int_{-\infty}^{\infty} \mathrm{e}^s f(\mathrm{e}^s k_\theta e_1)\mathrm{e}^{2\pi\mathrm{i}\zeta s}\,\mathrm{d}s \\ &= \int_0^{\infty} f(rk_\theta e_1) r^{2\pi\mathrm{i}\zeta}\,\mathrm{d}r = \widehat{f}^{\mathrm{rad}}(k_\theta, 2\pi\zeta) \end{aligned} \tag{9.23}$$

for all $\zeta \in \mathbb{R}$. Together, we obtain

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^2)} &= \int_0^{2\pi} \|\widecheck{F_\theta}\|^2_{L^2(\mathbb{R})}\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_{-\infty}^{\infty} |\widehat{f}^{\mathrm{rad}}(k_\theta, 2\pi\zeta)|\,\mathrm{d}\zeta\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty} |\widehat{f}^{\mathrm{rad}}(k_\theta, \xi)|\,\mathrm{d}\xi\,\mathrm{d}\theta = \|\widehat{f}^{\mathrm{rad}}|_{K\times\mathbb{R}}\|^2_{L^2(K\times\mathbb{R})} \end{aligned}$$

by using the substitution $\xi = 2\pi\zeta$. □

Definition 9.19 and Lemmas 9.26 and 9.27 suggest the following definition.

**Definition 9.28 (Integrals of principal series representations).** For any $\sigma$-finite measure $\mu$ on $\mathbb{R}$, we define the space $\mathcal{H}_\mu$ of all functions

$$F\colon \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{C}$$

satisfying the following properties:

(1) $F$ is measurable;
(2) $F(hb,\xi) = \overline{\chi_\xi(b)}\Delta_B(b)^{\frac{1}{2}}F(h,\xi)$ for all $h \in \mathrm{SL}_2(\mathbb{R})$, $b \in B$, $\xi \in \mathbb{R}$; and
(3) $\|f|_{K\times\mathbb{R}}\|_{L^2(K\times\mathbb{R},m_K\times\mu)} < \infty$.

The unitary representation[†]

[†] We did not discuss the integral of unitary representations, but believe that the notation is justified in this case.

$$\pi^\mu = \int_{\mathbb{R}} \pi^\xi \, \mathrm{d}\mu(\xi)$$

is defined by the left regular representation on the first component; that is,

$$\pi^\mu_g(F)(h,\xi) = F(g^{-1}h,\xi)$$

for all $F \in \mathcal{H}_\mu$, $g,h \in \mathrm{SL}_2(\mathbb{R})$, and $\xi \in \mathbb{R}$. Moreover, we also define

$$\pi^{\mu,\mathrm{e}} = \int_{\mathbb{R}} \pi^{\xi,\mathrm{e}} \, \mathrm{d}\mu(\xi)$$

and

$$\pi^{\mu,\mathrm{o}} = \int_{\mathbb{R}} \pi^{\xi,\mathrm{o}} \, \mathrm{d}\mu(\xi)$$

to be the restrictions of $\pi^\mu$ to the subspaces

$$\mathcal{H}^{\mathrm{even}}_\mu = \{F \in \mathcal{H}_\mu \mid F(-g,x) = F(g,x) \text{ for all } g,x\}$$

and

$$\mathcal{H}^{\mathrm{odd}}_\mu = \{F \in \mathcal{H}_\mu \mid F(-g,x) = -F(g,x) \text{ for all } g,x\}$$

respectively.

**Proposition 9.29 (Spectral decomposition of $\pi^{\mathbb{R}^2}$).** *The Koopman representation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$ is isomorphic to*

$$\pi^m = \pi^{m,\mathrm{e}} \oplus \pi^{m,\mathrm{o}} = \int_{\mathbb{R}} \pi^{\xi,\mathrm{e}} \, \mathrm{d}\xi \oplus \int_{\mathbb{R}} \pi^{\xi,\mathrm{o}} \, \mathrm{d}\xi,$$

*where we use the Lebesgue measure $\mu = m$ on $\mathbb{R}$.*

Proof. By Lemmas 9.26 and 9.27, the radial Fourier transform

$$C_c(\mathbb{R}^2) \ni f \longmapsto \widehat{f}^{\,\mathrm{rad}} \in \mathcal{H}_m$$

is well-defined, equivariant, and an isometry. Hence it extends by the density of $C_c(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$ to a well-defined, equivariant isometry

$$L^2(\mathbb{R}^2) \ni f \longmapsto \widehat{f}^{\,\mathrm{rad}} \in \mathcal{H}_m.$$

We keep referring to $\widehat{f}^{\,\mathrm{rad}}$ as the radial Fourier transform of $f \in L^2(\mathbb{R}^2)$.

It remains to show that this map is onto. For this, assume that $f_K \in C(K)$ and $f_{\mathbb{R}} \in C_c(\mathbb{R})$. We note that

$$f_K \otimes \widecheck{f_{\mathbb{R}}} \colon K \times \mathbb{R} \longrightarrow \mathbb{C}$$

can be extended using property (2) in Definition 9.28 to an element of $\mathcal{H}_m$. We claim that $f_K \otimes \widecheck{f_{\mathbb{R}}} = \widehat{f}^{\mathrm{rad}}$ for some $f \in C_c(\mathbb{R}^2)$. Also recall that the subspaces $C(K) \subseteq L^2(K)$ and $\widecheck{C_c(\mathbb{R})} \subseteq L^2(\mathbb{R}, m)$ are dense (for the latter, apply Theorem 2.15). Varying $f_K$ and $f_{\mathbb{R}}$, we can then, for example, approximate any function of the form $\mathbb{1}_{B_K} \otimes \mathbb{1}_{B_{\mathbb{R}}}$ extended to an element of $\mathcal{H}_\mu$, where $B_K \subseteq K$ and $B_{\mathbb{R}} \subseteq \mathbb{R}$ are measurable with finite measures. For this reason, the claim implies that the image of the radial Fourier transform (extended to $L^2(\mathbb{R}^2)$) is indeed all of $\mathcal{H}_\mu$.

To prove the claim, we reuse the argument from the proof of Lemma 9.27. Let $f_K \in C(K)$ and $f_{\mathbb{R}} \in C_c(\mathbb{R})$ be as above. We define $f \in C_c(\mathbb{R}^2)$ using polar coordinates by

$$f(rk_\theta e_1) = \tfrac{1}{2\pi} f_K(\theta) r^{-1} f_{\mathbb{R}}\left(\tfrac{1}{2\pi}\log r\right).$$

For this $f$, the function $F_\theta$ for $\theta \in [0, 2\pi)$ appearing in the proof of Lemma 9.27 becomes

$$F_\theta(s) = \mathrm{e}^s f(\mathrm{e}^s k_\theta e_1) = \tfrac{1}{2\pi} f_K(\theta) f_{\mathbb{R}}\left(\tfrac{1}{2\pi}s\right)$$

for $s \in \mathbb{R}$. Hence by (9.23) and the substitution $\widetilde{s} = \frac{1}{2\pi}s$ we have

$$\begin{aligned}\widehat{f}^{\mathrm{rad}}(k_\theta, 2\pi\zeta) = \widecheck{F_\theta}(\zeta) &= \tfrac{1}{2\pi} f_K(\theta) \int_R f_{\mathbb{R}}\left(\tfrac{1}{2\pi}s\right) \mathrm{e}^{2\pi \mathrm{i} s\zeta}\,\mathrm{d}s \\ &= f_K(\theta) \int_{\mathbb{R}} f_{\mathbb{R}}(\widetilde{s}) \mathrm{e}^{2\pi\mathrm{i}\widetilde{s}2\pi\zeta}\,\mathrm{d}\widetilde{s} = f_K(\theta)\widecheck{f_{\mathbb{R}}}(2\pi\zeta)\end{aligned}$$

for all $\theta \in [0, 2\pi)$ and $\zeta \in \mathbb{R}$. Equivalently, we have $\widehat{f}^{\mathrm{rad}} = f_K \otimes \widecheck{f_{\mathbb{R}}}$ as claimed, which gives the proposition. □

**Exercise 9.30.** (a) Show that $\pi^\mu$ as in Definition 9.28 is indeed a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ for any $\sigma$-finite measure $\mu$ on $\mathbb{R}$.
(b) Show that $\pi^\mu$ is tempered.

**Exercise 9.31.** Use Fourier inversion on $\mathbb{R}$ to prove a Fourier inversion formula that expresses $f \in C_c^\infty(\mathbb{R}^2)$ as an integral over values of $\widehat{f}^{\mathrm{rad}}$.

**Exercise 9.32.** Is the centralizer of the Koopman representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{R})$ abelian? Prove your claim. Can you identify the centralizer?

### 9.4.2 Moving a Loudspeaker to Infinity

To better understand the formulas required for the hyperbolic Fourier transform we wish to discuss a physical interpretation of the Fourier transform on the two planes $\mathbb{R}^2$ and $\mathbb{H}$. As this is just meant as a motivation for the formal definitions coming later, we leave the details of these calculations as exercises.

To begin with, we imagine a loudspeaker $L$, which we assume will produce the desired sound for any given frequency and amplitude. We imagine the sound wave being represented by a $\mathbb{C}$-valued function $f_L$ on the plane, where $|f_L|^2$ represents the energy of the wave, and the argument of $f_L$ represents the phase shift of the wave. We also imagine that there is no energy loss in the passage of the wave through the medium. This physical interpretation suggests that $|f_L(P_0)|^2$ is inversely proportional to the length of the circle with centre $L$ containing a point $P$, as illustrated in Figure 9.3.

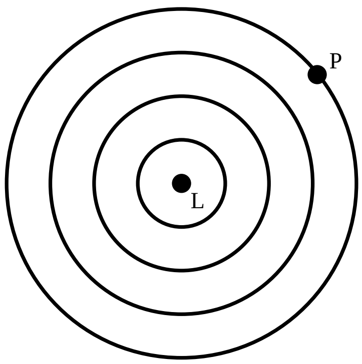


**Fig. 9.3:** The sound waves emanate from the loudspeaker $L$ and decrease in loudness. The concentric circles indicate the phase of $f_L$.

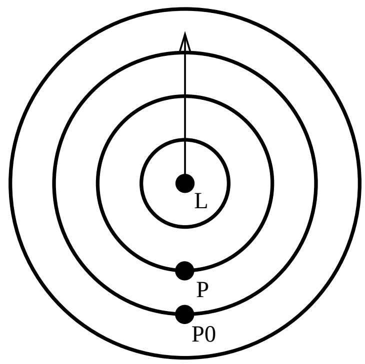


**Fig. 9.4:** We move the loudspeaker $L$ upwards, and turn up the volume.

We now fix some origin $P_0$ in the plane, and move the loudspeaker $L$ further away in some pre-determined direction, say upwards as in Figure 9.4. This of course means that we do not hear the sound much at $P_0$ if $L$ is already far away. To get round this problem, we simultaneously turn up the volume at $L$ so that $|f_L(P_0)|^2 = 1$. We now wish to move $L$ to infinity and describe what will happen to $f_L$ if we do so. However, to do this we have to distinguish between the cases of the Euclidean and hyperbolic planes.

**Euclidean plane:** In the Euclidean plane, the circle of radius $r$ has circumference $2\pi r$. If $P_0$ belongs to a circle of radius $r_0$ (that is, if $r_0 = \|P_0 - L\|$) and $P$ has distance $\|P - P_0\|$ to $P_0$, then $P$ belongs to a circle of radius $r$ with $\Delta r = r - r_0$ satisfying $|\Delta r| \leqslant \|P - P_0\|$ (by the triangle inequality). Hence

$$|f_L(P_0)|^2 \cdot 2\pi r_0 = |f_L(P)|^2 \cdot 2\pi r.$$

Letting $L$ go to infinity, we have $r_0 \to \infty$ and $\frac{2\pi r}{2\pi r_0} = \frac{r_0 + \Delta r}{r_0} \to 1$. Therefore the limiting sound distribution $f$ will have the property that $|f|^2$ is constant. Moreover, the concentric circles degenerate to equidistant parallel lines, so that in the limit we may obtain in this way the function

$$f(x,y) = \mathrm{e}^{\mathrm{i}\xi y}$$

for $P = (x,y) \in \mathbb{R}^2$, where $\xi$ represents the frequency of the wave. Allowing different frequencies and different directions along which $L$ is moved, one obtains in this way any character $\chi_{(\xi_1,\xi_2)}$ for $(\xi_1,\xi_2) \in \mathbb{R}^2$, which we may think of as the elementary waves on $\mathbb{R}^2$.

This suggests the following interpretation for the Fourier transform of a function on $\mathbb{R}^2$. Given $f$, we imagine infinitely many loudspeakers at infinity in all directions using various frequencies with well-chosen amplitudes. Together these then create the prescribed sound distribution $f$ by superposition of the resulting elementary waves.

**Exercise 9.33.** (a) For a given frequency $\xi \in \mathbb{R}$ and two points $P_0, L \in \mathbb{R}^2$, calculate the function $f_L$ representing a wave of frequency $\xi$ emanating from $L$ with $f_L(P_0) = 1$.
(b) Calculate the limit $f$ of $f_L$ as $L = (0,y) \to \infty$.

**Hyperbolic plane:** To get some intuition for the hyperbolic Fourier transform, we repeat the above discussion on $\mathbb{H}$, which will lead to the functions that will take over the role of characters in the more formal discussions of the following sections.

We again imagine the loudspeaker $L$ moving to infinity along the upward oriented geodesic and consider equidistant concentric circles with centre $L$, as in Figure 9.5. We note that in the limit we obtain circles (in the Euclidean sense, within $\mathbb{C} \supseteq \mathbb{D}$) touching the boundary. These are not hyperbolic geodesics; instead these curves are called *horocycles*.

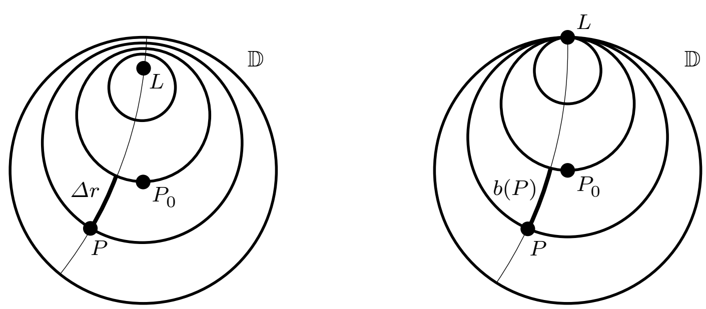


**Fig. 9.5:** A calculation reveals that any Möbius transformation $z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}$ on $\overline{\mathbb{C}}$ maps lines and circles to lines and circles. With this, it is straightforward to verify that concentric hyperbolic circles with centre $L$ appear in the disk model of the hyperbolic plane as circles, with $L$ appearing closer to the circle near the boundary. If $L$ is moved to the boundary, these circles degenerate to circles touching the boundary.

To understand the limit function $f$ of the sound distribution $f_L$ for $L$ going to the boundary, we need to calculate the circumference of a circle of radius $R$. To simplify matters, we let the centre be $0 \in \mathbb{D}$ as in Lemma 8.15. By (8.13) the Euclidean radius of this circle is given by $\rho = \tanh\left(\frac{R}{2}\right)$. Hence the circumference can be calculated using the path

$$[0, 2\pi] \ni \theta \longmapsto \rho e^{i\theta},$$

which, by definition of the Riemannian metric in (8.6), gives

$$\begin{aligned}\int_0^{2\pi} \frac{2}{(1-\rho^2)}\rho \,d\theta &= \frac{4\pi \tanh\left(\frac{R}{2}\right)}{\left(1-\tanh^2\left(\frac{R}{2}\right)\right)} = 4\pi \frac{\frac{\sinh(\frac{R}{2})}{\cosh(\frac{R}{2})}\cdot \cosh^2\left(\frac{R}{2}\right)}{\left(\cosh^2\left(\frac{R}{2}\right)-\sinh^2\left(\frac{R}{2}\right)\right)} \\ &= 4\pi \sinh\left(\tfrac{R}{2}\right)\cosh\left(\tfrac{R}{2}\right) = 2\pi \sinh R.\end{aligned}$$

We let $L$ go to infinity along the Northward geodesic, so that

$$r_0 = \mathsf{d}(P_0, L) \to \infty.$$

For a third point $P \in \mathbb{D}$, we let $r = \mathsf{d}(P, L)$. We also define the 'relative distance' from $L$ compared to $P_0$ by setting it equal to

$$\Delta r = \mathsf{d}(P, L) - \mathsf{d}(P_0, L) = r - r_0,$$

see also Figure 9.5. Note that $\Delta r = r - r_0$ satisfies $|\Delta r| \leqslant \mathsf{d}(P, P_0)$. With $\sinh R \sim e^R$ as $R \to \infty$, and

$$|f_L(P_0)|^2 2\pi \sinh r_0 = |f_L(P)|^2 2\pi \sinh r,$$

we obtain

$$\frac{|f_L(P)|^2}{|f_L(P_0)|^2} = \frac{\sinh r_0}{\sinh(r_0 + \Delta r)} \sim e^{-\Delta r}$$

as $r_0 \to \infty$. Since we normalize the loudness of $L$ along the way to have $|f_L(P_0)|^2 = 1$, we expect that the limit sound wave satisfies

$$|f(P)| = e^{-\frac{1}{2}b(P)}$$

for the limiting function $b$ of $\Delta r$. Putting the phase with frequency $\xi$ into the discussions, we expect that functions of the form[†]

$$f(P) = e^{(-\frac{1}{2}+\frac{i}{2}\xi)b(P)}$$

[†] As earlier, we normalize the meaning of frequency in the following discussions to simplify some of the formulas arising.

are the relevant elementary waves on the hyperbolic plane. This is indeed the case, so we will have to define the so-called Busemann function $b(P)$ more carefully (we refer to Busemann's monograph [6] for a thorough treatment).

By varying both the frequency and the position of the loudspeakers on the boundary of the hyperbolic plane, we again expect that any sound distribution on the plane can be produced as a superposition of elementary waves.

**Exercise 9.34.** Repeat (a) and (b) from Exercise 9.33 for $\mathbb{H}$.

### 9.4.3 The Busemann Function

In the upper half-plane model $\mathbb{H}$, the desired function takes a particularly easy form. Indeed, if we move the loudspeaker $L$ simply up to the point $\infty$ in $\partial\mathbb{H}$, then the concentric circles degenerate to horizontal lines near i, as in Figure 9.6.

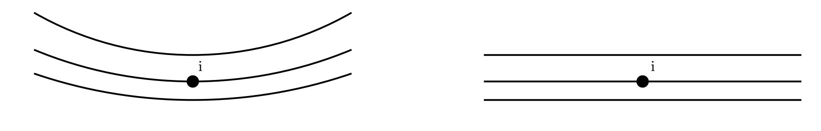


**Fig. 9.6:** On the left we see that the concentric circles with centre $L = y\mathrm{i}$ for large $y$ are almost horizontal Euclidean lines. On moving $y$ to $\infty$, these become horizontal Euclidean lines or *horizontal horocycles* in the hyperbolic plane $\mathbb{H}$.

**Definition 9.35 (Busemann function for $\infty \in \partial\mathbb{H}$).** The *Busemann function* on $\mathbb{H}$ with respect to $\infty \in \partial\mathbb{H}$ (and origin $\mathrm{i} \in \mathbb{H}$) is defined for $z \in \mathbb{H}$ by

$$b_\infty^{\mathbb{H}}(z) = -\log \Im(z).$$

We note that the point $\infty \in \partial\mathbb{H}$ should be thought of as being the point 'at infinity' that is higher up than any $z \in \mathbb{H}$. Roughly speaking, $b_\infty^{\mathbb{H}}(z)$ is comparing the distance of $z$ and of i to $\infty$ (both of which are of course infinite). More precisely, $b_\infty^{\mathbb{H}}(z)$ measures the hyperbolic distance between the horizontal horocycle at $z = x + \mathrm{i}y \in \mathbb{H}$ and the horizontal horocycle at our designated origin i, given by

$$\mathsf{d}(\mathrm{i}y, \mathrm{i}) = \left| \int_1^y \frac{\mathrm{d}y}{y} \right| = |\log y|.$$

We should think of $b_\infty^{\mathbb{H}}(z)$ as an oriented relative distance, since $b_\infty^{\mathbb{H}}(z) > 0$ means that $z$ is further from $\infty$ than i is, while $b_\infty^{\mathbb{H}}(z) < 0$ means that i is further away from $\infty$.

Using the discussion of Section 9.4.2 as in Figure 9.7, we are now led to the following definition.

**Definition 9.36 (Hyperbolic wave).** We define the *hyperbolic wave function* coming from $\infty$ with frequency $\xi \in \mathbb{R}$ (normalized for the origin $\mathrm{i} \in \mathbb{H}$) to be

$$\chi_{\infty,\xi}(z) = \mathrm{e}^{(-\frac{1}{2}+\frac{\mathrm{i}}{2}\xi)b_\infty^{\mathbb{H}}(z)} = \Im(z)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi}$$

for all $z \in \mathbb{H}$.

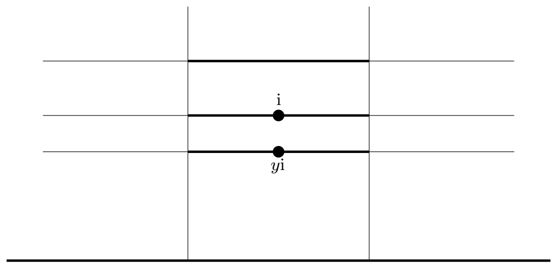


**Fig. 9.7:** We think of a hyperbolic wave coming from $\infty$ with horizontal horocycles being the wave fronts. Since an interval of Euclidean length 1 of the horizontal horocycle with vertical coordinate $y \in (0,\infty)$ has hyperbolic length $\frac{1}{y}$, the energy of the wave spreads over a larger region as it spreads down and so its intensity decreases.

We note that by definition of Möbius transformations in (8.3) and the description of $\Delta_B$ in (8.23) we have

$$\begin{aligned}\chi_{\infty,\xi}(b^{-1}\bullet z) &= \Im\bigl(\mathrm{e}^{-2t}(z-x)\bigr)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi} \\ &= \mathrm{e}^{-t(1-\mathrm{i}\xi)}\Im(z)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi} = \Delta_B(b)^{\frac{1}{2}}\chi_\xi(b)\chi_{\infty,\xi}(z)\end{aligned} \tag{9.24}$$

for all $b = u_x a_t \in B$ and $z \in \mathbb{H}$. These formulas suggest a possible link between the hyperbolic wave of frequency $\xi$ and the principal series representation $\pi^\xi$ corresponding to the frequency parameter $\xi$ (see Section 9.3), which we will explain after defining the hyperbolic Fourier transform.

We also note that the identifications

$$B \ni b = u_x a_t \longmapsto bK \in \mathrm{SL}_2(\mathbb{R})/K \longmapsto z = b\bullet\mathrm{i} = x + \mathrm{e}^{2t}\mathrm{i} \in \mathbb{H}$$

are measure-preserving by our choice of the Haar measure $m_B$ on $B$ in Section 8.3.5 and the normalization $m_K(K) = 1$. We will also simply write $m$ for the Haar measure $m = m_B \times m_K$ on $\mathrm{SL}_2(\mathbb{R}) = BK$, and will write $\int_G \cdot\,\mathrm{d}m$ for integration over $G = \mathrm{SL}_2(\mathbb{R})$.

### 9.4.4 The Hyperbolic Fourier Transform

We recall from Section 8.3.1 that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ by Möbius transformations preserves the hyperbolic area measure defined by

$$\mathrm{dvol} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

By Proposition 1.3 this gives rise to a Koopman unitary representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{H})$ defined by

$$\big(\pi_g^{\mathbb{H}}(f)\big)(z) = f(g^{-1}\bullet z)$$

for $g \in \mathrm{SL}_2(\mathbb{R})$, $f \in L^2(\mathbb{H})$, and $z \in \mathbb{H}$. We note that $\pi_{-I}^{\mathbb{H}} = I$ since

$$(-I)\bullet z = \frac{-1z+0}{0z-1} = z$$

for all $z \in \mathbb{H}$, so that $\pi^{\mathbb{H}}$ is an even representation. We also recall that we may use $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/K$ to identify $L^2(\mathbb{H})$ with the subspace of $L^2(\mathrm{SL}_2(\mathbb{R}))$ consisting of all right $K$-invariant functions. Under this identification, $\pi^{\mathbb{H}}$ becomes the restriction of the left regular representation $\lambda^{\mathrm{SL}_2(\mathbb{R})}$ to this subspace. In particular, $\pi^{\mathbb{H}}$ is tempered, and has uniform decay exponent 1 by Theorem 8.31.

In analogy to the definition of the radial Fourier transform in Definition 9.25, and motivated by the discussions in Sections 9.4.2 and 9.4.3, we are led to the following definition.

**Definition 9.37 (Hyperbolic Fourier transform).** We define the hyperbolic Fourier transform of $f$ at $(h,\xi)$ for $f \in L^2_{\mathrm{vol}}(\mathbb{H})$, $h \in \mathrm{SL}_2(\mathbb{R})$, and a frequency parameter $\xi \in \mathbb{R}$, by

$$\widehat{f}^{\,\mathrm{hyp}}(h,\xi) = \int_{\mathbb{H}} f(h\bullet z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z) = \int_0^\infty\int_{-\infty}^\infty f(h\bullet(x+\mathrm{i}y))y^{\frac{1}{2}+\frac{\mathrm{i}}{2}\xi}\,\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

As with the (radial) Fourier transform, this may not be a well-defined Lebesgue integral but, as we will see, can be defined for almost every $(h,\xi)$ by an isometric extension of the transform on $C_c(\mathbb{H})$. We note that the measure-preserving substitution $w = h\bullet z$ in the definition implies that

$$\widehat{f}^{\,\mathrm{hyp}}(h,\xi) = \int_{\mathbb{H}} f(w)\overline{\chi_{\infty,\xi}(h^{-1}\bullet w)}\,\mathrm{dvol}(w). \tag{9.25}$$

Hence we will think of $\widehat{f}^{\,\mathrm{hyp}}(h,\xi)$ as the *correlation* of $f$ with the hyperbolic wave function $\mathbb{H} \ni w \mapsto \chi_{\infty,\xi}(h^{-1}\bullet w)$ which we think of as 'coming from $h\bullet\infty$ normalized for the point $h\bullet\mathrm{i}$'.

**Lemma 9.38 (Equivariance and decay properties).** *For $f \in C_c(\mathbb{H})$ the hyperbolic Fourier transform $\widehat{f}^{\mathrm{hyp}}$ is well-defined and satisfies*

$$\widehat{\pi_g^{\mathbb{H}} f}^{\mathrm{hyp}}(h,\xi) = \widehat{f}^{\mathrm{hyp}}(g^{-1}h,\xi)$$

*and*

$$\widehat{f}^{\mathrm{hyp}}(hb,\xi) = \overline{\chi_\xi(b)} \Delta_B(b)^{\frac{1}{2}} \widehat{f}^{\mathrm{hyp}}(h,\xi)$$

*for all $g \in \mathrm{SL}_2(\mathbb{R})$, $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$ and $b \in B = AU$.*

In other words, the lemma says that for any $\xi \in \mathbb{R}$ the map

$$C_c(\mathbb{H}) \ni f \longmapsto \widehat{f}^{\mathrm{hyp}}(\cdot,\xi) \in \mathcal{H}_\xi^{\mathrm{even}}$$

is equivariant between the Koopman representation and the principal series representation $\pi^{\xi,\mathrm{e}}$.

Proof of Lemma 9.38. For $g,h \in \mathrm{SL}_2(\mathbb{R})$ and $\xi \in \mathbb{R}$ we have

$$\begin{aligned}
\widehat{\pi_g^{\mathbb{H}}(f)}^{\mathrm{hyp}}(h,\xi) &= \int_{\mathbb{H}} \bigl(\pi_g^{\mathbb{H}} f\bigr)(h\raisebox{0.1em}{\tiny$\bullet$} z) \overline{\chi_{\infty,\xi}(z)} \,\mathrm{dvol}(z) \\
&= \int_{\mathbb{H}} f(g^{-1}h\raisebox{0.1em}{\tiny$\bullet$} z) \overline{\chi_{\infty,\xi}(z)} \,\mathrm{dvol}(z) \\
&= \widehat{f}^{\mathrm{hyp}}(g^{-1}h,\xi).
\end{aligned}$$

Moreover, for $b = u_x a_t \in B$ we also have

$$\begin{aligned}
\widehat{f}^{\mathrm{hyp}}(hb,\xi) &= \int_{\mathbb{H}} f\bigl(h\raisebox{0.1em}{\tiny$\bullet$}(b\raisebox{0.1em}{\tiny$\bullet$} z)\bigr) \overline{\chi_{\infty,\xi}(z)} \,\mathrm{dvol}(z) \\
&= \int_{\mathbb{H}} f(h\raisebox{0.1em}{\tiny$\bullet$} w) \overline{\chi_{\infty,\xi}(b^{-1}\raisebox{0.1em}{\tiny$\bullet$} w)} \,\mathrm{dvol}(w) \\
&= \Delta_B(b)^{\frac{1}{2}} \overline{\chi_\xi(b)} \int_{\mathbb{H}} f(h\raisebox{0.1em}{\tiny$\bullet$} w) \overline{\chi_{\infty,\xi}(w)} \,\mathrm{dvol}(w)
\end{aligned}$$

by using the measure-preserving substitution $w = b\raisebox{0.1em}{\tiny$\bullet$} z$ and (9.24), which proves the lemma. □

We wish to explain Lemma 9.38 in another, more convenient, way using convolutions. For this, we let $g = kb \in KB = \mathrm{SL}_2(\mathbb{R})$ with $b = u_x a_t \in B$ and obtain

$$\chi_{\infty,\xi}(g^{-1}\raisebox{0.1em}{\tiny$\bullet$}\mathrm{i}) = \chi_{\infty,\xi}(b^{-1}\raisebox{0.1em}{\tiny$\bullet$}\mathrm{i}) = \mathrm{e}^{-t+\mathrm{i}t\xi} = \overline{F_\xi(b)} = \overline{F_\xi(g)}$$

where $F_\xi = F_{\xi,0} \in \mathcal{H}_\xi^{\mathrm{even}}$ is defined in (9.16). We also identify $\chi_{\infty,\xi}$ with the right $K$-invariant function

$$\chi_{\infty,\xi} \colon \mathrm{SL}_2(\mathbb{R}) \ni g \longmapsto \chi_{\infty,\xi}(g\raisebox{0.1em}{\tiny$\bullet$}\mathrm{i}).$$

Recalling that $\mathrm{SL}_2(\mathbb{R})$ is unimodular, we can use the involution of Section 1.4.1 to put the above into the form

$$\chi^*_{\infty,\xi} = F_\xi. \tag{9.26}$$

The identification between functions on $\mathbb{H}$ with right $\mathrm{SO}_2(\mathbb{R})$-invariant functions on $\mathrm{SL}_2(\mathbb{R})$ allows us to use convolutions in $L^1(\mathrm{SL}_2(\mathbb{R}))$ as discussed in Section 1.4.1 for functions on $\mathbb{H}$. We will however also use convolutions of functions in $C_c(\mathrm{SL}_2(\mathbb{R}))$ and $C(\mathrm{SL}_2(\mathbb{R}))$, giving rise to functions in $C(\mathrm{SL}_2(\mathbb{R}))$ (see Exercise 1.44).

**Lemma 9.39 (Convolution formula).** *For a function $f \in C_c(\mathbb{H})$ we have*

$$\widehat{f}^{\,\mathrm{hyp}}(h,\xi) = f * F_\xi(h) = \int_G f(g\bullet \mathrm{i}) F_\xi(g^{-1}h)\,\mathrm{d}m_G(g)$$

*for all $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$.*

We note that Lemma 9.39 implies both claims of Lemma 9.38. Indeed, $f * F_\xi \in \mathcal{H}_\xi$ since $F_\xi \in \mathcal{H}_\xi$ and $\mathcal{H}_\xi$ is defined by a formula using the right regular representation restricted to $B$, which commutes with the left convolution. Similarly, the equivariance under the Koopman representation (or, equivalently, under the left regular representation) also follows from the properties of convolutions.

Proof of Lemma 9.39. For a function $f \in C_c(\mathbb{H})$ and $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$ we have

$$\begin{aligned}
\widehat{f}^{\,\mathrm{hyp}}(h,\xi) &= \int_{\mathbb{H}} f(w)\overline{\chi_{\infty,\xi}}(h^{-1}\bullet w)\,\mathrm{dvol}(w) && \text{(by (9.25))}\\
&= \int_G f(g\bullet \mathrm{i})\chi^*_{\infty,\xi}(g^{-1}h)\,\mathrm{d}m(g)\\
&= \int_G f(g\bullet \mathrm{i})F_\xi(g^{-1}h)\,\mathrm{d}m(g) = f * F_\xi(h), && \text{(by (9.26))}
\end{aligned}$$

where we also extended integration from $w = g\bullet\mathrm{i} \in \mathbb{H}$ to $g \in \mathrm{SL}_2(\mathbb{R})$ using $m_K(K) = 1$. □

**Lemma 9.40 (Rapid decay of transform).** *For any function $f \in C_c^\infty(\mathbb{H})$, we have*

$$\left\|\mathbb{R} \ni \xi \longmapsto \xi^\ell \widehat{f}^{\,\mathrm{hyp}}(I,\xi)\right\|_\infty < \infty$$

*for any $\ell \in \mathbb{N}_0$.*

Proof. We first recall that for $F \in C_c^\infty(\mathbb{R})$ we have

$$\|\mathbb{R} \ni \xi \longmapsto \xi^\ell \check{F}(\xi)\|_\infty \ll_\ell \|F^{(\ell)}\|_1 \tag{9.27}$$

by partial integration and induction on $\ell$ (see, for example, [21, Prop. 9.43]). To apply this, we rewrite the definition of $\widehat{f}^{\mathrm{hyp}}$ using the substitution $y = \mathrm{e}^{2t}$ with $\frac{\mathrm{d}y}{y} = 2\,\mathrm{d}t$, which gives

$$\begin{aligned}\widehat{f}^{\mathrm{hyp}}(I,\xi) &= \int_{\mathbb{H}} f(z)\overline{\chi_{\infty,\xi}(z)}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= \int_0^\infty \int_{-\infty}^\infty f(x+\mathrm{i}y)\,\mathrm{d}x\,y^{-\frac{1}{2}}y^{\frac{\mathrm{i}}{2}\xi}\,\frac{\mathrm{d}y}{y}\\ &= \int_{-\infty}^\infty \underbrace{\int_{-\infty}^\infty f\big(x+\mathrm{i}\mathrm{e}^{2t}\big)\,\mathrm{d}x\,2\mathrm{e}^{-t}}_{=F(t)}\,\mathrm{e}^{\mathrm{i}t\xi}\,\mathrm{d}t = \widehat{F}\big(\tfrac{1}{2\pi}\xi\big).\end{aligned}$$

Differentiation under the integral sign shows that the function $F \in C_c(\mathbb{R})$ is indeed smooth, so that (9.27) proves the lemma. □

### 9.4.5 The Hyperbolic Fourier Inversion Formula

As explained at the end of Section 9.4.2, we expect to be able to write a given function on $\mathbb{H}$ as a superposition of elementary waves $z \mapsto \chi_{\infty,\xi}(k^{-1}\raisebox{0.3ex}{.}z)$ of various frequencies $\xi$ emanating from the boundary points $k\raisebox{0.3ex}{.}\infty \in \partial\mathbb{H}$. For this the hyperbolic Fourier transform $\widehat{f}^{\mathrm{hyp}}(k,\xi)$ for a pair $(k,\xi) \in K \times \mathbb{R}$ should be related to the desired volume at $k\raisebox{0.3ex}{.}\infty \in \partial\mathbb{H}$ for frequency $\xi$. Assuming smoothness and compact support of the original function ensures that the desired integral representation converges.

**Theorem 9.41 (Hyperbolic Fourier inversion).** *Let $f \in C_c^\infty(\mathbb{H})$. Then*

$$f(z) = \frac{1}{16\pi}\int_{\mathbb{R}}\int_K \widehat{f}^{\mathrm{hyp}}(k,\xi)\chi_{\infty,\xi}(k^{-1}\raisebox{0.3ex}{.}z)\,\mathrm{d}m_K(k)\xi\tanh\frac{\pi\xi}{2}\,\mathrm{d}\xi$$

*for all $z \in \mathbb{H}$.*

The proof will rely on some elementary integral manipulations, Fourier inversion on $\mathbb{R}$ (applied in a surprising way), and a contour integration to determine the correct volume amplification factor $\xi\tanh\big(\frac{\pi\xi}{2}\big)$ for $\xi \in \mathbb{R}$. To reduce the complexity of the problem, we first consider a special class of functions.

**Definition 9.42 (Spherical functions).** A function $f\colon \mathbb{H} \to \mathbb{C}$ is called *spherical* if $f(k\raisebox{0.3ex}{.}z) = f(z)$ for all $k \in K$ and $z \in \mathbb{H}$.

We note that due to the equivariance property in Lemma 9.38 the hyperbolic Fourier transform of a spherical function is again invariant under $K$.

Because of this, for a spherical function $f$ we will also use the simplified notation

$$\widehat{f}^{\text{hyp}}(\xi) = \int_{\mathbb{H}} f(z)\chi_{\infty,\xi}(z)\,\mathrm{dvol}(z)$$

satisfying

$$\widehat{f}^{\text{hyp}}(k,\xi) = \int_{\mathbb{H}} f(k\cdot z)\chi_{\infty,\xi}(z)\,\mathrm{dvol}(z) = \widehat{f}^{\text{hyp}}(\xi) \tag{9.28}$$

for all $(k,\xi) \in K \times \mathbb{R}$ by Definitions 9.37 and 9.42 above. Recall that Lemma 9.38 also shows that the function $\widehat{f}^{\text{hyp}}(\cdot,\xi)$ belongs to $\mathcal{H}_\xi^{\text{even}}$. With this, (9.28) becomes

$$\widehat{f}^{\text{hyp}}(\cdot,\xi) = \widehat{f}^{\text{hyp}}(\xi)F_\xi(\cdot), \tag{9.29}$$

where $F_\xi$ is the extension of $\mathbb{1}_K$ to an element of $\mathcal{H}_\xi^{\text{even}}$ (the case $n = 0$ in (9.16)).

The following lemma gives another connection between the hyperbolic Fourier transform and $\pi^\xi$, or more precisely its matrix coefficient $\phi_\xi = \varphi_{F_\xi}^{\pi^\xi}$.

**Lemma 9.43 (Matrix coefficient giving symmetry).** *Let $f \in C_c(\mathbb{H})$ be a spherical function. Then*

$$\widehat{f}^{\text{hyp}}(\xi) = \int_{\mathbb{H}} f\phi_\xi\,\mathrm{dvol}(z) \tag{9.30}$$

*for all $\xi \in \mathbb{R}$. Moreover, we have $\phi_\xi = \phi_{-\xi}$ and*

$$\widehat{f}^{\text{hyp}}(\xi) = \widehat{f}^{\text{hyp}}(-\xi)$$

*for all $\xi \in \mathbb{R}$.*

Proof. Using (9.28), the normalized Haar measure $m_K$ on $K$, the convolution formula in Lemma 9.39, and Fubini's theorem we have

$$\begin{aligned}
\widehat{f}^{\text{hyp}}(\xi) &= \int_K \widehat{f}^{\text{hyp}}(k,\xi)\,\mathrm{d}m_K(k)\\
&= \int_K\int_G f(g\cdot\mathrm{i})F_\xi(g^{-1}k)\,\mathrm{d}m_G(g)\,\mathrm{d}m_K(k)\\
&= \int_G f(g\cdot\mathrm{i}) \underbrace{\int_K F_\xi(g^{-1}k)F_\xi(k)\,\mathrm{d}m_K(k)}_{=\langle\pi_g^\xi F_\xi,F_\xi\rangle=\phi_\xi(g)}\,\mathrm{d}m_G(g),
\end{aligned}$$

which proves (9.30).

Finally, by Theorem 9.20, $\pi^{\xi,\mathrm{e}}$ and $\pi^{-\xi,\mathrm{e}}$ are unitarily isomorphic. Since the vector $F_{\pm\xi} \in \mathcal{H}_{\pm\xi}^{\text{even}}$ is, up to scalar multiples, the unique $K$-fixed vector and both have unit length, it follows that

$$\phi_{-\xi} = \varphi_{F_{-\xi}}^{\pi^{-\xi,\mathrm{e}}} = \varphi_{F_\xi}^{\pi^{\xi,\mathrm{e}}} = \phi_\xi.$$

Together with (9.30), this gives the lemma. Alternatively, one may also apply Proposition 9.15 and Lemma 9.22. □

We note that we are going to use the assumption that $f \in C_c^\infty(\mathbb{H})$ is spherical to reduce the number of free variables in the proof of the Fourier inversion formula. In fact, $f$ is spherical if and only if it can be written as

$$f(z) = F_d\bigl(\mathsf{d}_{\mathrm{hyp}}(z, \mathrm{i})\bigr)$$

for $z \in \mathbb{H}$, where $F_d\colon [0,\infty) \to \mathbb{C}$ is a function, and $\mathsf{d}_{\mathrm{hyp}}$ denotes the hyperbolic metric on $\mathbb{H}$ (see Lemma 8.16). This follows from the transitivity of the action of $K$ on every circle with centre i.

In terms of the upcoming integral substitutions, it is better to consider instead of $\mathsf{d}_{\mathrm{hyp}}(z,\mathrm{i})$ the closely related expression

$$r(z) = \cosh^2\bigl(\tfrac{1}{2}\mathsf{d}_{\mathrm{hyp}}(z,\mathrm{i})\bigr) = \tfrac{1}{2}\cosh\bigl(\mathsf{d}_{\mathrm{hyp}}(z,\mathrm{i})\bigr) + \tfrac{1}{2} \tag{9.31}$$

for $z \in \mathbb{H}$, where we have used the identity

$$\cosh^2 t = \left(\frac{\mathrm{e}^t + \mathrm{e}^{-t}}{2}\right)^2 = \frac{1}{2}\cosh(2t) + \frac{1}{2}$$

for $t = \frac{1}{2}\mathsf{d}_{\mathrm{hyp}}(z,\mathrm{i})$. Once more every spherical function $f\colon \mathbb{H} \to \mathbb{C}$ can be written in the form

$$f(z) = F_r(r(z))$$

for all $z \in \mathbb{H}$ and for some function $F_r\colon [1,\infty) \to \mathbb{C}$.

**Lemma 9.44 (Inversion at i for spherical functions).** *For a spherical function $f \in C_c^\infty(\mathbb{H})$ we have*

$$f(\mathrm{i}) = \frac{1}{16\pi^2}\int_{\mathbb{R}} \widehat{f}^{\mathrm{hyp}}(\xi)\xi \int_{\mathbb{R}} \frac{\sin \xi t}{\sinh t}\,\mathrm{d}t\,\mathrm{d}\xi.$$

Proof. As explained above, for the given spherical function $f \in C_c^\infty(\mathbb{H})$ there exists a continuous function $F \in C_c([1,\infty))$ with $f(z) = F(r(z))$ for all $z \in \mathbb{H}$. In fact we can define $F$ by restricting $f$ to $\{y\mathrm{i} \mid y > 0\}$ and an appropriate coordinate change. Indeed, if $y = \mathrm{e}^{2t}$, then $\mathsf{d}_{\mathrm{hyp}}(\mathrm{e}^{2t}\mathrm{i},\mathrm{i}) = 2|t|$ and $r(z) = \cosh^2 t$, which leads to the definition

$$F(r) = f\Bigl(\mathrm{i}\mathrm{e}^{2\operatorname{arcosh}\sqrt{r}}\Bigr)$$

for $r \in [1,\infty)$. Equivalently, we have $F(\cosh^2 t) = f(\mathrm{i}\mathrm{e}^{2t})$ for $t \in \mathbb{R}$, and, since $f$ is spherical, more generally, $F(r(z)) = f(z)$ for all $z \in \mathbb{H}$.

Using the formula for $\mathsf{d}_{\mathrm{hyp}}(\cdot,\cdot)$ in Lemma 8.16 we also have, for $r(z)$ as in (9.31),

$$\begin{aligned} r(z) &= \frac{1}{2}\left(1+\frac{\|z-\mathrm{i}\|^2}{2y}\right)+\frac{1}{2} \\ &= \frac{1}{2}\left(1+\frac{x^2}{2y}+\frac{(y-1)^2}{2y}\right)+\frac{1}{2} \\ &= \frac{1}{2}\left(1+\frac{x^2}{2y}+\frac{y+y^{-1}}{2}-1\right)+\frac{1}{2}=\frac{1}{2}+\frac{y+y^{-1}}{4}+\frac{x^2}{4y}. \end{aligned}$$

Below we will also use the coordinates $(x,t)\in\mathbb{R}^2$ for $z=x+\mathrm{i}\mathrm{e}^{2t}\in\mathbb{H}$, which gives

$$r(z)=\frac{1}{2}+\frac{\mathrm{e}^{2t}+\mathrm{e}^{-2t}}{4}+\frac{1}{4}x^2\mathrm{e}^{-2t}=\cosh^2 t+\frac{1}{4}x^2\mathrm{e}^{-2t}.$$

For our functions $f$ and $F$, this gives

$$F\left(\cosh^2 t+\tfrac{1}{4}x^2\mathrm{e}^{-2t}\right)=f(x+\mathrm{i}\mathrm{e}^{2t})$$

for $x,t\in\mathbb{R}$.

We now use this identity in the formula for the hyperbolic Fourier transform, and obtain

$$\begin{aligned} \widehat{f}^{\mathrm{hyp}}(\xi) &= \int_{\mathbb{H}} f(x+\mathrm{i}y)y^{\frac{1}{2}+\frac{1}{2}\mathrm{i}\xi}\,\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} F\left(\cosh^2 t+\tfrac{1}{4}x^2\mathrm{e}^{-2t}\right)\mathrm{e}^{t+\mathrm{i}\xi t}\mathrm{e}^{-4t}\,\mathrm{d}x\,2\mathrm{e}^{2t}\,\mathrm{d}t \\ &= 2\int_{\mathbb{R}}\int_{\mathbb{R}} F\left(\cosh^2 t+\tfrac{1}{4}x^2\mathrm{e}^{-2t}\right)\,\mathrm{d}x\,\mathrm{e}^{-t}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t \end{aligned}$$

by using $y=\mathrm{e}^{2t}$ and $\mathrm{d}y=2\mathrm{e}^{2t}\,\mathrm{d}t$. We now set $u=\frac{1}{2}\mathrm{e}^{-t}x$ with $2\,\mathrm{d}u=\mathrm{e}^{-t}\,\mathrm{d}x$ to arrive at

$$\widehat{f}^{\mathrm{hyp}}(\xi)=4\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\xi t}\int_{\mathbb{R}} F\left(\cosh^2 t+u^2\right)\,\mathrm{d}u\,\mathrm{d}t. \tag{9.32}$$

We use the inner integral to define the function

$$\varPhi(s)=\int_{\mathbb{R}} F\left(s+u^2\right)\,\mathrm{d}u$$

for $s\in[1,\infty)$ and the composition

$$\varPsi(t)=\varPhi(\cosh^2 t)=\int_{\mathbb{R}} F\left(\cosh^2 t+u^2\right)\,\mathrm{d}u \tag{9.33}$$

for $t \in \mathbb{R}$. We note that $\Phi \in C_c([1,\infty))$ and $\Psi \in C_c(\mathbb{R})$.

With the function $\Psi$ to hand, we realize that $\widehat{f}^{\text{hyp}}$ is essentially the Fourier transform of $\Psi$. More precisely, we may reformulate (9.32) as

$$\widehat{f}^{\text{hyp}}(\xi) = 4\check{\Psi}\left(\frac{1}{2\pi}\xi\right) \tag{9.34}$$

for all $\xi \in \mathbb{R}$.

By Lemma 9.40, we know that $\widehat{f}^{\text{hyp}}$ decays rapidly. Hence we may apply Fourier inversion on $\mathbb{R}$, and the substitution $\xi = 2\pi\zeta$ to obtain from (9.34) that

$$\Psi(t) = \int_{\mathbb{R}} \check{\Psi}(\zeta)\mathrm{e}^{-2\pi\mathrm{i}\zeta t}\,\mathrm{d}\zeta = \frac{1}{2\pi}\int_{\mathbb{R}} \check{\Psi}\left(\frac{1}{2\pi}\xi\right)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}\xi = \frac{1}{8\pi}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}\xi$$

for all $t \in \mathbb{R}$. By Lemma 9.43 we have $\widehat{f}^{\text{hyp}}(\xi) = \widehat{f}^{\text{hyp}}(-\xi)$, which turns the above into

$$\Phi(\cosh^2 t) = \Psi(t) = \frac{1}{8\pi}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\cos(\xi t)\,\mathrm{d}\xi$$

Using the rapid decay of $\widehat{f}^{\text{hyp}}$ in Lemma 9.40 again, we may differentiate under the integral to obtain

$$\Psi'(t) = -\frac{1}{8\pi}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\xi\sin(\xi t)\,\mathrm{d}\xi \tag{9.35}$$

for $t \in \mathbb{R}$. Using the chain rule for $\Psi(t) = \Phi(\cosh^2 t)$, we also have

$$\Psi'(t) = \Phi'(\cosh^2 t)2\cosh t\sinh t$$

for $t \in \mathbb{R}\smallsetminus\{0\}$. We divide this by $2\sinh t$ to obtain, with (9.35),

$$\Phi'(\cosh^2 t)\cosh t = \frac{\Psi'(t)}{2\sinh t} = -\frac{1}{16\pi}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\xi\frac{\sin\xi t}{\sinh t}\,\mathrm{d}\xi \tag{9.36}$$

for $t \in \mathbb{R}\smallsetminus\{0\}$.

Moreover, note that

$$\lim_{t\to 0}\frac{\sin\xi t}{\sinh t} = \xi$$

and

$$\left|\frac{\sin\xi t}{\sinh t}\right| = \left|\frac{\sin\xi t}{t}\right|\frac{t}{\sinh t} \leqslant |\xi|\frac{t}{\sinh t} \leqslant |\xi| \tag{9.37}$$

by the mean-value theorem applied to the function $\mathbb{R} \ni t \mapsto \sin\xi t$. Together with Lemma 9.40, this shows that $\Phi'$ extends continuously from $(1,\infty)$ to $[1,\infty)$. It follows from the mean value theorem for $\Phi$ that $\Phi$ also has a one-sided derivative as $s = 1$.

The Fourier inversion formulas above allow us to obtain the value

$$\Phi(1) = \Psi(0) = \int_{\mathbb{R}} F(1+u^2)\,\mathrm{d}u$$

from the hyperbolic Fourier transform, but we wish instead to obtain the value $F(1) = f(\mathrm{i})$ of the integrand $F$ in the definition of $\Phi$. To obtain this, we rely on some stunning but elementary integration trickery. In fact, we claim that one can recover $F$ from $\Phi$ by the formula

$$F(s) = -\frac{1}{\pi}\int_{\mathbb{R}} \Phi'(s+v^2)\,\mathrm{d}v \tag{9.38}$$

for all $s \geqslant 1$.

To see this, note first that $f \in C_c^\infty(\mathbb{H})$ implies that $F|_{(1,\infty)}$ is smooth, that

$$\Phi'(s) = \int_{\mathbb{R}} F'(s+u^2)\,\mathrm{d}u$$

and

$$\Phi'(s+v^2) = \int_{\mathbb{R}} F'(s+u^2+v^2)\,\mathrm{d}u$$

for all $s > 1$ and $v \in \mathbb{R}$. Integrating the latter over $v \in \mathbb{R}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \Phi'(s+v^2)\,\mathrm{d}v &= \int_{\mathbb{R}^2} F'(s+u^2+v^2)\underbrace{\mathrm{d}u\,\mathrm{d}v}_{R\,\mathrm{d}R\,\mathrm{d}\theta}\\ &= 2\pi\int_0^\infty F'(s+R^2)R\,\mathrm{d}R\\ &= \pi\int_0^\infty F'(s+\rho)\,\mathrm{d}\rho,\end{aligned}$$

where we used polar coordinates $(R,\theta)$ for $(u,v) \in \mathbb{R}^2$ and the substitution $\rho = R^2$. For the latter integral we may now apply the fundamental theorem of calculus. Since $F \in C_c([1,\infty))$, this takes the form

$$\int_0^\infty F'(s+\rho)\,\mathrm{d}\rho = -F(s)$$

for all $s > 1$, which proves the claim in (9.38) for $s > 1$. Since $F$ and $\Phi'$ both lie in $C_c([1,\infty))$, this extends by continuity to $s = 1$.

We now set $s = 1$ in (9.38), substitute $v = \sinh t$ with $\mathrm{d}v = \cosh t\,\mathrm{d}t$, and combine the resulting integral with (9.36), which leads to

$$\begin{aligned} F(1) &= -\frac{1}{\pi}\int_{\mathbb{R}} \Phi'(\underbrace{1+v^2}_{=\cosh^2 t})\,\mathrm{d}v \\ &= -\frac{1}{\pi}\int_{\mathbb{R}} \Phi'(\cosh^2 t)\cosh t\,\mathrm{d}t \\ &= \frac{1}{16\pi^2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\xi\frac{\sin\xi t}{\sinh t}\,\mathrm{d}\xi\right)\,\mathrm{d}t. \end{aligned}$$

Recall that by Lemma 9.38 we have

$$\left|\widehat{f}^{\text{hyp}}(\xi)\right| \ll_f \frac{1}{(1+\xi^2)^2}.$$

Together with (9.37), we obtain

$$\left|\widehat{f}^{\text{hyp}}(\xi)\xi\frac{\sin\xi t}{\sinh t}\right| \leqslant \frac{\xi^2}{(1+\xi^2)^2}\frac{t}{\sinh t},$$

which is easily seen to be integrable over $\mathbb{R}^2$. This implies that the integrand above lies in $L^1(\mathbb{R}^2)$. Hence we may apply Fubini's theorem and obtain, with

$$f(\mathrm{i}) = F(1) = \frac{1}{16\pi^2}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\xi\int_{\mathbb{R}}\frac{\sin\xi t}{\sinh t}\,\mathrm{d}t\,\mathrm{d}\xi,$$

the lemma. □

Lemma 9.44 explains why the following result is of interest to us.

**Lemma 9.45 (Volume factor).** *For $\xi\in\mathbb{R}$ we have*

$$\int_{\mathbb{R}} \frac{\sin\xi t}{\sinh t}\,\mathrm{d}t = \pi\tanh\left(\frac{\pi\xi}{2}\right). \tag{9.39}$$

Proof. To prove (9.39), we will apply the Cauchy integral formula to the meromorphic function $f$ defined by

$$f(z) = \frac{\mathrm{e}^{\mathrm{i}\xi z}}{\sinh z}. \tag{9.40}$$

We start with some elementary observations about the function $f$. For the point $z = x+\mathrm{i}y$ with $x, y\in\mathbb{R}$ we may apply the reverse triangle inequality to see that

$$|\sinh z| = \left|\frac{\mathrm{e}^x\mathrm{e}^{\mathrm{i}y}-\mathrm{e}^{-x}\mathrm{e}^{-\mathrm{i}y}}{2}\right| \geqslant \frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2} = \sinh x.$$

By symmetry, this gives

$$|\sinh z| \geqslant |\sinh x|, \tag{9.41}$$

which implies that $f(x+\mathrm{i}y)$ defined in (9.40) decays rapidly for $|x| \to \infty$ as long as $|y|$ is bounded. The estimate (9.41) also shows that $\sinh(x+\mathrm{i}y)$ can only vanish when $x=0$. Since $\sinh(\mathrm{i}y) = \mathrm{i}\sin y$ for $y \in \mathbb{R}$, we see that $f$ has poles precisely at the points in $\mathbb{Z}\pi\mathrm{i}$. At 0 the residue of $f$ is given by 1. Finally, we have

$$\sinh(z+\pi\mathrm{i}) = \frac{\mathrm{e}^{z+\pi\mathrm{i}} + \mathrm{e}^{-z-\pi\mathrm{i}}}{2} = -\sinh z$$

for $z \in \mathbb{C}$, which implies that

$$f(z+\pi\mathrm{i}) = \frac{\mathrm{e}^{\mathrm{i}\xi(z+\pi\mathrm{i})}}{-\sinh z} = -\mathrm{e}^{-\xi\pi} f(z) \tag{9.42}$$

for $z \in \mathbb{C} \smallsetminus (\mathbb{Z}\pi\mathrm{i})$.

We now integrate $f$ over the closed path

$$\gamma = \gamma_{\mathrm{bottom}} \sqcup \gamma_{\mathrm{right}} \sqcup \gamma_{\mathrm{top}} \sqcup \gamma_{\mathrm{top}} \sqcup \gamma_{\mathrm{left}}$$

indicated in Figure 9.8.

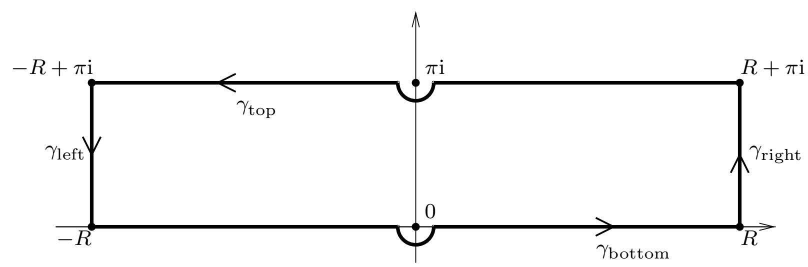


**Fig. 9.8:** The closed path $\gamma = \gamma_{R,\varepsilon}$ consists of four pieces. The first path $\gamma_{\mathrm{bottom}}$ goes from $-R$ to $R$ but avoids the pole at 0 (but including it inside the contour) by following a semi-circle of radius $\varepsilon$ around 0. The paths $\gamma_{\mathrm{right}}$, $\gamma_{\mathrm{top}}$, and $\gamma_{\mathrm{left}}$ go as indicated via $R+\pi\mathrm{i}$ and $-R+\pi\mathrm{i}$ back to $-R$, again avoiding the pole at $\pi\mathrm{i}$ (but leaving it outside the contour).

The description of the poles of $f$ given above now implies that

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}$$

independent of $R > 1$ and $\varepsilon \in (0,1)$. The decay properties of $f$ discussed above also imply that

$$\lim_{R\to\infty} \oint_{\gamma_{\mathrm{right}}} f(z)\,\mathrm{d}z = \lim_{R\to\infty} \oint_{\gamma_{\mathrm{left}}} f(z)\,\mathrm{d}z = 0.$$

Moreover, we defined the paths $\gamma_{\mathrm{bottom}}$ and $\gamma_{\mathrm{top}}$ so that $\gamma_{\mathrm{bottom}} + \pi\mathrm{i}$ is equal to $\gamma_{\mathrm{top}}$ except for the orientation, which is reversed. Together with (9.42) this gives

$$\oint_{\gamma_{\mathrm{top}}} f(z)\,\mathrm{d}z = -\oint_{\gamma_{\mathrm{bottom}}} f(z+\pi\mathrm{i})\,\mathrm{d}z = \mathrm{e}^{-\xi\pi} \oint_{\gamma_{\mathrm{bottom}}} f(z)\,\mathrm{d}z$$

again independently of $R$ and $\varepsilon$. Putting this together, we obtain

$$\begin{aligned} 2\pi\mathrm{i} &= \lim_{R\to\infty} \oint_{\gamma} f(z)\,\mathrm{d}z \\ &= \lim_{R\to\infty} \bigl(1+\mathrm{e}^{-\xi\pi}\bigr) \oint_{\gamma_{\mathrm{bottom}}} f(z)\,\mathrm{d}z \\ &= \bigl(1+\mathrm{e}^{-\xi\pi}\bigr) \left( \int_{-\infty}^{-\varepsilon} \frac{\mathrm{e}^{\mathrm{i}\xi t}}{\sinh t}\,\mathrm{d}t + \int_{\varepsilon}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\xi t}}{\sinh t}\,\mathrm{d}t + \oint_{\gamma_\varepsilon} f(z)\,\mathrm{d}z \right), \end{aligned} \tag{9.43}$$

where $\gamma_\varepsilon \colon [\pi, 2\pi] \ni t \mapsto \varepsilon \mathrm{e}^{\mathrm{i}t}$ is the semi-circular path appearing in $\gamma_{\mathrm{bottom}}$. To understand the asymptotics of

$$\oint_{\gamma_\varepsilon} f(z)\,\mathrm{d}z$$

as $\varepsilon$ decreases to 0, we note that $f(z) = \frac{1}{z} + h(z)$ for a function $h$ that is holomorphic at 0. By continuity of $h$, we have

$$\lim_{\varepsilon\searrow 0} \oint_{\gamma_\varepsilon} h(z)\,\mathrm{d}z = 0,$$

so we only have to calculate

$$\oint_{\gamma_\varepsilon} \frac{1}{z}\,\mathrm{d}z = \int_{\pi}^{2\pi} \frac{1}{\varepsilon \mathrm{e}^{\mathrm{i}t}} \varepsilon \mathrm{i} \mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t = \pi\mathrm{i}.$$

We now take the imaginary part in (9.43) and let $\varepsilon$ decrease to 0, which gives

$$2\pi = \bigl(1+\mathrm{e}^{-\xi\pi}\bigr) \left( \int_{-\infty}^{\infty} \frac{\sin \xi t}{\sinh t}\,\mathrm{d}t + \pi \right) = \bigl(1+\mathrm{e}^{-\xi\pi}\bigr) \int_{-\infty}^{\infty} \frac{\sin \xi t}{\sinh t}\,\mathrm{d}t + \pi + \mathrm{e}^{-\xi\pi}\pi.$$

Solving this equation for the integral gives

$$\int_{-\infty}^{\infty} \frac{\sin \xi t}{\sinh t} \, \mathrm{d}t = \frac{1 - \mathrm{e}^{-\xi\pi}}{1 + \mathrm{e}^{-\xi\pi}} \pi = \frac{\mathrm{e}^{\xi\pi/2} - \mathrm{e}^{-\xi\pi/2}}{\mathrm{e}^{\xi\pi/2} + \mathrm{e}^{-\xi\pi/2}} \pi,$$

and hence the lemma. □

Having obtained the hyperbolic Fourier inversion formula at i and for spherical functions in the last two lemmas, we are now in a position to prove the general case.

PROOF OF THEOREM 9.41. Let $f \in C_c^\infty(\mathbb{H})$. Combining Lemmas 9.44 and 9.45, we see that if $f \in C_c^\infty(\mathbb{H})$ is spherical, then

$$f(\mathrm{i}) = \frac{1}{16\pi} \int_{\mathbb{R}} \widehat{f}^{\,\mathrm{hyp}}(\xi) \xi \tanh\left(\frac{\pi\xi}{2}\right) \mathrm{d}\xi. \tag{9.44}$$

To use this for a general $f \in C_c^\infty(\mathbb{H})$, we define

$$f_{\mathrm{sph}}(z) = \int_K f(k\bullet z) \, \mathrm{d}m_K(k).$$

Since $k\bullet\mathrm{i} = \mathrm{i}$ for all $k \in K$, and we may differentiate under the integral sign, it follows that $f_{\mathrm{sph}} \in C_c^\infty(\mathbb{H})$ has $f_{\mathrm{sph}}(\mathrm{i}) = f(\mathrm{i})$. Applying Lemma 9.45 to this spherical function, we obtain

$$f(\mathrm{i}) = f_{\mathrm{sph}}(\mathrm{i}) = \frac{1}{16\pi} \int_{\mathbb{R}} \widehat{f}_{\mathrm{sph}}^{\,\mathrm{hyp}}(\xi) \xi \tanh\left(\frac{\pi\xi}{2}\right) \mathrm{d}\xi. \tag{9.45}$$

Using Fubini's theorem and the definition of the hyperbolic Fourier transform, we also have

$$\begin{aligned} \widehat{f}_{\mathrm{sph}}^{\,\mathrm{hyp}}(\xi) &= \int_{\mathbb{H}} f_{\mathrm{sph}}(z) \overline{\chi_{\infty,\xi}(z)} \, \mathrm{dvol}(z) \\ &= \int_K \int_{\mathbb{H}} f(k\bullet z) \overline{\chi_{\infty,\xi}(z)} \, \mathrm{dvol}(z) \, \mathrm{d}m_K(k) \\ &= \int_K \widehat{f}^{\,\mathrm{hyp}}(k,\xi) \, \mathrm{d}m_K(k). \end{aligned}$$

Putting this into (9.45), we obtain

$$f(\mathrm{i}) = \frac{1}{16\pi} \int_{\mathbb{R}} \int_K \widehat{f}^{\,\mathrm{hyp}}(k,\xi) \xi \tanh\left(\frac{\pi\xi}{2}\right) \mathrm{d}m_K(k) \, \mathrm{d}\xi.$$

Now let $h \in \mathrm{SL}_2(\mathbb{R})$ be arbitrary and define $\widetilde{f} = \pi_h^{\mathbb{H}} f$. Applying the previous formula to $\widetilde{f}$ and the equivariance claim in Lemma 9.38, we obtain

$$f(h^{-1}\bullet \mathrm{i}) = \big(\pi_h^{\mathbb{H}} f\big)(\mathrm{i}) = \frac{1}{16\pi}\int_{\mathbb{R}}\int_K \widehat{\pi_h^{\mathbb{H}} f}^{\,\mathrm{hyp}}(k,\xi)\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}m_K(k)\,\mathrm{d}\xi$$
$$= \frac{1}{16\pi}\int_{\mathbb{R}}\int_K \widehat{f}^{\mathrm{hyp}}(h^{-1}k,\xi)\,\mathrm{d}m_K(k)\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi.$$

Next we note that

$$\begin{aligned}
\int_K \widehat{f}^{\mathrm{hyp}}(h^{-1}k,\xi)\,\mathrm{d}m_K(k) &= \Big\langle \pi_h^{\xi,\mathrm{e}}\widehat{f}^{\mathrm{hyp}}(\cdot,\xi), F_\xi\Big\rangle_{\mathcal{H}_\xi} \\
&= \Big\langle \widehat{f}^{\mathrm{hyp}}(\cdot,\xi), \pi_{h^{-1}}^{\xi,\mathrm{e}} F_\xi\Big\rangle_{\mathcal{H}_\xi} \\
&= \int_K \widehat{f}^{\mathrm{hyp}}(k,\xi)\overline{F_\xi(hk)}\,\mathrm{d}m_K(k) \\
&= \int_K \widehat{f}^{\mathrm{hyp}}(k,\xi)\chi_{\infty,\xi}(k^{-1}h^{-1}\bullet \mathrm{i})\,\mathrm{d}m_K(k)
\end{aligned}$$

by (9.26). Combining this with the above, and setting $h^{-1}\bullet \mathrm{i} = z$, we obtain

$$f(z) = \frac{1}{16\pi}\int_{\mathbb{R}}\int_K \widehat{f}^{\mathrm{hyp}}(k,\xi)\chi_{\infty,\xi}(k^{-1}\bullet z)\,\mathrm{d}m_K(k)\xi\tanh\big(\tfrac{\pi\xi}{2}\big)\,\mathrm{d}\xi,$$

which gives the theorem. □

### 9.4.6 The Hyperbolic Fourier Transform in the Disc Model

We recall from Section 8.3.3 that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is conjugated to the action of $\mathrm{SU}_{1,1}(\mathbb{R})$ on $\mathbb{D}$ by the map

$$\Phi\colon \overline{\mathbb{C}} \ni w \longmapsto \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1\end{pmatrix}\bullet w = \frac{w+\mathrm{i}}{\mathrm{i}w+1} \in \overline{\mathbb{C}}$$

with $\Phi(\mathbb{D}) = \mathbb{H}$, $\Phi(0) = \mathrm{i}$ being our choice of origin in $\mathbb{H}$, and $\Phi(\mathrm{i}) = \infty$. Using this, we can move the Busemann function to $\mathbb{D}$.

**Definition 9.46 (Busemann function for $\mathrm{i}\in\partial\mathbb{D}$).** The Busemann function on $\mathbb{D}$ with respect to $\mathrm{i}\in\partial\mathbb{D}$ (and origin $0\in\mathbb{D}$) is defined by

$$b_{\mathrm{i}}^{\mathbb{D}}(w) = b_\infty^{\mathbb{H}}(\Phi(w)) = -\log\left(\frac{1-|w|^2}{|w-\mathrm{i}|^2}\right).$$

We now verify that the two formulas in Definition 9.46 above are in fact equivalent. Indeed, for $w = x+\mathrm{i}y$ we have

$$\Im\Phi(w) = \Im\frac{w+\mathrm{i}}{\mathrm{i}w+1} = \Im\frac{(x+\mathrm{i}y+\mathrm{i})(-\mathrm{i}x-y+1)}{(\mathrm{i}x-y+1)(-\mathrm{i}x-y+1)}$$
$$= \Im\frac{-\mathrm{i}x^2-\mathrm{i}y^2+\mathrm{i}y-\mathrm{i}y+\mathrm{i}}{x^2+(y-1)^2} = \frac{1-x^2-y^2}{x^2+(y-1)^2} = \frac{1-|w|^2}{|w-\mathrm{i}|^2}$$

as claimed.

Next we recall that

$$k_\theta = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \in K < \mathrm{SU}_{1,1}(\mathbb{R})$$

rotates $\mathbb{D}$ so that our origin $0 \in \mathbb{D}$ is fixed, and $k_\theta \bullet \mathrm{i} = \mathrm{e}^{-2\mathrm{i}\theta}\mathrm{i}$. This suggests the following more general definitions.

**Definition 9.47 (Busemann functions and hyperbolic waves on $\mathbb{D}$).** The *Busemann function on* $\mathbb{D}$ with respect to $p \in \partial\mathbb{D}$ (and origin $0 \in \mathbb{D}$) is defined by

$$b_p^{\mathbb{D}}(w) = -\log\left(\frac{1-|w|^2}{|w-p|^2}\right).$$

Moreover, the hyperbolic wave coming from $p$ and with frequency $\xi \in \mathbb{R}$ (normalized for the origin $0 \in \mathbb{D}$) is defined by the function

$$\chi_{p,\xi}(w) = \mathrm{e}^{(-\frac{1}{2}+\frac{\mathrm{i}}{2}\xi)b_p^{\mathbb{D}}(w)} = \left(\frac{1-|w|^2}{|w-p|^2}\right)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi}.$$

We imagine that the hyperbolic wave $\chi_{p,\xi}(w)$ is the sound produced by a loudspeaker at $p \in \partial\mathbb{D}$ using frequency $\xi \in \mathbb{R}$. The following reformulation of Theorem 9.41 establishes our goal to obtain any function $f \in C_c^\infty(\mathbb{D})$ as a superposition of such waves using loudspeakers at any point $p \in \partial\mathbb{D}$, and using all possible frequencies $\xi \in \mathbb{R}$.

**Theorem 9.48 (Fourier inversion on $\mathbb{D}$).** *Let $f \in C_c^\infty(\mathbb{D})$ and define the abbreviation*

$$\langle f, \chi_{p,\xi}\rangle = \int_{\mathbb{D}} f\overline{\chi_{p,\xi}}\,\mathrm{dvol}$$

*for $p \in \partial\mathbb{D}$ and $\xi \in \mathbb{R}$. Then*

$$f(z) = \frac{1}{16\pi}\int_{\mathbb{R}}\int_{\partial\mathbb{D}} \langle f, \chi_{p,\xi}\rangle \chi_{p,\xi}(z)\,\mathrm{d}p\,\xi\tanh\left(\tfrac{\pi\xi}{2}\right)\mathrm{d}\xi$$

*for all $z \in \mathbb{D}$, where* $\mathrm{d}p$ *denotes the normalized Lebesgue measure on $\partial\mathbb{D}$.*

### 9.4.7 The Unitary Isomorphism

We now show that the hyperbolic Fourier transform is in fact a unitary isomorphism between the Koopman representation $\pi^{\mathbb{H}}$ and an integral of all even principal series representations.

**Theorem 9.49 (Unitary isomorphism).** *The hyperbolic Fourier transform satisfies the identity*

$$\|f\|^2_{L^2(\mathbb{H})} = \frac{1}{8\pi}\int_{[0,\infty)} \big\|\widehat{f}^{\mathrm{hyp}}(k_\theta,\xi)\big\|^2_{\mathcal{H}_\xi}\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi$$

*for every $f\in C_c(\mathbb{H})$. Moreover, it extends to an equivariant unitary isomorphism between $\pi^{\mathbb{H}}$ and*

$$\pi^{\mu,\mathrm{e}} = \int_{[0,\infty)} \pi^{\xi,\mathrm{e}}\,\mathrm{d}\mu,$$

*where*

$$\mathrm{d}\mu = \frac{1}{8\pi}\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi,$$

*and* $\mathrm{d}\xi$ *denotes the Lebesgue measure on* $[0,\infty)$.

The proof of the theorem above will be split into two parts. We start the proof of the isometric property on p. 441, and surjectivity will be established on p. 447. We start with some preparatory material for the isometric property.

We note that the left regular representation and right convolutions commute, and that $\widehat{f}^{\mathrm{hyp}}(\cdot,\xi)$ belongs to the irreducible space $\mathcal{H}_\xi^{\mathrm{even}}$ for any functions $f\in C_c(\mathbb{H})$ and $\xi\in\mathbb{R}$ (see Lemma 9.38 and Theorem 9.20). Hence we expect $\widehat{f*\psi}^{\mathrm{hyp}}(\cdot,\xi)$ (if it is defined) to be a multiple of $\widehat{f}^{\mathrm{hyp}}(\cdot,\xi)$ by Schur's lemma (Theorem 1.27). The following result makes this precise. We refer to Figure 9.9 for the geometric meaning of $f*\psi$ for spherical $\psi$.

**Lemma 9.50 (Right convolution by spherical functions).** *Assume that $f\in C_c(\mathbb{H})$ and $\psi\in C(\mathbb{H})$. Then $f*\psi$ is again right $K$-invariant and so can be considered a function in $C(\mathbb{H})$. Moreover, if $\psi\in C_c(\mathbb{H})$ is spherical, then*

$$\widehat{f*\psi}^{\mathrm{hyp}}(h,\xi) = \widehat{\psi}^{\mathrm{hyp}}(\xi)\widehat{f}^{\mathrm{hyp}}(h,\xi)$$

*for all $(h,\xi)\in\mathrm{SL}_2(\mathbb{R})\times\mathbb{R}$.*

Proof of Lemma 9.50. By definition of convolution and the identification of right $K$-invariant functions on $\mathrm{SL}_2(\mathbb{R})$ and functions on $\mathbb{H}$, we have

$$f*\psi(g) = \int_G f(h\cdot\mathrm{i})\psi(h^{-1}g\cdot\mathrm{i})\,\mathrm{d}m(h)$$

for $g\in\mathrm{SL}_2(\mathbb{R})$. It follows that $f*\psi$ is also right $K$-invariant. This, together with Exercise 1.44, gives the first part of the lemma. We note that we may

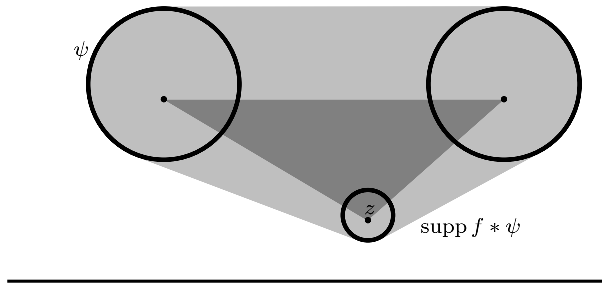


**Fig. 9.9:** Suppose $\psi$ is the normalized characteristic function of a ball around the point $\mathrm{i} \in \mathbb{H}$ (or a continuous approximation of it). The geometric meaning of the value $f * \psi(z)$ is, in this case, that the ball is moved to $z \in \mathbb{H}$ and $f$ is averaged over it. This creates a blurred-out version of $f$ with support in a neighbourhood (drawn in light grey) of the original support (drawn in dark grey).

now also write

$$f * \psi(z) = \int_G f(h\bullet\mathrm{i})\psi(h^{-1}\bullet z)\,\mathrm{d}m(h) \tag{9.46}$$

by using $z = g\bullet\mathrm{i} \in \mathbb{H}$ as the argument instead of $g \in \mathrm{SL}_2(\mathbb{R})$.

We now suppose in addition that $\psi$ is spherical, and calculate the hyperbolic Fourier transform of the convolution $f * \psi$. By our definition, the convolution formula in Lemma 9.39, and (9.29) on p. 428, we have

$$\begin{aligned}\widehat{f * \psi}^{\,\mathrm{hyp}}(g,\xi) = f * \psi * F_\xi(g) &= \int_G f(h)\underbrace{(\psi * F_\xi)(h^{-1}g)}_{=\widehat{\psi}^{\,\mathrm{hyp}}(\xi)F_\xi(h^{-1}g)}\,\mathrm{d}m(g)\\ &= \widehat{\psi}^{\,\mathrm{hyp}}(\xi) f * F_\xi(g) = \widehat{\psi}^{\,\mathrm{hyp}}(\xi)\widehat{f}^{\,\mathrm{hyp}}(g,\xi).\end{aligned}$$

□

**Lemma 9.51 (Symmetry on $C_c(\mathbb{H})$).** *Let $f \in C_c(\mathbb{H})$ and $\xi \in \mathbb{R}$. Then*

$$\int_K \bigl|\widehat{f}^{\,\mathrm{hyp}}(k,\xi)\bigr|^2\,\mathrm{d}m_K(k) = \int_{\mathbb{H}} f * \phi_\xi \overline{f}\,\mathrm{dvol} = \int_K \bigl|\widehat{f}^{\,\mathrm{hyp}}(k,-\xi)\bigr|^2\,\mathrm{d}m_K(k).$$

Proof. By Lemma 9.39 we have the convolution formula $\widehat{f}^{\,\mathrm{hyp}}(\cdot,\xi) = f * F_\xi$. Combining this with Fubini's theorem, we obtain for $\int_K |\widehat{f}^{\,\mathrm{hyp}}(k,\xi)|^2\,\mathrm{d}m_K(k)$ the formula

$$\int\limits_K\int\limits_G f(h_1\raisebox{0.2ex}{.}\mathrm{i})F_\xi(h_1^{-1}k)\,\mathrm{d}m(h_1)\int\limits_G \overline{f(h_2\raisebox{0.2ex}{.}\mathrm{i})F_\xi(h_2^{-1}k)}\,\mathrm{d}m(h_2)\,\mathrm{d}m_K(k)$$

$$= \int\limits_G\int\limits_G f(h_1\raisebox{0.2ex}{.}\mathrm{i})\underbrace{\int\limits_K F_\xi(h_1^{-1}k)\overline{F_\xi(h_2^{-1}k)}\,\mathrm{d}m_K(k)}_{=\langle\pi^\xi_{h_1}F_\xi,\pi^\xi_{h_2}F_\xi\rangle_{\mathcal{H}_\xi}}\,\mathrm{d}m(h_1)\overline{f(h_2\raisebox{0.2ex}{.}\mathrm{i})}\,\mathrm{d}m(h_2)$$

$$= \int\limits_G\int\limits_G f(h_1\raisebox{0.2ex}{.}\mathrm{i})\phi_\xi(h_2^{-1}h_1)\,\mathrm{d}m(h_1)\overline{f(h_2\raisebox{0.2ex}{.}\mathrm{i})}\,\mathrm{d}m(h_2)$$

$$= \int\limits_G\int\limits_G f(h_1\raisebox{0.2ex}{.}\mathrm{i})\phi_\xi(h_1^{-1}h_2)\,\mathrm{d}m(h_1)\overline{f(h_2\raisebox{0.2ex}{.}\mathrm{i})}\,\mathrm{d}m(h_2)$$

$$= \int_G f*\phi_\xi\overline{f}\,\mathrm{d}m,$$

where we used the fact that

$$\phi_\xi(g) = \overline{\phi_\xi(g^{-1})} = \phi_\xi(g^{-1})$$

is real-valued (see Lemma 9.14). Since $\phi_\xi = \phi_{-\xi}$ by Lemma 9.43, the lemma follows. □

Proof of isometry formula in Theorem 9.49. Let $f \in C_c^\infty(\mathbb{H})$. Applying the hyperbolic Fourier inversion formula (Theorem 9.41) to $f$ and Fubini's theorem , we see that

$$\|f\|^2_{L^2(\mathbb{H})} = \int\limits_{\mathbb{H}} f(z)\overline{f(z)}\,\mathrm{dvol}(z)$$

$$= \frac{1}{16\pi}\int\limits_{\mathbb{H}}\int\limits_{\mathbb{R}}\int\limits_K \widehat{f}^{\mathrm{hyp}}(k,\xi)\chi_{\infty,\xi}(k^{-1}\raisebox{0.2ex}{.}z)\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}m_K(k)\,\mathrm{d}\xi\overline{f(z)}\,\mathrm{dvol}(z)$$

$$= \frac{1}{16\pi}\int\limits_{\mathbb{R}}\int\limits_K \widehat{f}^{\mathrm{hyp}}(k,\xi)\underbrace{\int\limits_{\mathbb{H}}\chi_{\infty,\xi}(k^{-1}\raisebox{0.2ex}{.}z)\overline{f(z)}\,\mathrm{dvol}(z)}_{=\overline{\widehat{f}^{\mathrm{hyp}}(k,\xi)}}\mathrm{d}m_K(k)\xi\tanh(\tfrac{\pi\xi}{2})\mathrm{d}\xi$$

$$= \frac{1}{16\pi}\int_{\mathbb{R}}\|\widehat{f}^{\mathrm{hyp}}(\cdot,\xi)\|^2_{\mathcal{H}_\xi}\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi.$$

Applying Lemma 9.51, we can also write this in the form

$$\|f\|^2_{L^2(\mathbb{H})} = \frac{1}{8\pi}\int\limits_{[0,\infty)}\int\limits_K \left|\widehat{f}^{\mathrm{hyp}}(k,\xi)\right|^2\mathrm{d}m_K(k)\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi.$$

It remains to show that this formula not only holds for all $f \in C_c^\infty(\mathbb{H})$ but also for $f \in C_c(\mathbb{H})$.

For this we let $(B_n)$ be a decreasing sequence of compact neighbourhoods of $I \in \mathrm{SL}_2(\mathbb{R})$ that form a basis of neighbourhoods of $I$. Using compactness of $K$ and continuity of conjugation, we may also suppose that $B_n$ is invariant under conjugation by all $k \in K$, meaning that

$$B_n = \{khk^{-1} \mid k \in K, h \in B_n\}.$$

Let $(\psi_n)$ be an approximate identity in $C_c^\infty(G)$ as in Proposition 1.42, with $\operatorname{supp}\psi_n \subseteq B_n$ for all $n \geqslant 1$. Replacing $\psi_n$ by the function

$$\mathrm{SL}_2(\mathbb{R}) \ni g \longmapsto \int_K \psi_n\bigl(\ell g \ell^{-1}\bigr)\,\mathrm{d}m_K(\ell)$$

if necessary, we may also suppose that

$$\psi_n\bigl(kgk^{-1}\bigr) = \psi_n(g) \tag{9.47}$$

for all $k \in K$, $g \in \mathrm{SL}_2(\mathbb{R})$, and $n \in \mathbb{N}$.

Now let $f \in C_c(\mathbb{H})$. We note that $f * m_K$ (in the sense of Section 1.4.2) is equal to $f$ (see Exercise 9.52). We define

$$f_n = f * \psi_n \in C_c^\infty(G)$$

for $n \in \mathbb{N}$, and note that by continuity of $f$ we have

$$\begin{gathered} f_n \to f \text{ as } n \to \infty; \\ \operatorname{supp} f_n \subseteq (\operatorname{supp} f)B_1 \text{ for } n \in \mathbb{N}. \end{gathered} \tag{9.48}$$

Finally we define $\widetilde{\psi_n} = m_K * \psi_n$ for $n \in \mathbb{N}$, so that

$$f_n = f * \psi_n = (f * m_K) * \psi_n = f * \widetilde{\psi_n}.$$

We claim that $\widetilde{\psi_n}$ is a right $K$-invariant continuous function, so that we have $f_n \in C_c^\infty(\mathbb{H})$ for all $n \in \mathbb{N}$ by Lemma 9.50. Using the claim and (9.48), dominated convergence implies that

$$\widehat{f_n}^{\mathrm{hyp}}(h,\xi) \longrightarrow \widehat{f}^{\mathrm{hyp}}(h,\xi) \tag{9.49}$$

as $n \to \infty$ for all $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$.

To prove the claim, we show that $\widetilde{\psi_n}$ is the function

$$\widetilde{\psi_n}(g) = \int_K \psi_n(k^{-1}g)\,\mathrm{d}m_K(k). \tag{9.50}$$

To see this, let $F \geqslant 0$ be measurable on $\mathrm{SL}_2(\mathbb{R})$ and use the definitions, Fubini's theorem, and the measure-preserving substitution $h = kg$ to obtain

$$\begin{aligned}\int_G F \,\mathrm{d}m_K * \psi_n &= \int_K \int_G F(kg)\psi_n(g)\,\mathrm{d}m_K(k)\,\mathrm{d}m(g) \\ &= \int_G F(h) \int_K \psi_n(k^{-1}h)\,\mathrm{d}m_K(k)\,\mathrm{d}m(h).\end{aligned}$$

Using (9.47) in (9.50), we also have

$$\widetilde{\psi_n}(g) = \int_K \psi_n(gk^{-1})\,\mathrm{d}m_K(k),$$

which easily implies the claim that $\widetilde{\psi_n}(g\ell) = \widetilde{\psi_n}(g)$ for $g \in \mathrm{SL}_2(\mathbb{R})$ and $\ell \in K$.

For the smooth functions $f_n \in C_c^\infty(\mathbb{H})$, we already established the isometry formula. Moreover, (9.48) shows that $f_n \to f$ as $n \to \infty$ in $L^2(\mathbb{H})$ by dominated convergence. Together, these show that $\widehat{f_n}^{\mathrm{hyp}}|_{K\times\mathbb{R}}$, considered as an element of $\mathcal{H}_\mu^{\mathrm{even}} \cong L^2(K \times [0,\infty), \mu)$ forms a Cauchy sequence, which will have an $L^2$ limit $F$ with

$$\begin{aligned}\|F\|_{L^2(K\times\mathbb{R},\mu)} &= \lim_{n\to\infty} \|\widehat{f_n}^{\mathrm{hyp}}|_{K\times[0,\infty)}\|_{L^2(K\times[0,\infty),\mu)} \\ &= \lim_{n\to\infty} \|f_n\|_{L^2(\mathbb{H},\mathrm{vol})} = \|f\|_{L^2(\mathbb{H},\mathrm{vol})}.\end{aligned}$$

This implies that along a subsequence $\widehat{f_n}^{\mathrm{hyp}}|_{K\times\mathbb{R}}$ converges to $F$, which, together with (9.49), implies that $F = \widehat{f}^{\mathrm{hyp}}$ and hence the isometry formula

$$\|\widehat{f}^{\mathrm{hyp}}|_{K\times[0,\infty)}\|_{L^2(K\times[0,\infty),\mu)} = \|f\|_{L^2(\mathbb{H},\mathrm{vol})}.$$

□

**Exercise 9.52.** Show that $f \in C_c(\mathbb{H})$ implies that $f * m_K = f$.

It follows from the now established isometry formula for the $L^2$-norms of $f \in C_c(\mathbb{H})$ and $\widehat{f}^{\mathrm{hyp}}|_{K\times[0,\infty)}$ and Lemma 9.38 that the hyperbolic Fourier transform can be extended uniquely to an equivariant isometry from $L^2(\mathbb{H})$ into $\mathcal{H}_\mu^{\mathrm{even}}$. Moreover, the image $\mathcal{V}$ is then a closed $\pi^\mu$-invariant subspace of $\mathcal{H}_\mu^{\mathrm{even}}$. This brings up the question of whether such subspaces can be classified. We answer this in a slightly more general case in the following proposition.

**Proposition 9.53 (Invariant subspaces).** *Let $\mu$ be a $\sigma$-finite measure on $[0,\infty)$ and define $\mathcal{H}_\mu^{\mathrm{even}}$ as in Definition 9.28. Then, for any closed $\pi^{\mu,\mathrm{e}}$-invariant subspace $\mathcal{V} \subseteq \mathcal{H}_\mu^{\mathrm{even}}$, there exists a measurable set $S_\mathcal{V} \subseteq [0,\infty)$, that may be thought of as the 'support' of the subspace, so that*

$$\mathcal{V} = \left\{F \in \mathcal{H}_\mu^{\text{even}} \mid F(\cdot, \xi) = 0 \textit{ for } \mu\textit{-almost every } \xi \in [0,\infty) \smallsetminus S_{\mathcal{V}}\right\}. \quad (9.51)$$

**Exercise 9.54.** Let $\mu$ be as above, and let $\mathcal{V} \subseteq \mathcal{H}_\mu^{\text{even}}$ be a closed subspace. Show that a measurable subset $S_{\mathcal{V}} \subseteq [0,\infty)$ satisfying (9.51) is uniquely determined up to a null set by this property, assuming it exists.

Before starting the formal proof, we outline the structure of the argument. For simplicity, we write $\pi = \pi^{\mu,\mathrm{e}}$ and $\mathcal{H}_\pi = \mathcal{H}_\mu^{\text{even}}$.

We will show, in turn, the following statements.

(a) The operator $T_\pi = \pi_\partial(\Omega)$ from Corollary 9.8 is given by

$$T_\pi = M_{-\mathrm{id}^2} \quad (9.52)$$

where $M_{-\mathrm{id}^2}$ is the multiplication operator defined by

$$M_{-\mathrm{id}^2}(F)(h,\xi) = -\xi^2 F(h,\xi)$$

for all $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times [0,\infty)$ and $F$ in the domain

$$D_{M_{-\mathrm{id}^2}} = \left\{F \in \mathcal{H}_\pi \mid \int_0^\infty \int_K |\xi^2 F(k,\xi)|^2 \,\mathrm{d}m_K(k)\,\mathrm{d}\mu(\xi) < \infty\right\}.$$

(b) We define

$$f_0\colon [0,\infty) \ni \xi \longmapsto (1+\xi^2)^{-1}$$

and obtain from the above that $M_{f_0} = (I - T_\pi)^{-1}$. We claim that this multiplication operator $M_{f_0}$ commutes with the centralizer $C(\pi)$ of $\pi$.

(c) Using the functional calculus of $M_{f_0}$, we will prove that all multiplication operators $M_f$ for $f \in L^\infty_\mu([0,\infty))$ commute with $C(\pi)$.

(d) Applying this to the orthogonal projection operator $P_{\mathcal{V}}$ of a $\pi$-invariant subspace $\mathcal{V} \subseteq \mathcal{H}_\pi$, we then obtain that $\mathcal{V}$ has $L^\infty_\mu([0,\infty))\mathcal{V} \subseteq \mathcal{V}$, which will allow us to conclude the proof.

We now discuss these four steps in detail.

Proof that $T_\pi = \pi_\partial(\Omega) = M_{\mathrm{id}^2}$ as claimed in (a). Let $T > 0$ and assume that $f$ is a function in $L^2_\mu([0,\infty))$ with $f(\xi) = 0$ for $\mu$-almost every $\xi > T$. For an integer $n \in 2\mathbb{Z}$ we now define

$$F_n(h,\xi) = f(\xi) F_{\xi,n}(h)$$

for $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times [0,\infty)$, where $F_{\xi,n} \in \mathcal{H}_\xi^{\text{even}}$ is as defined in Lemma 9.22. By (9.16), we have that $F_{\xi,n}(h)$ depends smoothly on $h \in \mathrm{SL}_2(\mathbb{R})$ for any value of $\xi \in [0,\infty)$. Moreover, for a fixed $n$ and for $\xi \in [0,T]$ the derivatives are uniformly bounded. Applying the mean value theorem, dominated convergence, and (9.17), it follows that

$$\pi_\partial^\xi(\mathbf{a}) F_n(h,\xi) = \frac{n+1+\mathrm{i}\xi}{2} F_{n+2}(h,\xi) + \frac{-n+1+\mathrm{i}\xi}{2} F_{n-2}(h,\xi). \quad (9.53)$$

As in the proof of Lemma 9.22, we can now use Proposition 9.9 to conclude that the first summand is equal to $\pi_\partial(\mathbf{r}^+)F_n$ and second is equal to $\pi_\partial(\mathbf{r}^-)F_n$. Using the fact that $\mathbf{k}, \mathbf{r}^+, \mathbf{r}^-$ span $\mathfrak{sl}_2(\mathbb{C})$, it follows that $F_n$ is a smooth vector for $\pi$, and using the formula for $\Omega$ in (9.11), we also obtain

$$\pi_\partial(\Omega)F_n(h,\xi) = -\xi^2 F_n(h,\xi)$$

for $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times [0,\infty)$. Equivalently, the closed self-adjoint operator $T_\pi$ satisfies (9.52) for the function $F_n$ as above.

We now extend (9.52) to other functions $F \in \mathcal{H}_\pi$. To begin with, we may vary $n \in 2\mathbb{Z}$ over a finite set (using different functions $f_n \in L^2_\mu([0,\infty))$ with support in $[0,T]$). Note that $M_{-\mathrm{id}^2}$ is bounded on

$$\mathcal{H}_{\pi,\leqslant T} = \bigl\{F \in \mathcal{H}_\pi \mid F(\cdot,\xi) = 0 \text{ for } \mu\text{-almost every } \xi \in (T,\infty)\bigr\},$$

and these finite sums are precisely the $K$-finite vectors in $\mathcal{H}_{\pi,\leqslant T}$. Hence, it follows by continuity of $M_{-\mathrm{id}^2}$ and closedness of $T_\pi$ that (9.52) also holds for all $F \in \mathcal{H}_{\pi,\leqslant T}$.

Next let $F \in D_{M_{-\mathrm{id}^2}}$ and define

$$F_{\leqslant T} = \mathbb{1}_{[0,T]}F \in \mathcal{H}_{\pi,\leqslant T}$$

so that

$$T_\pi(F_{\leqslant T}) = M_{-\mathrm{id}^2}(F_{\leqslant T}).$$

For $T \to \infty$ we have $F_{\leqslant T} \to F$ and $M_{-\mathrm{id}^2}(F_{\leqslant T}) \to M_{-\mathrm{id}^2}(F)$ by dominated convergence. Since $T_\pi$ is a closed operator, we see that (9.52) holds for all functions $F \in D_{M_{-\mathrm{id}^2}}$. Equivalently, we have $M_{-\mathrm{id}^2} \subseteq T_\pi$. It is easy to see that $M_{-\mathrm{id}^2}$ is a self-adjoint (and, depending on $\operatorname{supp}\mu$, possibly unbounded) operator. By Corollary 9.8 the same is true for $T_\pi$. For two self-adjoint operators, the inclusion $M_{-\mathrm{id}^2} \subseteq T_\pi$ actually implies equality of the operators. Indeed, for $v \in D_{M_{-\mathrm{id}^2}}$ and $w \in D_{T_\pi}$, we have

$$\langle M_{-\mathrm{id}^2}v, w\rangle = \langle T_\pi v, w\rangle = \langle v, T_\pi w\rangle.$$

This shows that

$$D_{M_{-\mathrm{id}^2}} \ni v \longmapsto \langle M_{-\mathrm{id}^2}v, w\rangle$$

is a bounded linear function, which implies that $w$ belongs to the domain of $M^*_{-\mathrm{id}^2} = M_{-\mathrm{id}^2}$, and $M_{-\mathrm{id}^2}w = T_\pi w$.

To summarize, we have shown that $T_\pi = M_{-\mathrm{id}^2}$ as claimed in (a). □

Proof that $M_{f_0}$ commutes with $C(\pi)$ as claimed in (b). We define $f_0(\xi) = (1+\xi^2)^{-1}$ for $\xi \in [0,\infty)$, and first show that (a) implies that

$$M_{f_0} = (I - T_\pi)^{-1}.$$

Indeed,

$$I - T_\pi = M_{(1+\mathrm{id}^2)}$$

is injective on its domain, maps onto $\mathcal{H}_\pi$, and has $M_{f_0}$ as its (bounded) inverse operator. We now show that $M_{f_0}$ commutes with every equivariant bounded operator $B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$. For this, we first note that $B$ maps every smooth vector $v$ to a smooth vector $Bv$, and we have

$$\pi_\partial(\Omega)Bv = B\pi_\partial(\Omega)v.$$

If now $v \in D_{T_\pi}$, then there exists a sequence of smooth vectors $(v_n)$ with $v_n \to v$ and $\pi_\partial(\Omega)v_n \to T_\pi v$ as $n \to \infty$. However, this implies $Bv_n \to Bv$ and $\pi_\partial(\Omega)Bv_n \to BT_\pi v$ as $n \to \infty$, and hence also $T_\pi B \supseteq BT_\pi$. Using $M_{f_0} = (I - T_\pi)^{-1}$, we now obtain the claim. Indeed, let $v \in \mathcal{H}_\pi$ and $(I - T_\pi)^{-1}v = w$ so that $v = (I - T_\pi)w$. Then $Bv = (I - T_\pi)Bw$, which implies that

$$(I - T_\pi)^{-1}Bv = Bw = B(I - T_\pi)^{-1}v$$

as claimed in (b). □

Proof that $M_f$ commutes with $C(\pi)$ as claimed in (c). Since

$$(I - T_\pi)^{-1} = M_{f_0}$$

is already realized as a multiplication operator and $f_0$ is injective, the following are now relatively easy claims to prove. The measurable functional calculus for $M_{f_0}$ (see [21, Sec. 12.6]) gives rise to other multiplication operators. By the injectivity of $f_0$, every multiplication operator $M_f$ for $f \in L^\infty_\mu([0,\infty))$ can be obtained from the measurable functional calculus of $M_{f_0}$. Together with the previous claim and the properties of the measurable functional calculus (see [21, Prop. 12.68]), this implies that every equivariant bounded operator $B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ commutes with $M_f$ for $f \in L^\infty_\mu([0,\infty))$. □

Conclusion of the proof of Proposition 9.53, as outlined in (d). Now let $\mathcal{V} \subseteq \mathcal{H}_\pi$ be a closed $\pi$-invariant subspace, and let $B = P_{\mathcal{V}}$ be the orthogonal projection onto $\mathcal{V}$. By invariance of $\mathcal{V}$, the projection is equivariant and, by (c) above, we have $M_f\mathcal{V} \subseteq \mathcal{V}$ for all $f \in L^\infty_\mu([0,\infty))$.

Now let $F \in \mathcal{V}$ and define

$$S_F = \{\xi \in [0,\infty) \mid F(\cdot,\xi) \neq 0\}.$$

Using invariance of $\mathcal{V}$ under $K$, we can split $F$ into a sum of $K$-eigenfunctions as

$$F = \sum_{n\in 2\mathbb{Z}} F_n$$

satisfying $F_n \in \mathcal{V}_n$ for all $n \in 2\mathbb{Z}$, and

$$S_F = \bigcup_{n\in 2\mathbb{Z}} S_{F_n}.$$

Since $\mathcal{H}_\xi$ contains (up to scalars) only one $K$-eigenfunction of weight $n \in 2\mathbb{Z}$ (namely $F_{\xi,n}$ in (9.16)) there exists some $f_n \in L^2_\mu([0,\infty))$ such that

$$F_n(h,\xi) = f_n(\xi)F_{\xi,n}(h)$$

for $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times [0,\infty)$. In particular,

$$S_{F_n} = \{\xi \in [0,\infty) \mid f_n(\xi) \neq 0\}.$$

Using $M_f\mathcal{V} \subseteq \mathcal{V}$ for all $f \in L^\infty_\mu([0,\infty))$, the fact that $F_n \in \mathcal{V}$, and dominated convergence, it follows that the function

$$\mathrm{SL}_2(\mathbb{R}) \times [0,\infty) \ni (h,\xi) \longmapsto f(\xi)F_{\xi,n}(h)$$

belongs to $\mathcal{V}$ for any $f \in L^2_\mu([0,\infty))$ with $\{\xi \mid f(\xi) \neq 0\} \subseteq S_{F_n}$. For $T > 0$ we define

$$S_{n,T} = S_{F_n} \cap [0,T].$$

By (9.53) and the argument directly following it, we see that $\pi_\partial(\mathbf{r}^\pm)$ are bounded operators on

$$\{F \in \mathcal{H}_{\pi,\leqslant T} \mid F \text{ has } K\text{-weight } n\}.$$

Since $\mathcal{V}$ is invariant under $\pi_\partial(\mathbf{r}^\pm)$ (where it is defined) it follows that the function

$$\mathrm{SL}_2(\mathbb{R}) \times [0,\infty) \ni (h,\xi) \longmapsto f(\xi)F_{\xi,m}(h)$$

belongs to $\mathcal{V}$ for any $f \in L^2_\mu([0,\infty))$ with $\{\xi \mid f(\xi) \neq 0\} \subseteq S_{F_n}$, where $m,n$ lie in $2\mathbb{Z}$ are arbitrary. Varying $m,n \in 2\mathbb{Z}$ and the functions $f \in L^2_\mu([0,\infty))$, we can write any $F \in \mathcal{H}_\pi$ with $\{\xi \mid F(\cdot,\xi) \neq 0\} \subseteq S_F$ as a convergent sum of elements of $\mathcal{V}$ and obtain $F \in \mathcal{V}$.

Since $\mathcal{V}$ is separable, we can find a dense set $\{F_{(k)} \mid k \in \mathbb{N}\}$ of vectors, apply the above argument to each $F_{(k)}$ and obtain the same statement for

$$B = \bigcup_{k\in\mathbb{N}} B_{F_{(k)}} = \{\xi \in [0,\infty) \mid \text{there exists a } k \in \mathbb{N} \text{ with } F_{(k)}(\cdot,\xi) \neq 0\}.$$

This proves the proposition. □

**Exercise 9.55.** Let $\mu$ be a $\sigma$-finite measure on $[0,\infty)$ as in Proposition 9.53. Show that the centralizer of $\pi^{\mu,\mathrm{e}}$ is given by all multiplication operators $M_f$ with $f \in L^\infty_\mu([0,\infty))$.

The following finishes our discussions of the hyperbolic Fourier transform.

Concluding the proof of Theorem 9.49. Let $\mu$ be the measure on $[0,\infty)$ defined by $\frac{1}{8\pi}\xi\tanh\xi\,\mathrm{d}\xi$. By the first part of the proof on p. 441, we know that

$$C_c(\mathbb{H}) \ni f \longmapsto \widehat{f}^{\,\mathrm{hyp}} \in \mathcal{H}_\mu^{\mathrm{even}}$$

is an equivariant isometry between the Koopman representation $\pi^{\mathbb{H}}$ and the integral $\pi^{\mu,\mathrm{e}}$ of the even principal series representation. Hence it can be extended to an equivariant isometry from $L^2(\mathbb{H})$ to a closed $\pi^{\mu,\mathrm{e}}$-invariant subspace $\mathcal{V} \subseteq \mathcal{H}_\mu^{\mathrm{even}}$. By Proposition 9.53, this subspace can be defined by a measurable subset $S_{\mathcal{V}} \subseteq [0,\infty)$ and the formula (9.51). We show that $S_{\mathcal{V}} = [0,\infty)$ (up to null sets) by finding a sequence $(f_n)$ in $C_c(\mathbb{H})$, so that for every $\xi \in [0,\infty)$ there exists some $n \in \mathbb{N}$ with $\widehat{f_n}^{\,\mathrm{hyp}}(\cdot,\xi) \neq 0$.

In fact we let $(f_n)$ be a sequence of spherical functions in $C_c(\mathbb{H})$ with

$$\int_{\mathbb{H}} f_n \,\mathrm{dvol} = 1$$

for all $n \in \mathbb{N}$, so that $\operatorname{supp} f_n$ is a shrinking neighbourhood of $\mathrm{i} \in \mathbb{H}$. For every $\xi \in [0,\infty)$ it now follows that

$$\widehat{f_n}^{\,\mathrm{hyp}}(k,\xi) = \int_{\mathbb{H}} f_n(z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z) \longrightarrow 1$$

as $n \to \infty$ and for all $k \in K$ by $K$-invariance of $f_n$ and continuity of the function $z \mapsto \chi_{\infty,\xi}(z) = \Im(z)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi}$.

Hence we have $S_{\mathcal{V}} = [0,\infty)$ (up to null sets), which gives $\mathcal{V} = \mathcal{H}_\mu^{\mathrm{even}}$ by (9.51), and Theorem 9.49 follows. □

## 9.5 The Complementary Series Representation

We recall from Section 9.3 that for $\xi \in \mathbb{R}$ the principal series representation $\pi^{\xi,\mathrm{e}}$ is constructed from the unitary character $\chi_\xi$ defined by

$$\chi_\xi(a_t u_x) = \mathrm{e}^{\mathrm{i}\xi t}$$

for $a_t u_x \in B$. Moreover, $\pi^{\xi,\mathrm{e}}$ then turned out to be an irreducible unitary representation with Casimir eigenvalue $\alpha_{\pi^{\xi,\mathrm{e}}} = -\xi^2$. According to the proof of Theorem 9.18 in Section 9.2.5, there could (and, according to the statement of Theorem 9.18, there should) be another type $\gamma^s$ of even irreducible unitary representation with Casimir eigenvalues $\alpha_{\gamma^s} = s^2$ for $s \in (0,1)$ that we have not yet seen. To construct $\gamma^s$ we try to mimic the construction of $\pi^{\xi,\mathrm{e}}$ in Definition 9.19 while attempting to 'replace $\mathrm{i}\xi$ by $s$'.

**Definition 9.56 (Complementary series representation).** For $s \in (0,1)$ we define the non-unitary character $\chi_{(s)}$ on $B = \{a_t u_x \mid t, x \in \mathbb{R}\}$ by