Chapter 2
Abelian Groups

In this chapter we completely classify unitary representations of locally compact σ-compact metric abelian groups. The assumptions that the group is locally compact and abelian are critical for the material developed in this chapter, but as before we will also assume that the group G is σ-compact and metric. We will keep the topological assumptions on G implicit and refer to G as *the abelian group* G.

2.1 Pontryagin Dual

In Corollary 1.30 we saw that every irreducible representation of the abelian group G is one-dimensional and hence defines a (continuous unitary) character. In the context of abelian groups the following terminology is often used instead of the phrase 'unitary dual' as in Definition 1.23.

Definition 2.1 (Pontryagin dual of the abelian group). The *dual group* (or *Pontryagin dual*) of the abelian group G is defined (as an abstract group) by

$$\widehat{G} = \{\chi\colon G \longrightarrow \mathbb{S}^1 \mid \chi \text{ is a continuous character}\}$$

with the group operations being pointwise product and inverse.

The reader not familiar with this notion of duality may use the following exercise as a warmup for (the conclusions of which will be special cases of the theory developed in this chapter).

Exercise 2.2 (Generalization of Fourier series). Suppose that the abelian group G is compact, and normalize the Haar measure m to satisfy $m(G) = 1$. Assume furthermore that $\widehat{G}$ separates points[†] in G, meaning that for every $g_1 \neq g_2$ in G there exists some $\chi \in \widehat{G}$ with $\chi(g_1) \neq \chi(g_2)$.

[†] We note that this assumption is in fact always satisfied. This can be checked directly in many examples, and in general follows from Corollary 1.75 or the more specialized Theorem 2.15.

(a) Show that $\widehat{G}$ forms a countable orthonormal basis of $L^2(G)$.
(b) Show, for any unitary representation π, that $\mathcal{H}_\pi$ is the orthogonal direct sum of the *eigenspaces* $\mathcal{H}_\pi^\chi$ *of weight* $\chi \in \widehat{G}$ defined by

$$\mathcal{H}_\pi^\chi = \big\{w \in \mathcal{H}_\pi \mid \pi_g w = \chi(g) w \text{ for all } g \in G\big\}.$$

For many concrete groups it is not at all difficult to describe the dual group explicitly.

Essential Exercise 2.3 (Basic examples). Prove the following isomorphisms of (abstract) groups: (a) $\widehat{\mathbb{Z}} \cong \mathbb{T}$; (b) $\widehat{\mathbb{T}} \cong \mathbb{Z}$; and (c) $\widehat{\mathbb{R}} \cong \mathbb{R}$.

It is easy to give concrete examples of non-compact groups where the description of a unitary representation is in general more complicated than it is in Exercise 2.2(b). In fact, let λ denote the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ and let χ denote a unitary character of $\mathbb{R}$. If now $\lambda_g f = \chi(g) f$ for all $g \in \mathbb{R}$ and some $f \in L^2(\mathbb{R})$, then $|f|$ would be constant, and so $f = 0$. The same applies to any non-compact abelian group G and shows that $L^2(G)$ cannot be a direct sum of irreducible representations. Nonetheless, we will arrive at a complete understanding of all unitary representations of abelian groups using only characters in this chapter.

2.1.1 Characters as Algebra Homomorphisms

Given explicit presentations of a group and its dual, many of the theorems in this chapter have alternative and often simpler proofs. However, to handle the general case seamlessly the following is an important tool. We recall from Section 1.4.1 that $L^1(G)$ is a commutative Banach algebra.

Proposition 2.4 (Algebra homomorphisms of $L^1(G)$). *For the abelian group G every character $\chi \in \widehat{G}$ induces a continuous non-trivial algebra homomorphism from $\mathcal{A} = L^1(G)$ to $\mathbb{C}$ via the formula*

$$\chi_{\mathcal{A}}(f) = \int f \chi \,\mathrm{d}m$$

for $f \in L^1(G)$. Moreover, every continuous non-trivial algebra homomorphism from $\mathcal{A}$ to $\mathbb{C}$ is of this form for some uniquely determined $\chi \in \widehat{G}$.

Proof. Given a unitary character $\chi \in \widehat{G}$ and $f_1, f_2 \in L^1(G)$ we have

$$\begin{aligned}\int f_1 * f_2(g)\chi(g)\,\mathrm{d}m(g) &= \iint f_1(h)f_2(g-h)\chi(g)\,\mathrm{d}m(h)\,\mathrm{d}m(g)\\ &= \iint f_1(h)f_2(k)\chi(h+k)\,\mathrm{d}m(k)\,\mathrm{d}m(h)\\ &= \Bigl(\int f_1\chi\,\mathrm{d}m\Bigr)\Bigl(\int f_2\chi\,\mathrm{d}m\Bigr)\end{aligned}$$

by Fubini's theorem, which shows that $\chi_{\mathcal{A}}$ is indeed an algebra homomorphism. By the identification of the dual of $L^1(G)$ with $L^\infty(G)$ (see [21, Prop. 7.34]), $\chi_{\mathcal{A}}$ is continuous and non-trivial since $\|\chi_{\mathcal{A}}\| = \|\chi\|_\infty = 1$.

The converse is more involved. Let $\chi_{\mathcal{A}}\colon \mathcal{A} = L^1(G) \to \mathbb{C}$ be a non-trivial continuous algebra homomorphism. Then there is an element $\chi \in L^\infty(G)$ with $\|\chi\|_\infty < \infty$ with

$$\chi_{\mathcal{A}}(f) = \int_G f\chi\,\mathrm{d}m$$

for all $f \in L^1(G)$. We have to show that χ can be chosen in $C_b(G)$ and with the property that $\chi(gh) = \chi(g)\chi(h)$ for all $g, h \in G$.

By assumption $\chi_{\mathcal{A}} \neq 0$, so there exists some $f_0 \in L^1(G)$ with $\chi_{\mathcal{A}}(f_0) \neq 0$. Also, by the assumption on $\chi_{\mathcal{A}}$ and Fubini's theorem

$$\begin{aligned}\chi_{\mathcal{A}}(f)\chi_{\mathcal{A}}(f_0) = \chi_{\mathcal{A}}(f * f_0) &= \int_G\int_G f(g)f_0(h-g)\,\mathrm{d}m(g)\chi(h)\,\mathrm{d}m(h)\\ &= \int_G f(g)\int_G \lambda_g(f_0)\chi\,\mathrm{d}m\,\mathrm{d}m(g)\end{aligned}$$

for all $f \in L^1(G)$. We now define

$$\chi'(g) = (\chi_{\mathcal{A}}(f_0))^{-1}\int_G \lambda_g(f_0)\chi\,\mathrm{d}m,$$

so that $\chi_{\mathcal{A}}(f) = \int_G f\chi'\,\mathrm{d}m$ for all $f \in L^1(G)$, and in particular $\chi' = \chi$ almost everywhere.

Notice that χ' is defined using f_0 and χ essentially by convolution. This implies that $\chi' \in C_b(G)$, since

$$\begin{aligned}\bigl|\chi'(g) - \chi'(g_0)\bigr| &= |\chi_{\mathcal{A}}(f_0)|^{-1}\left|\int_G(\lambda_g f_0 - \lambda_{g_0}f_0)\chi\,\mathrm{d}m\right|\\ &\leqslant |\chi_{\mathcal{A}}(f_0)|^{-1}\|\lambda_g f_0 - \lambda_{g_0}f_0\|_1\|\chi\|_\infty \longrightarrow 0\end{aligned}$$

as $g \to g_0$, since $G \ni g \mapsto \lambda_g f_0 \in L^1(G)$ is continuous. Simplifying the notation, we suppose that $\chi = \chi' \in C_b(G)$.

Now choose a sequence (B_n) of decreasing open neighbourhoods of $0 \in G$ that form a basis of the neighbourhoods at 0. Then the sequence (ψ_n) defined by

$$\psi_n = \frac{1}{m(B_n)} \mathbb{1}_{B_n}$$

for all $n \geqslant 1$ forms an approximate identity (see Proposition 1.42).

Now let $g_1, g_2 \in G$ be arbitrary. Then $\lambda_{g_1}\psi_n = \frac{1}{m(B_n)}\mathbb{1}_{B_n+g_1}$ and so

$$\chi_{\mathcal{A}}\left(\lambda_{g_1}\psi_n\right) = \frac{1}{m(B_n)}\int_{B_n+g_1} \chi \,\mathrm{d}m \longrightarrow \chi(g_1)$$

as $n \to \infty$ by continuity of χ, and similarly $\chi_{\mathcal{A}}(\lambda_{g_2}\psi_n) \to \chi(g_2)$ as $n \to \infty$. Moreover, (1.14) shows (by identifying $f \in L^1(G)$ with $\nu_f \in M(G)$) that

$$\begin{aligned}\lambda_{g_1}\psi_n * \lambda_{g_2}\psi_n &= \delta_{g_1} * \psi_n * \delta_{g_2} * \psi_n \\ &= \delta_{g_1+g_2} * \psi_n * \psi_n = \lambda_{g_1+g_2}(\psi_n * \psi_n)\end{aligned}$$

and using $\psi_n * \psi_n$ (which also forms an approximate identity by Proposition 1.42) in the same way as ψ_n above we obtain from this

$$\begin{aligned}\chi_{\mathcal{A}}\left(\lambda_{g_1}\psi_n\right)\chi_{\mathcal{A}}\left(\lambda_{g_2}\psi_n\right) &= \chi_{\mathcal{A}}\left(\lambda_{g_1}\psi_n * \lambda_{g_2}\psi_n\right) \\ &= \chi_{\mathcal{A}}\left(\lambda_{g_1+g_2}\left(\psi_n * \psi_n\right)\right) \longrightarrow \chi(g_1+g_2)\end{aligned}$$

as $n \to \infty$. Thus with the above we obtain

$$\chi(g_1+g_2) = \chi(g_1)\chi(g_2)$$

for $g_1, g_2 \in G$. In other words, $\chi\colon G \to \mathbb{C}$ is a continuous homomorphism to the multiplicative structure of $\mathbb{C}$. Since χ is bounded and non-zero, it follows that χ is non-zero everywhere, and that it takes values in $\mathbb{S}^1$. This shows that $\chi_{\mathcal{A}}$ is defined by $\chi \in \widehat{G}$. □

Using the correspondence in Proposition 2.4 as an identification, we can and will consider the weak* topology on $\widehat{G} \subseteq L^\infty(G)$. The following corollary defines the important notion of *Fourier transform*, and gives the fundamental properties of the Fourier transform. These will be used frequently, and often without an explicit reference to this corollary.

Corollary 2.5 (Topology, structure, and functions on $\widehat{G}$). *For the abelian group G the dual group $\widehat{G}$ is a locally compact σ-compact metric group in the weak* topology (equivalently the compact-open topology). The Fourier (back) transform*

$$\check{f}(\chi) = \int f\chi \,\mathrm{d}m \tag{2.1}$$

for $f \in L^1(G)$ satisfies $\check{f} \in C_0(\widehat{G})$, $\widecheck{f_1 * f_2} = \check{f}_1\check{f}_2$, $\widecheck{f^*} = \overline{\check{f}}$, and $\|\check{f}\|_\infty \leqslant \|f\|_1$ for all $f, f_1, f_2 \in L^1(G)$. Moreover, $\left\{\check{f} \mid f \in L^1(G)\right\} \subseteq C_0(\widehat{G})$ is a dense sub-algebra with respect to the supremum norm on $\widehat{G}$.

We note that (2.1) is often referred to as the *Fourier back transform* (which is the reason for the notation). However, we decided to use this formula as a starting point, as we will see in the next section that it has (with this sign convention) important and conveniently described relationships to unitary representations. Because of the duality results that we will see later, the designation of one transform as the 'back' one and the other, implicitly, as the 'forward' one is rather arbitrary, and refers to a choice of a sign.

PROOF. Let $\sigma(L^1(G))$ denote the set of all non-trivial continuous algebra homomorphisms $L^1(G) \to \mathbb{C}$. By Proposition 2.4 these all have norm one. Hence

$$\Omega = \sigma\bigl(L^1(G)\bigr) \cup \{0\}$$

may be described as the set of all $\chi \in L^\infty(G)$ with $\|\chi\|_\infty \leqslant 1$ and with

$$\int f_1 * f_2 \chi \,\mathrm{d}m = \int f_1 \chi \,\mathrm{d}m \int f_2 \chi \,\mathrm{d}m$$

for all $f_1, f_2 \in L^1(G)$. This implies that Ω is weak* closed, and hence by Banach–Alaoglu (see [21, Th. 8.10]) also weak* compact. Since $L^1(G)$ is separable, the weak* topology is metrizable when restricted to the closed unit ball in $L^\infty(G)$ (see [21, Prop. 8.11]). This implies that $\sigma(L^1(G)) = \Omega \smallsetminus \{0\}$ is locally compact, σ-compact and metrizable. We identify $\sigma(L^1(G))$ with $\widehat{G}$ as in Proposition 2.4. Moreover, recall from Proposition 1.74 that the weak* topology on $\mathcal{P}^1(G) \supseteq \widehat{G}$ coincides with the compact-open topology, which proves that $\widehat{G}$ is locally compact, σ-compact and metric with respect to the compact-open topology. In this topology it is easy to see that the group operations are continuous.

Let $f \in L^1(G)$ and define the Fourier transform as in the corollary. Equivalently, we may think of $\check{f}$ as the evaluation map

$$\mathrm{ev}_f \colon \chi \longmapsto \chi(f) = \int f \chi \,\mathrm{d}m$$

from $L^1(G)' \cong L^\infty(G)$ to $\mathbb{C}$ restricted to $\sigma(L^1(G)) \cong \widehat{G}$. Hence by the definition of the weak* topology we see that $\check{f}$ is continuous on $\widehat{G}$. Moreover, by setting

$$\check{f}(0) = \int f 0 \,\mathrm{d}m = 0$$

we see that $\check{f}$ has a continuous extension to the compact space

$$\Omega = \sigma(L^1(G)) \cup \{0\},$$

with $\check{f}(0) = 0$. In other words, $0 \in \Omega$ plays the role of ∞ in the one-point compactificiation of $\widehat{G}$, and $\check{f} \in C_0(\widehat{G})$.

The property

$$\widecheck{f_1 * f_2}(\chi) = \chi(f_1 * f_2) = \chi(f_1)\chi(f_2) = \check{f}_1\check{f}_2(\chi)$$

for $f_1, f_2 \in L^1(G)$ is precisely the algebra homomorphism property of the map $\chi \in \sigma(L^1(G))$. In particular, this implies that the image of the Fourier transform is a sub-algebra of

$$C_0(\widehat{G}) \cong \{F \in C(\Omega) \mid F(0) = 0\}.$$

For $f \in L^1(G)$ we defined f^* in Section 1.4.1, which in the additive notation becomes $f^*(g) = \overline{f(-g)}$ for all $g \in G$. Since

$$\widecheck{f^*}(\chi) = \int_G f^* \chi \,\mathrm{d}m = \int_G \overline{f}(g)\chi(-g)\,\mathrm{d}m = \overline{\check{f}(\chi)}$$

for all $\chi \in \Omega = \widehat{G} \cup \{0\}$, we see that the image of $L^1(G)$ in $C_0(\widehat{G})$ is closed under conjugation. The inequality $\|\check{f}\|_\infty \leqslant \|f\|_1$ follows, since $\chi \in \widehat{G}$ satisfies $\|\chi\|_\infty = 1$ which gives $|\check{f}(\chi)| = |\int f\chi\,\mathrm{d}m| \leqslant \|f\|_1$.

Finally we note that the sub-algebra $\{\widehat{f} \mid f \in L^1(G)\} \subseteq C(\Omega)$ also separates points: if $\check{f}(\chi_1) = \check{f}(\chi_2)$ for some $\chi_1, \chi_2 \in \Omega$ and all $f \in L^1(G)$, then χ_1 and χ_2 define the same functional on $L^1(G)$ and so $\chi_1 = \chi_2$. Applying the Stone–Weierstrass theorem ([21, Thm. 2.40]) we see that the algebra $\mathcal{A} = \left\{\check{f} \mid f \in L^1(G)\right\} + \mathbb{C}\mathbb{1}$ is dense in $C(\Omega)$. This shows that for a given $F \in C_0(\widehat{G})$ and $\varepsilon > 0$ there exists some $f \in L^1(G)$ and $\alpha \in \mathbb{C}$ such that $\|\check{f} + \alpha\mathbb{1} - F\|_\infty < \varepsilon$. Since $F(0) = 0$ and $\check{f}(0) = 0$ we see that $|\alpha| < \varepsilon$, and hence $\|\check{f} - F\|_\infty < 2\varepsilon$ as required. □

Essential Exercise 2.6 (Basic examples). Show that the isomorphisms in Exercise 2.3 are also isomorphisms of topological groups. Show furthermore that for any $d \in \mathbb{N}$ we have the isomorphism $\widehat{\mathbb{R}^d} \cong \mathbb{R}^d$ as topological groups.

Exercise 2.7 (Continuity and bound). Let $\mathcal{A}$ be a commutative Banach algebra over $\mathbb{C}$. Show that any algebra homomorphism $\chi\colon \mathcal{A} \to \mathbb{C}$ is continuous and satisfies $\|\chi\| \leqslant 1$.

2.2 Spectral Theory, First Formulations

Due to the results (and exercises) of the last section it is natural to have a more symmetric notation for the group and its Pontryagin dual. Hence, we also use additive notation for the abelian group $\widehat{G}$. In fact, we will write $t, t_0, t_1, \ldots$ for the elements of the additive group $\widehat{G}$ and for $t \in \widehat{G}$ we write $\chi_t\colon G \to \mathbb{S}^1$ for the associated multiplicative character. Furthermore we define the dual pairing $\langle\cdot,\cdot\rangle\colon G \times \widehat{G} \to \mathbb{S}^1$ by

$$G \times \widehat{G} \ni (g, t) \longmapsto \langle g, t\rangle = \chi_t(g) \in \mathbb{S}^1.$$

In particular, in this notation we also write

$$\check{f}(t) = \int_G f\chi_t \,\mathrm{d}m = \int_G f(g)\,\langle g,t\rangle\,\mathrm{d}m(g)$$

for the Fourier transform of $f \in L^1(G)$ at $t \in \widehat{G}$.

2.2.1 Bochner's Theorem

We now return to the discussion of general unitary representations of the abelian group G, and recall from Section 1.5 that for this we need to understand positive-definite functions.

Theorem 2.8 (Bochner's theorem). *Let ϕ be a positive-definite function on the abelian group G. Then there exists a uniquely determined finite measure μ on $\widehat{G}$ such that*

$$\phi(g) = \int_{\widehat{G}} \langle g,t\rangle\,\mathrm{d}\mu(t) \tag{2.2}$$

for all $g \in G$.

We refer to Exercise 2.9 for a quicker proof of Bochner's theorem, but prefer the following argument as the ideas in the proof will be used again in Section 4.3 in a more general context.

Proof of Theorem 2.8. Suppose the finite measures μ_1, μ_2 both satisfy (2.2). For $f \in L^1(G)$ we may then use Fubini's theorem to see that

$$\begin{aligned}\int_G f(g)\phi(g)\,\mathrm{d}m(g) &= \int_G \int_{\widehat{G}} f(g)\langle g,t\rangle\,\mathrm{d}\mu_j(t)\,\mathrm{d}m(g)\\ &= \int_{\widehat{G}} \check{f}(t)\,\mathrm{d}\mu_j(t)\end{aligned}$$

for $j = 1, 2$. However, $\mathcal{A} = \{\check{f} \mid f \in L^1(G)\} \subseteq C_0(\widehat{G})$ is dense with respect to the supremum norm by Corollary 2.5. Therefore, we obtain

$$\int_{\widehat{G}} F\,\mathrm{d}\mu_1 = \int_{\widehat{G}} F\,\mathrm{d}\mu_2$$

for all $F \in C_0(\widehat{G})$ and the uniqueness of the measure in the Riesz representation theorem implies that $\mu_1 = \mu_2$.

We will now prove the existence of the measure μ by defining a linear functional Λ on $C_0(\widehat{G})$ and applying the Riesz representation theorem. By density of $\mathcal{A} \subseteq C_0(\widehat{G})$, it is sufficient to define $\Lambda(\check{f})$ for $f \in L^1(G)$ and show that

$$|\Lambda(\check{f})| \leqslant \phi(0)\|\check{f}\|_\infty. \tag{2.3}$$

To this end, define

$$\Lambda(\check{f}) = \int f\phi \,\mathrm{d}m \tag{2.4}$$

for $f \in L^1(G)$ and notice that (2.3), once established, will in particular imply that $\Lambda(\check{f})$ is indeed well-defined for any $\check{f} \in C_0(\widehat{G})$, meaning that it does not depend on the choice of f, but only on its Fourier transform $\check{f}$.

For the proof of the bound (2.3) for the functional in (2.4) we recall from Proposition 1.67 that $\phi = \varphi_{v_0}$ is a matrix coefficient of some $v_0 \in \mathcal{H}_\pi$ for some unitary representation π of G. With this we have

$$\int f\phi \,\mathrm{d}m = \int f\varphi_{v_0} \,\mathrm{d}m = \int f(g) \left\langle \pi(g)v_0, v_0 \right\rangle \,\mathrm{d}m(g) = \left\langle \pi_*(f)v_0, v_0 \right\rangle$$

for all $f \in L^1(G)$. Therefore

$$\left| \int f\phi \,\mathrm{d}m \right| \leqslant \|\pi_*(f)\|_{\mathrm{op}} \|v_0\|^2 = \|\pi_*(f)\|_{\mathrm{op}} \phi(0)$$

and so we need to estimate $\|\pi_*(f)\|_{\mathrm{op}}$. For this, we recall from Section 1.4.3 that $\pi_*(f)^* = \pi_*(f^*)$. Also recall that $L^1(G)$ is an abelian Banach algebra (since G is assumed to be abelian), which implies that $\pi_*(f)$ is a normal operator, and so its operator norm is equal to its spectral radius

$$\|\pi_*(f)\|_{\mathrm{op}} = \lim_{n\to\infty} \left\| \pi_*(f)^{2^n} \right\|_{\mathrm{op}}^{2^{-n}} = \lim_{n\to\infty} \left\| \pi_*\bigl(f^{*2^n}\bigr) \right\|_{\mathrm{op}}^{2^{-n}}.$$

Combining this with the bound

$$\|\pi_*(f^{*2^n})\|_{\mathrm{op}} \leqslant \|f^{*2^n}\|_1$$

we obtain

$$\|\pi_*(f)\|_{\mathrm{op}} \leqslant \lim_{n\to\infty} \|f^{*2^n}\|_1^{2^{-n}}.$$

Here the limit exists because of the spectral radius formula in a commutative Banach algebra (see [21, Cor. 11.29]) and equals

$$\max_{\chi \in \sigma(L^1(G)) \cup \{0\}} |\chi(f)| = \|\check{f}\|_\infty$$

by the identification $\sigma(L^1(G)) = \widehat{G}$ in Proposition 2.4 and the proof of Corollary 2.5. To summarize, we have shown that (2.4) defines a continuous linear functional Λ on $C_0(\widehat{G})$ satisfying (2.3).

Applying the Riesz representation theorem (see [21, Th. 7.54]) to Λ, we find a finite measure μ on $\widehat{G}$ such that

$$\int_G f\phi \,\mathrm{d}m = \Lambda(\check{f}) = \int_{\widehat{G}} \check{f}\,\mathrm{d}\mu \tag{2.5}$$

for all $f \in L^1(G)$ and $\|\mu\| = \|\Lambda\| \leqslant \phi(0)$. Here μ is potentially complex-valued, and we may write $\mathrm{d}\mu = F\,\mathrm{d}|\mu|$ for some (positive) finite Borel measure $|\mu|$ on $\widehat{G}$ and some measurable function F satisfying

$$\|F\|_{L^1_{|\mu|}} = \|\Lambda\| \leqslant \phi(0).$$

Now set $f = \psi_n$ in (2.5) for some approximate identity (ψ_n) and let $n \to \infty$ to obtain

$$\phi(0) = \lim_{n\to\infty} \int \psi_n \phi \,\mathrm{d}m = \lim_{n\to\infty} \int \widecheck{\psi_n} F \,\mathrm{d}|\mu| = \int F\,\mathrm{d}|\mu|$$

since the bound $\|\widecheck{\psi_n}\|_\infty \leqslant 1$ and the convergence $\widecheck{\psi_n}(t) \to 1$ as $n \to \infty$ for every $t \in \widehat{G}$ allows us to apply dominated convergence. However,

$$\int |F|\,\mathrm{d}|\mu| = \|F\|_{L^1_{|\mu|}} = \|\Lambda\| \leqslant \phi(0) = \int F\,\mathrm{d}|\mu|$$

implies that F is real-valued and non-negative almost everywhere with respect to $|\mu|$, so that $\mathrm{d}\mu = F\,\mathrm{d}|\mu|$ is in fact a (positive) finite Borel measure.

Finally, fix some $g \in G$, set $f = \lambda_g \psi_n$ in (2.5) and let $n \to \infty$ to obtain

$$\phi(g) = \lim_{n\to\infty} \int \psi_n(h)\phi(g+h)\,\mathrm{d}m(h) = \lim_{n\to\infty} \int \big(\lambda_g \psi_n\big)\,\phi\,\mathrm{d}m = \lim_{n\to\infty} \int \widecheck{\lambda_g \psi_n}\,\mathrm{d}\mu.$$

Since $\|\widecheck{\lambda_g \psi_n}\|_\infty \leqslant \|\lambda_g \psi_n\|_1 = 1$ and

$$\widecheck{\lambda_g \psi_n}(t) = \int (\lambda_g \psi_n)\chi_t \,\mathrm{d}m = \int \psi_n(h)\chi_t(g+h)\,\mathrm{d}m(h) \longrightarrow \chi_t(g) = \langle g, t\rangle$$

as $n \to \infty$ for any $t \in \widehat{G}$, we may again apply dominated convergence to obtain

$$\phi(g) = \int \langle g, t\rangle \,\mathrm{d}\mu(t)$$

as claimed. □

Exercise 2.9. Let $\mathcal{P}^{\leqslant 1}(G)$ be as in Corollary 1.73. Let

$$\mathcal{M}^{\leqslant 1}(\widehat{G}) = \big\{\mu \mid \mu \text{ is a measure on } \widehat{G} \text{ with } \mu(\widehat{G}) \in [0,1]\big\}.$$

Define

$$\Phi \colon \mathcal{M}^{\leqslant 1}(\widehat{G}) \ni \mu \longmapsto \phi_\mu \in \mathcal{P}^{\leqslant 1}(G)$$

by

$$\phi_\mu(g) = \int \langle g, t\rangle \,\mathrm{d}\mu(t)$$

for $g \in G$. Show that Φ is continuous with respect to the weak* topologies on $\mathcal{M}^{\leqslant 1}(\widehat{G})$ (resp. $\mathcal{P}^{\leqslant 1}(G)$). Conclude from this that

$$\Phi\big(\mathcal{M}^{\leqslant 1}(\widehat{G})\big) = \mathcal{P}^{\leqslant 1}(G),$$

and deduce Theorem 2.8.

The following exercise may help to develop an intuitive understanding of spectral measures.

Exercise 2.10. Fix a sequence $(t_n)_{n\in\mathbb{N}}$ in $\widehat{G}$, and define a unitary representation π of G on $\mathcal{H}_\pi = \ell^2(N)$ by

$$\big(\pi_g v\big)_n = \langle g, t_n\rangle v_n$$

for all $n \in \mathbb{N}$, $g \in G$, and $v = (v_n) \in \mathcal{H}_\pi$. Calculate μ_v for $v \in \mathcal{H}_\pi$, and interpret it as giving weights to the eigenvalues in $\widehat{G}$.

2.2.2 The Spectral Theorems

Using Bochner's theorem (Theorem 2.8) it is now quite straightforward to completely describe cyclic as well as general unitary representations of abelian groups.

Corollary 2.11 (Spectral theorem for cyclic representations). *Let π be a unitary representation of the abelian group G, and let $v \in \mathcal{H}_\pi$. Applying Bochner's theorem for the positive-definite function φ_v defined by*

$$\varphi_v(g) = \big\langle \pi_g v, v\big\rangle$$

for $g \in G$, we obtain the spectral measure *μ_v for v. Then the cyclic representation on*

$$\langle v\rangle_\pi = \overline{\big\langle \pi_g v \mid g \in G\big\rangle_{\mathbb{C}}}$$

is unitarily isomorphic to the unitary multiplication representation *M of G on $L^2_{\mu_v}(\widehat{G})$ defined by*

$$M_g(w)(t) = \langle g, t\rangle\, w(t) \tag{2.6}$$

for $g \in G$, $w \in L^2_{\mu_v}(\widehat{G})$, and $t \in \widehat{G}$. Moreover, the equivariant isometry sends $v \in \mathcal{H}_\pi$ to $\mathbb{1} \in L^2_{\mu_v}(\widehat{G})$ and in particular,

$$\|v\|^2 = \varphi_v(0) = \mu_v(\widehat{G}).$$

Finally, π is cyclic if and only if there exists a finite μ on $\widehat{G}$ such that π is isomorphic to the multiplication representation M on $L^2_\mu(\widehat{G})$.

Proof. First note that the multiplication representation M_g for $g \in G$ defines a unitary representation on $L^2_\mu(\widehat{G})$ for any σ-finite measure μ on $\widehat{G}$. Indeed, since $\langle g,t\rangle \in \mathbb{S}^1$ for all $(g,t) \in G \times \widehat{G}$ the operator $M_g\colon L^2_\mu(X) \to L^2_\mu(\widehat{G})$ is unitary for any $g \in G$. Moreover, if (g_n) is a sequence in G with $g_n \to g_0$ as $n \to \infty$, then continuity of the character χ_t implies that $\langle g_n, t\rangle \to \langle g,t\rangle$ as $n \to \infty$ for all $t \in \widehat{G}$. Given some $w \in L^2_\mu(\widehat{G})$, this then implies that

$$\left\|M_{g_n} w - M_{g_0} w\right\|_2^2 = \int_{\widehat{G}} |\left\langle g_n, t\right\rangle - \left\langle g_0, t\right\rangle|^2 |w(t)|^2 \,\mathrm{d}\mu(t) \longrightarrow 0$$

as $n \to \infty$ by dominated convergence.

We now consider the case of a multiplication representation defined by a finite measure μ on $\widehat{G}$. We will show that the representation is cyclic and has the vector $\mathbb{1} \in L^2_\mu(\widehat{G})$ as a generator. Indeed, if $f \in L^1(G)$ and $w \in L^2_\mu(\widehat{G})$ then

$$\begin{aligned}
\langle M_*(f)\mathbb{1}, w\rangle_{L^2_\mu(\widehat{G})} &= \int_G f(g) \left\langle M_g \mathbb{1}, w\right\rangle_{L^2_\mu(\widehat{G})} \,\mathrm{d}m(g)\\
&= \int_G f(g) \int_{\widehat{G}} \langle g,t\rangle\, \overline{w(t)} \,\mathrm{d}\mu(t) \,\mathrm{d}m(g)\\
&= \int_{\widehat{G}} \check{f}(t)\overline{w(t)} \,\mathrm{d}\mu(t) = \left\langle \check{f}, w\right\rangle_{L^2_\mu(\widehat{G})}
\end{aligned}$$

by definition of the convolution operator $M_*(f)$ using weak integration, Fubini's theorem, and the definition of the Fourier transform $\check{f}$. Since the function $w \in L^2_\mu(\widehat{G})$ was arbitrary, this shows that $M_*(f)\mathbb{1} = \check{f} \in \langle \mathbb{1}\rangle_M$. By density of the set of these functions in $C_0(\widehat{G})$ with respect to the supremum norm (see Corollary 2.5) and since μ is assumed to be a finite measure, we see that $C_0(\widehat{G}) \subseteq \langle \mathbb{1}\rangle_M$. By density of $C_c(\widehat{G}) \subseteq L^2_\mu(\widehat{G})$ we obtain $\langle \mathbb{1}\rangle_M = L^2_\mu(\widehat{G})$ as claimed.

Now consider a unitary representation π and a vector $v \in \mathcal{H}_\pi$. Then the function defined by $\varphi_v(g) = \left\langle \pi_g v, v\right\rangle$ is positive-definite. Applying Bochner's theorem (Theorem 2.8) we find a finite measure μ_v on $\widehat{G}$ such that

$$\varphi_v(g) = \int_{\widehat{G}} \langle g,t\rangle \,\mathrm{d}\mu_v(t).$$

Notice that the matrix coefficient of $\mathbb{1} \in L^2_\mu(\widehat{G})$ is given by

$$\left\langle M_g \mathbb{1}, \mathbb{1}\right\rangle_{L^2_\mu(\widehat{G})} = \int_{\widehat{G}} \langle g,t\rangle \,\mathrm{d}\mu_v(t) = \varphi_v(g) = \left\langle \pi_g v, v\right\rangle$$

for all $g \in G$. By Proposition 1.60 this shows that π restricted to the cyclic representation $\langle v\rangle_\pi$ generated by v is unitarily isomorphic to $\langle \mathbb{1}\rangle_M = L^2_\mu(\widehat{G})$, and that we may assume that v is sent to $\mathbb{1}$.

The last statement in the corollary follows now too. If π is cyclic then we have found the isomorphism. If π is isomorphic to the multiplication representation on $L^2_\mu(\widehat{G})$ for a finite measure μ, then it must be cyclic since we already showed that $L^2_\mu(\widehat{G})$ is cyclic (with generator $\mathbb{1}$). □

Using the cyclic case, we easily obtain a similar description of a general representation.

Corollary 2.12 (Spectral theorem). *Let π be a unitary representation of the abelian group G. Then there exists a finite measure μ on $X = \widehat{G} \times \mathbb{N}$ and a unitary isomorphism between π and the unitary* multiplication representation M *of G on $L^2_\mu(X)$ defined by*

$$M_g(w)(t,n) = \langle g,t\rangle\, w(t,n) \tag{2.7}$$

for $g \in G$, $w \in L^2_\mu(X)$, and $(t,n) \in X$.

Moreover, given a multiplication representation M of G defined by (2.7) *and a σ-finite measure on $X = \widehat{G} \times \mathbb{N}$ the convolution operator $M_*(f)$ associated to some $f \in L^1(G)$ is given by the multiplication operator*

$$\begin{aligned} M_*(f) = M_{\check{f}}\colon L^2_\mu(X) &\longrightarrow L^2_\mu(X) \\ w &\longmapsto \check{f}w. \end{aligned}$$

Proof. That the multiplication representation is indeed a unitary representation follows by the same argument as in the beginning of the proof of Corollary 2.11. Next we claim that if μ is a σ-finite measure on X, then the multiplication representation is unitarily isomorphic to a multiplication representation defined by a finite measure μ' on X. Indeed, since μ is σ-finite there exists some strictly positive measurable function F with the property that F^2 is integrable. Now define the finite measure μ' by $\mathrm{d}\mu' = F^2\,\mathrm{d}\mu$ and $U = M_{F^{-1}}$ so that for $w \in L^2_\mu(X)$ we have $U(w) = F^{-1}w$ and

$$\|Uw\|^2_{L^2_{\mu'}(X)} = \int F^{-2}|w|^2\,\mathrm{d}\mu' = \int F^{-2}|w|^2 F^2\,\mathrm{d}\mu = \|w\|^2_{L^2_\mu(X)},$$

which implies the claim since $M_{F^{-1}}$ commutes with M_g for any $g \in G$.

Let now π be a unitary representation of G. Applying Lemma 1.56 we can split

$$\mathcal{H}_\pi = \bigoplus_{n\geqslant 1}\langle v_n\rangle_\pi$$

into a direct sum of cyclic representations. Applying the cyclic case in Corollary 2.11, we find a sequence of finite measures (μ_n) such that

$$\mathcal{H}_\pi = \bigoplus_{n\geqslant 1}\langle v_n\rangle_\pi \cong \bigoplus_{n\geqslant 1} L^2_{\mu_n}(\widehat{G}).$$

We use these measures to define a σ-finite measure μ on $X = \widehat{G}\times\mathbb{N}$ by setting

$$\mu(B) = \sum_{n=1}^{\infty} \mu_n\big(\{t \in \widehat{G} \mid (t,n) \in B\}\big)$$

for any Borel set $B \subseteq X$. This measure satisfies

$$L^2_\mu(X) \cong \bigoplus_{n \geqslant 1} L^2_{\mu_n}(\widehat{G})$$

where $f \in L^2_\mu(X)$ corresponds to the sequence (f_n) defined by $f_n(t) = f(t,n)$ for $t \in \widehat{G}$ and $n \geqslant 1$. Combining the above isomorphisms, we see that

$$\mathcal{H}_\pi \cong L^2_\mu(X)$$

and the main claim in the corollary follows (using also the reduction to finite measures proven above).

The argument for the last part of the corollary we have already seen in the proof of Corollary 2.11. Indeed for $f \in L^1(G)$ and $v, w \in L^2_\mu(X)$ we have

$$\begin{aligned}
\langle M_*(f)v, w\rangle_{L^2_\mu(X)} &= \int_G f(g) \langle M_g v, w\rangle_{L^2_\mu(X)} \,\mathrm{d}m(g) \\
&= \int_G f(g) \int_X \langle g, t\rangle\, v(t,n)\overline{w(t,n)}\,\mathrm{d}\mu(t,n)\,\mathrm{d}m(g) \\
&= \int_X \check{f}(t) v(t,n)\overline{w(t,n)}\,\mathrm{d}\mu(t,n) = \left\langle \check{f} v, w\right\rangle_{L^2_\mu(X)}
\end{aligned}$$

by definition of the convolution operator $M_*(f)$ using weak integration, Fubini's theorem, and the definition of the Fourier transform $\check{f}$. As $v, w \in L^2_\mu(X)$ were arbitrary the corollary follows. □

Spectral measures as in Corollaries 2.11 and 2.12 carry complete information about containment of one representation in another. We leave the following special case as an exercise and return to this question more generally in Sections 2.5.1 and 2.7, where we will also prove more refined versions of the spectral theorem.

Essential Exercise 2.13 (Containment for characters). Let π be a unitary representation of the abelian group G, let $t_0 \in \widehat{G}$, and denote the corresponding character by χ_{t_0}. Characterize, in terms of spectral measures μ_v for $v \in \mathcal{H}_\pi$, when χ_{t_0} is contained in π.

Exercise 2.14 (Example with infinite multiplicity). Give an example of a unitary representation of the abelian group G for which we really have to use the space $X = \widehat{G} \times \mathbb{N}$ in Corollary 2.12, and could not have used $\widehat{G} \times \{1, \ldots, n\}$ for some $n \in \mathbb{N}$.

2.3 Plancherel Formula

We show in this section that by applying the spectral theorem (Corollary 2.11 and Corollary 2.12) to the regular representation of G we obtain a general formulation of the Fourier transform. We will use this in the next section to establish a duality principle between the abelian group G and its Pontryagin dual $\widehat{G}$.

For $t \in \widehat{G}$ we write M_t for the multiplication operator

$$M_t(v)(g) = \langle g, t \rangle v(g) \tag{2.8}$$

for $v \in L^2(G)$ and $g \in G$. Moreover, we will write $\widehat{\lambda}$ for the regular representation of $\widehat{G}$ on functions f on $\widehat{G}$, so that

$$\widehat{\lambda}_{t_0}(f)(t) = f(t - t_0)$$

for all $t, t_0 \in \widehat{G}$.

Theorem 2.15 (Plancherel formula). *Given the abelian group G and a normalization of its Haar measure $m = m_G$ there exists a normalization of the Haar measure $m_{\widehat{G}}$ on $\widehat{G}$ and a unitary isomorphism*

$$U \colon L^2(G) \longrightarrow L^2(\widehat{G})$$

which extends the map $f \mapsto \check{f}$ for $f \in L^1(G) \cap L^2(G)$ to all of $L^2(G)$ and has the equivariance properties

$$U \circ \lambda_g = M_g \circ U,$$
$$U \circ M_t = \widehat{\lambda}_{-t} \circ U$$

for all $g \in G$ and $t \in \widehat{G}$. Moreover, the inverse $U^{-1} \colon L^2(\widehat{G}) \to L^2(G)$ is the unique isometric extension of the map

$$L^1(\widehat{G}) \cap L^2(\widehat{G}) \ni F \longmapsto \widehat{F} \in C_0(G) \cap L^2(G), \tag{2.9}$$

where

$$\widehat{F}(g) = \int_{\widehat{G}} F(t) \overline{\langle g, t \rangle} \, \mathrm{d}m_{\widehat{G}}(t)$$

for any $g \in G$.

We split the proof of the theorem into several steps.

Lemma 2.16 (Convolution on $L^2(G)$). *For the regular representation λ of the abelian group G on the space $L^2(G)$ and functions $f \in L^1(G)$, $v \in L^2(G)$, we have that*

$$\lambda_*(f)v = f * v \in L^2(G)$$

can be calculated almost everywhere by the integral formula defining convolution.

Proof. Let $f \in L^1(G)$ and $v \in L^2(G)$. Fixing another $w \in L^2(G)$ we may use Fubini's theorem to see

$$\begin{aligned}\langle \lambda_*(f)v, w\rangle &= \int f(h)\,\langle \lambda_h v, w\rangle \,\mathrm{d}m(h)\\ &= \iint f(h)v(g-h)\overline{w(g)}\,\mathrm{d}m(g)\,\mathrm{d}m(h)\\ &= \int f * v(g)\overline{w(g)}\,\mathrm{d}m(g) = \langle f*v, w\rangle.\end{aligned}$$

This shows that the integral defining $f * v$ exists almost everywhere, defines a function in $L^2(G)$, and equals $\lambda_*(f)v$. (We also refer to Exercise 1.53 and the hint on p. 475 for a different argument.) □

Lemma 2.17 (Generators of $L^2(G)$). *The regular representation λ of the abelian group G on $L^2(G)$ is cyclic. In fact, there exists some*

$$\psi \in \mathcal{V} = L^1(G) \cap L^2(G)$$

with $\check{\psi} > 0$ on all of $\widehat{G}$ and every such ψ is a generator for the regular representation.

Proof. We first claim that there exists some function $\psi \in \mathcal{V} = L^1(G)\cap L^2(G)$ such that $\check{\psi} > 0$ on all of $\widehat{G}$. For this we let $\psi_n \in \mathcal{V}$ be an approximate identity as in Proposition 1.42 so that $\widecheck{\psi_n}(t) \to 1$ as $n \to \infty$ for every $t \in \widehat{G}$ (by continuity of χ_t). Now choose $c_n > 0$ decaying sufficiently rapidly so that

$$\psi = \sum_{n=1}^{\infty} c_n \psi_n^* * \psi_n$$

converges both in $L^1(G)$ and in $L^2(G)$. Since $\widecheck{\psi_n^* * \psi_n} = \left|\widecheck{\psi_n}\right|^2$ this implies together that $\check{\psi} > 0$ as claimed.

For the claim in the lemma that any such $\psi \in \mathcal{V}$ is a generator, we apply the spectral theorem in the form of Corollary 2.12 to the unitary representation λ (which we do not know to be cyclic yet). Hence we obtain a finite measure μ on $X = \widehat{G} \times \mathbb{N}$ and a unitary isomorphism

$$U\colon L^2(G) \longrightarrow L^2_\mu(X)$$

such that $U \circ \lambda_*(f) = M_{\check{f}} \circ U$ for all $f \in L^1(G)$ (by the last claim in Corollary 2.12). To see that ψ is a generator suppose that $v \in L^2(G)$ satisfies $v \in \langle\psi\rangle_\lambda^\perp$ and let $f \in \mathcal{V}$. Together with the identity $\lambda_*(f)\psi = f * \psi$ from

Lemma 2.16 and commutativity of convolution we obtain

$$0 = \langle \lambda_*(f)\psi, v\rangle = \langle f * \psi, v\rangle = \langle \psi * f, v\rangle = \langle \lambda_*(\psi) f, v\rangle = \left\langle \check{\psi} U(f), U(v)\right\rangle.$$

Now vary f in the dense subspace $\mathcal{V} \subseteq L^2(G)$ and use continuity of the multiplication operator $M_{\check{\psi}}\colon w \in L^2_\mu(X) \mapsto \check{\psi} w \in L^2_\mu(X)$ to obtain that $U(v)$ is orthogonal to $\check{\psi} L^2_\mu(X)$. However, since $\check{\psi} > 0$ the image $\check{\psi} L^2_\mu(X)$ of the multiplication operator $M_{\check{\psi}}$ is dense, which in turn implies that $U(v) = 0$ and hence also $v = 0$. This implies the remaining claim of the lemma, namely that $L^2(G)$ is cyclic and is generated by any $\psi \in \mathcal{V}$ with $\check{\psi} > 0$ on all of $\widehat{G}$. □

Lemma 2.18 (Flattening the measure). *If we apply the spectral theorem in the form of Corollary* 2.11 *to the regular representation* λ *of the abelian group* G*, it is possible to replace the original measure* ν *on* $\widehat{G}$ *by a* σ*-finite measure* μ *defining the same measure class as* ν*, such that the map* $U\colon L^2(G) \to L^2_\mu(G)$ *also satisfies*

$$U(f) = \check{f} \tag{2.10}$$

for all $f \in \mathcal{V} = L^1(G) \cap L^2(G)$*.*

We note that the measure arising in Corollary 2.11 is not at all canonical, since it depends on the chosen generator. Lemma 2.18 ‘corrects’ this issue.

Proof of Lemma 2.18. By Lemma 2.17 we can apply Corollary 2.11 and assume that the unitary isomorphism between the regular representation λ and the multiplication representation M has the form $U_0\colon L^2(G) \to L^2_\nu(\widehat{G})$ for a finite measure ν on $\widehat{G}$. Using Lemma 2.16 and the last part of Corollary 2.12, we obtain

$$\check{\psi} U_0(f) = U_0(\psi * f) = U_0(f * \psi) = \check{f} U_0(\psi) \tag{2.11}$$

almost everywhere (with respect to ν) and for any $f \in \mathcal{V} = L^1(G) \cap L^2(G)$. In particular,

$$U_0(f) = \check{\psi}^{-1} U_0(\psi) \check{f}.$$

As $U_0(\mathcal{V}) \subseteq L^2_\nu(\widehat{G})$ is dense, it follows that $U_0(\psi) \neq 0$ almost everywhere. With this we define the complex-valued measurable and non-vanishing function

$$F = \check{\psi} U_0(\psi)^{-1}$$

on $\widehat{G}$, the σ-finite measure μ on $\widehat{G}$ by

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = |F|^{-2},$$

and the map $U = M_F \circ U_0\colon L^2(G) \to L^2_\mu(\widehat{G})$ (with inverse $U_0^{-1} \circ M_{F^{-1}}$). The latter satisfies

$$\|U(f)\|^2_{L^2_\mu(\widehat{G})} = \int_{\widehat{G}} |F|^2 |U_0(f)|^2 \,\mathrm{d}\mu = \int_{\widehat{G}} |U_0(f)|^2 |F|^2 \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\,\mathrm{d}\nu$$
$$= \|U_0(f)\|^2_{L^2_\nu(\widehat{G})} = \|f\|_2$$

for all $f \in L^2(G)$, and by (2.11) also $U(f) = \check{f}$ as in (2.10) for $f \in \mathcal{V}$. Since any two multiplication operators commute, the new unitary isomorphism still satisfies the conclusion of the spectral thereom. □

Lemma 2.19 (Translation invariance). *Let μ be a σ-finite measure on $\widehat{G}$ satisfying $\|\check{f}\|_{L^2_\mu(\widehat{G})} = \|f\|_{L^2(G)}$ for all $f \in \mathcal{V} = L^1(G) \cap L^2(G)$. Then $\mu = m_{\widehat{G}}$ is a Haar measure on $\widehat{G}$.*

PROOF. By Lemma 2.16 we have $L^1(G) * L^2(G) \subseteq L^2(G)$, which implies that $\mathcal{V} * \mathcal{V} \subseteq \mathcal{V}$. Together with Corollary 2.5 we then see that $\check{\mathcal{V}} = U(\mathcal{V})$ is contained in $C_0(\widehat{G}) \cap L^2_\mu(\widehat{G})$ and is a sub-algebra of $C_0(\widehat{G})$. Also recall from Lemma 2.17 that there exists some $\psi \in \mathcal{V}$ with $\check{\psi} > 0$.

Next we claim that μ is locally finite. Using $\check{\psi} \in C_0(\widehat{G})$ we see that

$$O = \left\{t \in \widehat{G} \mid \check{\psi}^2(t) > \tfrac{1}{2}\check{\psi}^2(t_0)\right\}$$

is a neighbourhood of $t_0 \in \widehat{G}$. Together with $\check{\psi}^2 \in L^1_\mu(\widehat{G})$, which follows from the fact that $\check{\psi} = U(\psi) \in L^2_\mu(G)$, it follows that $\mu(O) < \infty$. Since $t_0 \in \widehat{G}$ was arbitrary, it follows that μ is locally finite as claimed.

For $f \in \mathcal{V}$ and $t, t_0 \in \widehat{G}$ we have

$$\widecheck{(\chi_{t_0} f)}(t) = \int_G (\chi_{t_0} f)\chi_t \,\mathrm{d}m = \check{f}(t_0 + t) = \widehat{\lambda}_{-t_0}\check{f}(t). \tag{2.12}$$

Notice that

$$L^2(G) \ni f \longmapsto \chi_{t_0} f \in L^2(G)$$

is unitary, which implies that

$$\check{\mathcal{V}} \ni \check{f} \longmapsto \widehat{\lambda}_{-t_0}\check{f} \tag{2.13}$$

preserves the norm and inner products for all $t_0 \in \widehat{G}$. This is already a translation invariance claim for μ, but restricted to the class of functions $\big(\check{\mathcal{V}}\big)^2$.

To prove that μ is translation-invariant, we show that

$$\int_{\widehat{G}} F(t)\,\mathrm{d}\mu(t) = \int_{\widehat{G}} F(t_0 + t)\,\mathrm{d}\mu(t)$$

for any $F \in C_c(\widehat{G})$ and $t_0 \in \widehat{G}$, which in turn we show by approximating F with elements of $\big(\check{\mathcal{V}}\big)^2$.

Indeed, by density of $\mathcal{V}$ in $L^1(G)$ and by Corollary 2.5 we see that $\check{\mathcal{V}}$ is dense in $C_0(\widehat{G})$ with respect to the supremum norm. Clearly this approximation may be of little use since $\mu(\widehat{G})$ might be infinite. To overcome this we recall that $\check{\psi}^2 \in L^1_\mu(\widehat{G})$. Given some $F \in C_c(\widehat{G})$ and $\varepsilon > 0$, we can apply the density claim and find a function $f \in \mathcal{V}$ such that

$$\|\check{f} - \check{\psi}^{-2} F\|_\infty < \varepsilon.$$

We multiply this by $\check{\psi}^2$ and obtain

$$|\check{\psi}^2 \check{f} - F| < \varepsilon \check{\psi}^2 \tag{2.14}$$

on all of $\widehat{G}$. Integrating this inequality we obtain

$$\left| \int \check{\psi}^2 \check{f} \,\mathrm{d}\mu - \int F \,\mathrm{d}\mu \right| < \varepsilon \int \check{\psi}^2 \,\mathrm{d}\mu.$$

As noted in (2.13), the integrals of the functions $\check{\psi}^2 \check{f}, \check{\psi}^2 \in \big(\check{\mathcal{V}}\big)^2$ remain unchanged when these are shifted by t_0. Shifting the estimate in (2.14) by t_0 and integrating again, this gives

$$\left| \int F(t_0 + t) \,\mathrm{d}\mu(t) - \int F(t) \,\mathrm{d}\mu(t) \right| < 2\varepsilon \int \check{\psi}^2 \,\mathrm{d}\mu.$$

As $\varepsilon > 0$ was arbitrary, we deduce that

$$\int F(t_0 + t) \,\mathrm{d}\mu(t) = \int F(t) \,\mathrm{d}\mu(t)$$

for any $F \in C_c(\widehat{G})$. Since $\mu \neq 0$ we see that μ is a Haar measure on $\widehat{G}$. □

We now show that the combination of the above lemmas gives the Plancherel formula for G.

Proof of Theorem 2.15. By Lemma 2.17 and Lemma 2.18 we may apply the spectral theorem in the form of Corollary 2.11 and assume that the unitary isomorphism

$$U \colon L^2(G) \to L^2_\mu(\widehat{G})$$

satisfies $U(f) = \check{f}$ for all $f \in \mathcal{V} = L^1(G) \cap L^2(G)$. Applying Lemma 2.19, we also see that the measure is a Haar measure $\mu = m_{\widehat{G}}$. The formula

$$U \circ \lambda_g = M_g \circ U$$

holds by Corollary 2.11. Finally,

$$U(M_{t_0} f) = \widehat{\lambda}_{-t_0}(U(f))$$

holds initially only for $f \in \mathcal{V}$ (see (2.12)), but knowing that μ is the Haar measure on $\widehat{G}$ (so that $\widehat{\lambda}_{-t_0}$ is unitary) this extends to all of $L^2(G)$.

It remains to prove the description of the inverse of U on $L^1(\widehat{G}) \cap L^2(\widehat{G})$. So let $F \in L^1(\widehat{G}) \cap L^2(\widehat{G})$ and let (ψ_n) with $\psi_n = \frac{1}{m(B_n)} \mathbb{1}_{B_n}$ for $n \in \mathbb{N}$ be again an approximate identity (as in Proposition 1.42) for a basis (B_n) of the neighbourhood of $0 \in G$ with $B_n = -B_n$ for all $n \in \mathbb{N}$. For $g \in G$ we have

$$\begin{aligned}\left\langle U^{-1} F, \lambda_g \psi_n \right\rangle &= \left\langle F, \widecheck{\lambda_g \psi_n} \right\rangle \\ &= \left\langle F, M_g \widecheck{\psi_n} \right\rangle \\ &= \int F(t) \overline{\langle g, t\rangle} \overline{\widecheck{\psi}_n(t)} \, \mathrm{d}t \longrightarrow \int F(t) \overline{\langle g, t\rangle} \, \mathrm{d}t = \widehat{F}(g)\end{aligned}$$

as $n \to \infty$ by using the isomorphism U and by dominated convergence (since we have $|\widecheck{\psi}_n(t)| \leqslant \|\psi_n\|_1 = 1$ and $\widecheck{\psi}_n(t) \to 1$ as $n \to \infty$ for every $t \in \widehat{G}$).

To obtain the desired conclusion, we interpret $\langle U^{-1} F, \lambda_g \psi_n \rangle$ as a convolution. Indeed, we have

$$\begin{aligned}\left\langle U^{-1} F, \lambda_g \psi_n \right\rangle &= \int U^{-1}(F)(h) \underbrace{\psi_n(h-g)}_{=\psi_n(g-h)} \, \mathrm{d}m(h) \\ &= \int \psi_n(k) U^{-1}(F)(g-k) \, \mathrm{d}m(k) \\ &= \psi_n * U^{-1}(F)(g) = \lambda_*(\psi_n)\big(U^{-1}(F)\big)(g)\end{aligned}$$

for almost every $g \in G$ by using the substitution $k = g - h$ and by Lemma 2.16. Using Proposition 1.47 (for the regular representation), we see that

$$G \ni g \longmapsto \langle U^{-1} F, \lambda_g \psi_n \rangle$$

converges as the function $\lambda_*(\psi_n)(U^{-1} F)$ in $L^2(G)$ to $U^{-1} F$ as $n \to \infty$.

Even though the two notions of convergence above are different, we now obtain $U^{-1} F = \widehat{F}$ almost everywhere. Indeed, L^2-convergence implies the existence of a subsequence along which there is pointwise convergence, which gives the desired inequality. □

2.4 Pontryagin Duality

We note that the Plancherel formula in Theorem 2.15 already expresses some symmetry between G and $\widehat{G}$ (apart from a choice of sign). Using this, we will now establish a complete symmetry between G and its dual group $\widehat{G}$. For this we let $\widehat{\widehat{G}}$ denote the *Pontryagin bi-dual* (that is, the Pontryagin dual of the Pontryagin dual $\widehat{G}$) of the abelian group G. Moreover, for any $g \in G$ we define a map on $\widehat{G}$ by $\imath(g)(t) = \langle g, t \rangle = \chi_t(g)$ for $t \in \widehat{G}$. The properties of the Pontryagin bi-dual, the maps $\imath(g)$ for $g \in G$, and the map $\imath$ are given in the following result.

Theorem 2.20 (Pontryagin duality). *For the abelian group G the canonical map $\imath\colon G \to \widehat{\widehat{G}}$ is an isomorphism of topological groups.*

Let us start with some observations about the dual pairing $\langle \cdot, \cdot \rangle$.

Lemma 2.21. (1) *The map $\langle \cdot, \cdot \rangle : G \times \widehat{G} \longrightarrow \mathbb{S}^1$ is continuous.*
(2) *$\imath(g) \in \widehat{\widehat{G}}$ for any $g \in G$.*
(3) *The canonical map $\imath\colon G \to \widehat{\widehat{G}}$ is injective and continuous.*

Proof. For the proof of (1) suppose (g_n) in G converges to $g \in G$ and (t_n) in $\widehat{G}$ converges to $t \in \widehat{G}$ as $n \to \infty$. Then $K = \{g_n \mid n \in \mathbb{N}\} \cup \{g\}$ is a compact subset of G. By Corollary 2.5 the convergence $t_n \to t$ implies in particular that (χ_{t_n}) converges uniformly to χ_t on K as $n \to \infty$. This implies that $\langle g_n, t_n \rangle = \chi_{t_n}(g_n) \to \chi_t(g) = \langle g, t \rangle$ as $n \to \infty$.

Property (2) now follows quickly from (1) since $\imath(g)(t) = \langle g, t \rangle \in \mathbb{S}^1$ depends continuously on $t \in \widehat{G}$, and (by the definition of the group structure on $\widehat{G}$) defines a homomorphism from $\widehat{G}$ to $\mathbb{S}^1$.

It remains to prove (3). For $g_1, g_2 \in G$ and $t \in \widehat{G}$, we have

$$\imath(g_1 - g_2)(t) = \chi_t(g_1 - g_2) = \chi_t(g_1)\overline{\chi_t(g_2)} = \bigl(\imath(g_1)\imath(g_2)^{-1}\bigr)(t),$$

which shows that $\imath\colon G \to \widehat{G}$ is a homomorphism. Let g be in $G \smallsetminus \{0\}$. By the Gelfond–Raikov theorem (Corollary 1.75) there exists some $t \in \widehat{G}$ with $\langle g, t \rangle \neq 1$. (Alternatively, we could also note that $\lambda_g \neq I$ and apply the equivariance formulas in Theorem 2.15 to obtain the same conclusion.) In other words, $\imath(g) \in \widehat{\widehat{G}} \smallsetminus \{0\}$ and we see that $\imath\colon G \to \widehat{\widehat{G}}$ is injective.

To prove continuity of $\imath$, suppose that (g_n) converges to $g \in G$ as $n \to \infty$ and let $K \subseteq \widehat{G}$ be a compact subset. Then

$$L = (\{g_n \mid n \in \mathbb{N}\} \cup \{g\}) \times K \subseteq G \times \widehat{G}$$

is compact, and $\langle \cdot, \cdot \rangle$ restricted to L is uniformly continuous. This in particular implies that for every $\varepsilon > 0$ there exists some $N \geqslant 1$ such that

$$|\, \langle g_n, t \rangle - \langle g, t \rangle \,| < \varepsilon$$

for all $n \geqslant N$ and $t \in K$. In other words, the functions $\imath(g_n)\colon \widehat{G} \to \mathbb{S}^1$ and $\imath(g)\colon \widehat{G} \to \mathbb{S}^1$ are uniformly ε-close on K. Since the compact set $K \subseteq \widehat{G}$ and $\varepsilon > 0$ were arbitrary, Corollary 2.5 implies that $\imath(g_n) \to \imath(g)$ as $n \to \infty$, and it follows that $\imath$ is continuous. □

Exercise 2.22. Show (without using Theorem 2.20) that $\imath\colon G \to \widehat{\widehat{G}}$ is proper. Indeed, show first that if (g_n) is a sequence in G with $g_n \to \infty$ as $n \to \infty$ then $\lambda_{g_n}(v) \to 0$ in the weak* topology as $n \to \infty$ for any $v \in L^2(G)$. Now try to push this statement to a claim about the regular representation $\widehat{\lambda}_{\imath(g_n)}$ on $\widehat{G}$ and the elements $\imath(g_n) \in \widehat{\widehat{G}}$.

For the proof of Pontryagin duality, we will use the following general result concerning homomorphisms of topological groups in the special case of the canonical homomorphism $\imath\colon G \to \widehat{\widehat{G}}$.

Lemma 2.23. *Let G and G' be locally compact σ-compact metric abelian groups, and let $\theta\colon G \to G'$ be a continuous injective group homomorphism with $\theta_*(m_G) = m_{G'}$. Then θ is an isomorphism of topological groups.*

Proof. Let U be an open neighbourhood of $0 \in G$, and let $V = -V$ be a compact neighbourhood of 0 with $V + V \subseteq U$. Using $\theta_* m_G = m_{G'}$, we see that the characteristic function $f = \mathbb{1}_{\theta(V)}$ belongs to $L^1(G') \cap L^2(G')$. Moreover,

$$\begin{aligned} f * f(g') &= \int_{G'} \mathbb{1}_{\theta(V)}(h) \underbrace{\mathbb{1}_{\theta(V)}(g'-h)}_{=\mathbb{1}_{\theta(V)}(h-g')} \,\mathrm{d}m_{G'}(h) \\ &= \langle \lambda_{g'} \mathbb{1}_{\theta(V)}, \mathbb{1}_{\theta(V)} \rangle_{L^2(G')} = m_{G'}\bigl((\theta(V)+g') \cap \theta(V)\bigr) \end{aligned}$$

for all $g' \in G'$. This realises the function $f*f$ as the diagonal matrix coefficient for $\mathbb{1}_{\theta(V)} \in L^2(G')$, and gives $f * f \in C_b(G')$ and $f * f(0) > 0$. Therefore

$$(f*f)^{-1}\bigl((0,\infty)\bigr) \subseteq \theta(V) - \theta(V) = \theta(V - V) \subseteq \theta(U)$$

is a neighbourhood of $0 \in G'$. As U was an arbitrary neighbourhood of $0 \in G$ and θ is a homomorphism, this shows that θ maps open sets in G to open sets in G' (that is, θ is an open map).

In particular, $\theta(G) \subseteq G'$ is an open subgroup. However, this implies that $\theta(G)$ is also closed, since the continuity of the group operation implies that its complement

$$G' \smallsetminus \theta(G) = \bigcup_{g' \in G' \smallsetminus \theta(G)} \bigl(g' + \theta(G)\bigr)$$

is open. Finally, using $\theta_* m_G = m_{G'}$ in the form

$$m_{G'}\bigl(G' \smallsetminus \theta(G)\bigr) = m_G\bigl(\theta^{-1}(G' \smallsetminus \theta(G))\bigr) = m_G(\emptyset) = 0$$

shows, by the properties of Haar measure, that the open set $G' \smallsetminus \theta(G)$ must be empty. In other words, the continuous injective homomorphism $\theta\colon G \to G'$ is open and surjective, so is a homeomorphism between G and G'. $\square$

We will now use the Plancherel formula and its equivariance properties to prove Pontryagin duality.

Proof of Theorem 2.20. Let $\imath\colon G \to \widehat{\widehat{G}}$ be the canonical homomorphism studied in Lemma 2.21. We define the measure $\mu = \imath_*(m_G)$ on $\imath(G) \subseteq \widehat{\widehat{G}}$. Because of Lemma 2.23, our main goal is to show that μ is the Haar measure on $\widehat{\widehat{G}}$.

For this, we let $U\colon L^2(G) \to L^2(\widehat{G})$ be the unitary isomorphism from the Plancherel formula (Theorem 2.15 applied to G). In particular, we have that $U^{-1}(F) = \widehat{F}$ is given by the Fourier transform for

$$F \in \mathcal{V}_{\widehat{G}} = L^1(\widehat{G}) \cap L^2(\widehat{G}).$$

With this and the substitution $h = -g$ on G, we obtain

$$\begin{aligned}
\|\check{F}\|^2_{L^2_\mu(\widehat{G})} &= \int \bigl|\check{F}(\imath(g))\bigr|^2 \,\mathrm{d}m_G(g) \\
&= \int \Bigl|\int F(t) \underbrace{\langle t, \imath(g)\rangle}_{=\langle g,t\rangle} \,\mathrm{d}m_{\widehat{G}}(t)\Bigr|^2 \,\mathrm{d}m_G(g) \\
&= \int \Bigl|\int F(t)\langle -h, t\rangle \,\mathrm{d}m_{\widehat{G}}(t)\Bigr|^2 \,\mathrm{d}m_G(h) \\
&= \int |\widehat{F}|^2 \,\mathrm{d}m_G = \|F\|^2_{L^2(\widehat{G})}.
\end{aligned}$$

In other words, the measure μ on $\widehat{\widehat{G}}$ satisfies the assumptions of Lemma 2.19 when applied to $\widehat{G}$. Therefore $\mu = m_{\widehat{\widehat{G}}}$ is a Haar measure on $\widehat{\widehat{G}}$.

Recalling that $\mu = \imath_*(m_G)$ and the fact that $\imath\colon G \to \widehat{\widehat{G}}$ is an injective continuous group homomorphism, Lemma 2.23 shows that $\imath\colon G \to \widehat{\widehat{G}}$ is in fact an isomorphism of topological groups. □

The automatic reflexivity of the abelian group G in the Pontryagin duality theorem (Theorem 2.20) allows us to prove many duality statements that are reminiscent of finite dimensional linear algebra. In fact these duality statements go much further, as we will see in the following subsections.

Exercise 2.24. Lemma 2.23 was phrased for abelian groups to avoid changing notation in the middle of the chapter. Show that the assumption that the groups G and G' are abelian can be dropped.

2.4.1 First Duality Results, Sums and Products

Proposition 2.25 (Compactness and discreteness). *Let G be an abelian locally compact metric group. If G is compact, then $\widehat{G}$ is discrete. If G is discrete, then $\widehat{G}$ is compact.*

PROOF. Recall from Corollary 2.5 that the topology on $\widehat{G}$ is the compact-open topology. Suppose that G is compact and $t_0 \in \widehat{G}$ belongs to the neighbourhood

$$\left\{t \in \widehat{G} \mid |\langle g, t\rangle - 1| < 1 \text{ for all } g \in G\right\}$$

of $0 \in \widehat{G}$. Then $\{\langle g, t_0\rangle \mid g \in G\}$ is a subgroup of $\mathbb{S}^1$ contained in

$$\{z \in \mathbb{S}^1 \mid |z - 1| < 1\}.$$

As the only such subgroup is the trivial subgroup $\{1\} \subseteq \mathbb{S}^1$, we see that $t_0 = 0$ and hence that $\widehat{G}$ is discrete.

Suppose now that G is discrete. Then the compact-open topology on $\widehat{G}$ is equal to the product topology inherited from $(\mathbb{S}^1)^G$ and

$$\widehat{G} = \left\{\chi \in (\mathbb{S}^1)^G \mid \chi(g_1 + g_2) = \chi(g_1)\chi(g_2) \text{ for all } g_1, g_2 \in G\right\}$$

is a closed subset of the compact space $(\mathbb{S}^1)^G$ and hence is compact. □

It will be convenient to use the notation

$$N_{\widehat{G}}(K, \varepsilon) = \left\{t \in \widehat{G} \mid |\langle g, t\rangle - 1| < \varepsilon \text{ for all } g \in K\right\}$$

for the neighbourhood of $0 \in \widehat{G}$ defined by a compact subset $K \subseteq G$ and $\varepsilon > 0$ in the compact-open topology of $\widehat{G}$. In the following, G_1, $G_2, \dots$ will always denote abelian groups that are as usual locally compact, σ-compact, and metric, and we will refer to them simply as abelian groups.

For discrete abelian groups G_1, $G_2, \dots$ we define the *direct sum* by

$$\sum_{n=1}^{\infty} G_n = \Big\{(g_n) \in \prod_{n\in\mathbb{N}} G_n \mid g_n = 0 \text{ for all sufficiently large } n \in \mathbb{N}\Big\},$$

which we again endow with the discrete topology.

Proposition 2.26 (Products and sums). *The Pontryagin dual $\widehat{G_1 \times G_2}$ of the direct product $G_1 \times G_2$ of the abelian groups G_1 and G_2 is canonically isomorphic to $\widehat{G_1} \times \widehat{G_2}$. If the abelian groups G_n for $n \in \mathbb{N}$ are compact, then the Pontryagin dual of the direct product $\prod_{n=1}^{\infty} G_n$ is canonically isomorphic to the direct sum $\sum_{n=1}^{\infty} \widehat{G_n}$. Finally, if the abelian groups G_n for $n \in \mathbb{N}$ are discrete, then the Pontryagin dual of the direct sum $\sum_{n=1}^{\infty} G_n$ is canonically isomorphic to $\prod_{n=1}^{\infty} \widehat{G_n}$.*

We note that the isomorphisms are indeed quite natural. For instance, in the first statement we can use $(t_1, t_2) \in \widehat{G_1} \times \widehat{G_2}$ to induce a character on $G_1 \times G_2$ by the formula

$$\chi_{(t_1,t_2)}(g_1, g_2) = \langle (g_1, g_2), (t_1, t_2)\rangle = \langle g_1, t_2\rangle \, \langle g_2, t_2\rangle \tag{2.15}$$

for all $(g_1, g_2) \in G_1 \times G_2$. The isomorphism in the second and third are of the same nature. Hence we may and will interpret the above claims as equalities written

$$\widehat{G_1 \times G_2} = \widehat{G_1} \times \widehat{G_2},$$

$$\widehat{\prod_{n=1}^{\infty} G_n} = \sum_{n=1}^{\infty} \widehat{G_n}$$

if all the G_n are compact, and

$$\widehat{\sum_{n=1}^{\infty} G_n} = \prod_{n=1}^{\infty} \widehat{G_n}$$

if all the G_n are discrete.

Proof of Proposition 2.26. It is easy to see that the natural group operations make $G_1 \times G_2$ again into a locally compact, σ-compact, metric abelian group. If $(t_1, t_2) \in \widehat{G_1} \times \widehat{G_2}$ then (2.15) defines an element $\chi_{(t_1,t_2)}$ of $\widehat{G_1 \times G_2}$. If, on the other hand, $\chi \in \widehat{G_1 \times G_2}$ we may compose χ with the embedding of G_1 into $G_1 \times G_2$, which defines a continuous homomorphism $G_1 \to G_1 \times G_2 \to \mathbb{S}^1$ by

$$g_1 \longmapsto (g_1, 0) \longmapsto \chi(g_1, 0).$$

Hence there exists a uniquely determined $t_1 \in \widehat{G_1}$ with $\chi(g_1, 0) = \langle g_1, t_1 \rangle$ for all $g_1 \in G_1$. Similarly there exists a uniquely determined $t_2 \in \widehat{G_2}$ with $\chi(0, g_2) = \langle g_2, t_2 \rangle$ for all $g_2 \in \widehat{G_2}$. Since χ is a homomorphism, this gives

$$\chi(g_1, g_2) = \chi(g_1, 0)\chi(0, g_2) = \chi_{(t_1,t_2)}(g_1, g_2)$$

for all $(g_1, g_2) \in G_1 \times G_2$. It follows that (2.15) defines an isomorphism between $\widehat{G_1} \times \widehat{G_2}$ and $\widehat{G_1 \times G_2}$, which is easily seen to be an isomorphism of groups.

To see continuity of this isomorphism in both directions, it is sufficient to consider neighbourhoods of the identity. So suppose first that $K_1 \subseteq G_1$ and $K_2 \subseteq G_2$ are compact and $\varepsilon > 0$. If now $\chi_{(t_1,t_2)} \in N_{\widehat{G_1 \times G_2}}(K_1 \times K_2, \varepsilon)$ then by restriction to $G_1 \times \{0\}$ and $\{0\} \times G_2$ we obtain $t_1 \in N_{\widehat{G_1}}(K_1, \varepsilon)$ and $t_2 \in N_{\widehat{G_2}}(K_2, \varepsilon)$. For the converse, we note that if $K \subseteq G_1 \times G_2$ is compact, then $K \subseteq K_1 \times K_2$ for the compact projections K_1 and K_2 of K to G_1 and G_2. If now $t_1 \in N_{\widehat{G_1}}(K_1, \frac{\varepsilon}{2})$ and $t_2 \in N_{\widehat{G_2}}(K_2, \frac{\varepsilon}{2})$ then

$$\left|\chi_{(t_1,t_2)}(g_1, g_2) - 1\right| = \left|\langle g_1, t_1 \rangle\bigl(\langle g_2, t_2 \rangle - 1\bigr) + \langle g_1, t_1 \rangle - 1\right| < \varepsilon$$

for all $(g_1, g_2) \in K$, and hence $\chi_{(t_1,t_2)} \in N_{\widehat{G_1 \times G_2}}(K, \varepsilon)$.

Suppose now that $G_1, G_2, \dots$ are all compact and $G = \prod_{n=1}^{\infty} G_n$ is equipped with the product topology. Any $\chi \in \widehat{G}$ restricts as before to any factor G_m for $m \in \mathbb{N}$, and we obtain that there exists a uniquely determined $t_m \in \widehat{G_m}$ with

$$\chi(0, \dots, 0, g_m, 0, \dots) = \langle g_m, t_m \rangle$$

for all $g_m \in G_m$, where $(0, \dots, 0, g_m, 0, \dots)$ denotes the element of $\prod_{n=1}^{\infty} G_n$ that has g_m as its mth entry, and otherwise only zeroes. By continuity of χ there exists a neighbourhood U of $0 \in G$ such that $|\chi(g)-1| < 1$ for all $g \in U$. By definition of the product topology there exists some $N \in \mathbb{N}$ such that

$$\{0\}^N \times \prod_{n=N+1}^{\infty} G_n \subseteq U.$$

Therefore,

$$\chi\Big(\{0\}^N \times \prod_{n=N+1}^{\infty} G_n\Big)$$

is a subgroup of $\mathbb{S}^1$ contained in $\{z \in \mathbb{S}^1 \mid |z-1| < 1\}$, which must therefore be trivial. Hence $t_m = 0$ for all $m > N$ and χ is given by

$$\chi\big((g_n)_n\big) = \prod_{n=1}^{N} \langle g_n, t_n \rangle \tag{2.16}$$

for all $(g_n)_n \in G$. Using the projections from G to its factors G_n for $n \in \mathbb{N}$ it follows that (2.16) defines an element of $\widehat{G}$ for any $(t_n)_n \in \prod_{n=1}^{N} \widehat{G_n}$ and N in $\mathbb{N}$. Since $\widehat{G}$ and $\sum_{n=1}^{\infty} \widehat{G_n}$ are both discrete by Proposition 2.25, this proves the second claim.

The third claim follows from the second by applying it to $\widehat{G_n}$ and using Pontryagin duality (Theorem 2.20). □

Exercise 2.27. Prove the last claim in Proposition 2.26 directly, without relying on Pontryagin duality.

2.4.2 Dual Homomorphisms

Let now G_1 and G_2 be locally compact, σ-compact, metric, abelian groups and let $\theta\colon G_1 \to G_2$ be a continuous group homomorphism. Then there exists a dual map $\widehat{\theta}\colon \widehat{G_2} \to \widehat{G_1}$ defined by

$$\langle g, \widehat{\theta}(t) \rangle = \langle \theta(g), t \rangle \tag{2.17}$$

for $g \in G_1$ and $t \in \widehat{G_2}$.

Lemma 2.28 (Dual homomorphisms). *With the assumptions above, equation* (2.17) *defines a continuous group homomorphism* $\widehat{\theta}\colon \widehat{G_2} \to \widehat{G_1}$*, called the* dual homomorphism *of* θ*. Under the canonical isomorphism between the biduals and the original groups in Theorem* 2.20 *the dual homomorphism of* $\widehat{\theta}$ *is given by* θ*. Moreover, if* $\theta'\colon G_2 \to G_3$ *is another continuous group homomorphism with values in a locally compact,* σ*-compact, metric abelian group* G_3*, then* $\widehat{\theta' \circ \theta} = \widehat{\theta} \circ \widehat{\theta'}\colon \widehat{G_3} \to \widehat{G_1}$.

Proof. Clearly the right hand side of (2.17) belongs to $\mathbb{S}^1$ for every $g \in G_1$ and $t \in \widehat{G_2}$. Fixing t and varying g we can use the fact that θ is a continuous group homomorphism to see that the map $G_1 \ni g \mapsto \langle \theta(g), t\rangle$ indeed defines an element $\widehat{\theta}(t) \in \widehat{G_1}$.

For $g \in G_1$ and $t_1, t_2 \in G_2$ we have

$$\begin{aligned}\big\langle g, \widehat{\theta}(t_1 + t_2)\big\rangle &= \langle \theta(g), t_1 + t_2\rangle \\ &= \langle \theta(g), t_1\rangle\langle \theta(g), t_2\rangle \\ &= \big\langle g, \widehat{\theta}(t_1)\big\rangle\big\langle g, \widehat{\theta}(t_2)\big\rangle = \big\langle g, \widehat{\theta}(t_1) + \widehat{\theta}(t_2)\big\rangle,\end{aligned}$$

which shows that $\widehat{\theta}\colon \widehat{G_2} \to \widehat{G_1}$ is a homomorphism.

As $\widehat{\theta}$ is a homomorphism, it suffices to prove continuity at the identity of $\widehat{G_2}$ to obtain continuity on all of $\widehat{G_2}$. So let $K \subseteq G_1$ be some compact subset and $\varepsilon > 0$ and use these to define the neighbourhood $N_{\widehat{G_1}}(K, \varepsilon)$. Then $\theta(K) \subseteq G_2$ is compact (since θ is continuous), so it defines a neighbourhood $N_{\widehat{G_2}}(\theta(K), \varepsilon)$. It is now easy to see that $t \in N_{\widehat{G_2}}(\theta(K), \varepsilon)$ and $g \in K$ implies that

$$\big|\big\langle g, \widehat{\theta}(t)\big\rangle - 1\big| = |\langle \theta(g), t\rangle - 1| < \varepsilon$$

and hence $\widehat{\theta}(t) \in N_{\widehat{G_1}}(K, \varepsilon)$, which gives continuity of $\widehat{\theta}$ at $0 \in \widehat{G_2}$, as required.

For $g \in G_1$ and $t \in \widehat{G_2}$ we have[†]

$$\langle \widehat{\widehat{\theta}}(g), t\rangle = \big\langle g, \widehat{\theta}(t)\big\rangle = \langle \theta(g), t\rangle,$$

which proves that $\widehat{\widehat{\theta}} = \theta$.

Finally for θ' and $t \in \widehat{G_3}$ as in the last part of the lemma we have

$$\langle g, \widehat{\theta' \circ \theta}(t)\rangle = \langle \theta'(\theta(g)), t\rangle = \langle \theta(g), \widehat{\theta'}(t)\rangle = \langle g, \widehat{\theta} \circ \widehat{\theta'}(t)\rangle$$

for all $g \in G_1$, and the lemma follows. □

Recall from Exercise 2.6 that $\widehat{\mathbb{R}^d} \cong \mathbb{R}^d$ by the formula $\langle g, t\rangle = \mathrm{e}^{2\pi \mathrm{i} g\cdot t}$ for $g, t \in \mathbb{R}^d$, where $g \cdot t = \sum_{j=1}^d g_j t_j$ is the standard inner product on $\mathbb{R}^d$

[†] This is to be interpreted as $\langle t, \widehat{\widehat{\theta}}(\imath(g))\rangle = \langle \widehat{\theta}(t), \imath(g)\rangle = \langle g, \widehat{\theta}(t)\rangle$.

for $d \geqslant 1$. Suppose now $d, e \geqslant 1$ and that $A\colon \mathbb{R}^d \to \mathbb{R}^e$ is a linear map. Then the dual homomorphism $\widehat{A}$ is equal to the dual map A^{t} in the sense of linear algebra, and so is defined by the transpose of the matrix defining A if we use the standard basis on $\mathbb{R}^d$ and $\mathbb{R}^e$. In fact,

$$\left\langle g, \widehat{A}t \right\rangle = \langle Ag, t \rangle = \mathrm{e}^{2\pi \mathrm{i}(Ag\cdot t)} = \mathrm{e}^{2\pi \mathrm{i}(g\cdot A^{\mathrm{t}} t)} = \left\langle g, A^{\mathrm{t}} t \right\rangle$$

for all $g \in \mathbb{R}^d$ and $t \in \mathbb{R}^e$, which proves the claim.

We finish the subsection by stating another duality claim for homomorphisms, which we will prove at the end of the next subsection as a corollary of our discussion regarding quotients.

Corollary 2.29 (Injectivity and dense image). *Let $\theta\colon G_1 \to G_2$ be a continuous group homomorphism from the abelian group G_1 to the abelian group G_2. Then*

(1) *θ is injective if and only if $\widehat{\theta}$ has dense image; and*
(2) *θ has dense image if and only if $\widehat{\theta}$ is injective.*

2.4.3 Subgroups and Quotients*

For a closed subgroup H of the abelian group G, we define

$$G/H = \{g + H \mid g \in G\}$$

as the quotient group, and equip G/H with the quotient topology. Recall that G/H is abelian, and by the more general Proposition B.3 it also follows that G/H in the quotient topology is a locally compact σ-compact metric abelian group.

The *annihilator* $H^\perp$ of a closed subgroup (or even a subset) H of the abelian group G is defined by

$$H^\perp = \left\{t \in \widehat{G} \mid \langle h, t\rangle = 1 \text{ for all } h \in H\right\}.$$

Proposition 2.30 (Duality of subgroups and quotients). *Let $H \subseteq G$ be a closed subgroup of the abelian group G. Then $H^\perp \subseteq \widehat{G}$ is a closed subgroup which, together with the quotient group, satisfies the following duality statements.*

(1) *$\widehat{G/H} = H^\perp$ via the pairing defined by $\langle g + H, t\rangle = \langle g, t\rangle$ for $t \in H^\perp$ and $g + H \in G/H$.*
(2) *$\widehat{H} = \widehat{G}/H^\perp$ via the pairing defined by $\langle h, t + H^\perp\rangle = \langle h, t\rangle$ for $h \in H$ and $t + H^\perp \in \widehat{G}/H^\perp$.*
(3) *$\left(H^\perp\right)^\perp = H$, where we identify $\widehat{\widehat{G}}$ with G using Theorem 2.20.*

Proof. By definition,

$$H^{\perp} = \bigcap_{h \in H} \ker\big(\widehat{G} \ni t \longmapsto \langle h, t \rangle\big),$$

and so by Lemma 2.21 we see that $H^{\perp}$ is a closed subgroup of $\widehat{G}$.

We note that any $t \in H^{\perp}$ induces a well-defined homomorphism

$$\chi \colon G/H \ni g + H \longmapsto \langle g, t \rangle \in \mathbb{S}^1,$$

which by definition of the quotient topology is also continuous. On the other hand, a character χ on G/H induces by composition a character

$$\chi \circ p \colon G \to \mathbb{S}^1$$

on G which must correspond to some $t \in H^{\perp}$. This gives the identification between $\widehat{G/H}$ and $H^{\perp}$, which is clearly also compatible with the group structures. It remains to show that this identification is a homeomorphism.

By Proposition B.3 compact subsets $K' \subseteq G/H$ are precisely of the form $K' = p(K)$ for some compact set $K \subseteq G$. This shows that the neighbourhood $N_{\widehat{G/H}}(K', \varepsilon)$ of $0 \in \widehat{G/H}$ corresponds to the neighbourhood

$$H^{\perp} \cap N_{\widehat{G}}(K, \varepsilon)$$

of $0 \in H^{\perp}$ for any $K' = p(K)$ and $\varepsilon > 0$, which completes the proof of (1).

Next we prove (3). Clearly we have $H \subseteq \big(H^{\perp}\big)^{\perp}$. Suppose that $g \in G \smallsetminus H$. Recall from Lemma 2.21(3) that $\imath \colon G/H \to \widehat{\widehat{G/H}}$ is injective which, by definition, means that for $g+H \neq 0+H$ in G/H there exists a character on G/H that maps $g + H$ to $z \neq 1$ in $\mathbb{S}^1$. By (1), this implies the existence of $t \in H^{\perp}$ with $\langle g, t \rangle = z \neq 1$. However, this implies that $g \notin \big(H^{\perp}\big)^{\perp}$, proving (3).

The isomorphism in (2) now follows from Pontryagin duality (Theorem 2.20) by applying (1) to the subgroup $H^{\perp}$ in $\widehat{G}$. □

Example 2.31 (Duality between projection and embedding). Let $H \subseteq G$ be a closed subgroup of the abelian group G. Using the first isomorphism in Proposition 2.30 the dual of the canonical map $p \colon G \to G/H$ is the canonical embedding map from $\widehat{G/H} = H^{\perp}$ to $\widehat{G}$ since

$$\langle g, \widehat{p}(t) \rangle = \langle p(g), t \rangle = \langle g + H, t \rangle = \langle g, t \rangle.$$

for $g \in G$ and $t \in H^{\perp}$. Similarly we may use the second isomorphism to conclude that the dual of the embedding $\imath \colon H \to G$ is given by the canonical projection from $\widehat{G}$ to $\widehat{H} = \widehat{G}/H^{\perp}$ as

$$\langle h, \widehat{\imath}(t) \rangle = \langle h, t \rangle = \langle h, t + H^{\perp} \rangle$$

for all $h \in H$ and $t \in \widehat{G}$.

Using Proposition 2.30 we can now prove the duality claim regarding injectivity and dense image for homomorphisms.

Proof of Corollary 2.29. Suppose that θ has dense image and $t \in \widehat{G_2}$ satisfies $\widehat{\theta}(t) = 0$. Then we have $1 = \langle g, \widehat{\theta}(t)\rangle = \langle \theta(g), t\rangle$ for all $g \in G_1$, or equivalently $\chi_t(\operatorname{Im}\theta) = 1$. Since $\overline{\operatorname{Im}\theta} = G_2$ and χ_t is continuous, this implies that $t = 0$ and hence that $\widehat{\theta}$ is injective.

Suppose now that $\overline{\operatorname{Im}\theta} \neq G_2$. Then by Proposition 2.30 there exists a non-trivial $t \in (\overline{\operatorname{Im}\theta})^{\perp} \subseteq \widehat{G_2}$. For this t and all $g \in G_1$ we then have

$$\langle g, \widehat{\theta}(t)\rangle = \langle \theta(g), t\rangle = 1,$$

which implies that $\widehat{\theta}(t) = 0$ and hence that $\widehat{\theta}$ is not injective.

This proves (2), which together with Pontryagin duality and the identity $\widehat{\widehat{\theta}} = \theta$ in Lemma 2.28 also implies (1). □

Exercise 2.32 (Kernel and closure of image). Let $\theta\colon G_1 \to G_2$ be a homomorphism as in Corollary 2.29.
(a) Show that $\left(\overline{\operatorname{Im}\theta}\right)^{\perp} = \ker\widehat{\theta}$.
(b) Show that $\left(\ker\theta\right)^{\perp} = \overline{\operatorname{Im}\widehat{\theta}}$.

Exercise 2.33 (Connectedness and torsion). Let G be a compact metric abelian group. Show that G is connected if and only if $\widehat{G}$ has no torsion elements (that is, if and only if $t \in \widehat{G}$ with $nt = 0$ and $n \geqslant 1$ implies that $t = 0$).

Exercise 2.34. Let $A \subseteq G$ be an arbitrary subset of a topological group. Prove that $(A^{\perp})^{\perp}$ is the closure of the group generated by A.

2.4.4 Projective and Direct Limits*

We wish to discuss two more constructions that are once again dual to each other under Pontryagin duality. We start with the projective limit, which in a sense generalizes the direct product, and is defined as follows. Suppose that (G_n) is a sequence of abelian groups, and $\theta_n\colon G_{n+1} \to G_n$ is a continuous surjective homomorphism with compact kernel for every $n \in \mathbb{N}$. Then the *projective limit* of the system (G_n, θ_n) is defined to be the closed subgroup

$$G = \varprojlim(G_n, \theta_n) = \left\{ (g_n) \in \prod_{n=1}^{\infty} G_n \mid \theta_n(g_{n+1}) = g_n \text{ for every } n \in \mathbb{N} \right\}$$

of the product $\prod_{n=1}^{\infty} G_n$ equipped with the product topology.

The second construction is the direct limit, which in a sense generalizes the direct sum. For this, suppose that (H_n) is a sequence of abelian groups and $\imath_n\colon H_n \to H_{n+1}$ is a homomorphism such that $\imath_n\colon H_n \to \imath_n(H_n)$ is an isomorphism of topological groups between H_n and the open subgroup $\imath(H_n) \subseteq H_{n+1}$ for every $n \in \mathbb{N}$. We use $\imath_n$ to identify H_n with the subgroup $\imath_n(H_n)$ and define the *direct limit* of the system $(H_n, \imath_n)$ as

$$H = \varinjlim(H_n, \imath_n) = \bigcup_{n=1}^{\infty} H_n. \tag{2.18}$$

This is a small cheat, as we use the identification to suppress the set-theoretic construction of the direct limit in the category of sets,[†] that is of a set H_∞ and maps $\phi_n\colon H_n \to H_\infty$ with $H_\infty = \bigcup_{n=1}^{\infty} \phi_n(H_n)$ and $\phi_{n+1} \circ \imath_n = \phi_n$ for all $n \in \mathbb{N}$. As we want to focus here on the algebraic and topological properties of $\varinjlim(H_n, \imath_n)$, this will simplify our discussion a little. In concrete examples this step may need to be treated more carefully (see Exercise 2.35).

Exercise 2.35. Let $q > 1$ be an integer, define $H_n = \mathbb{Z}$ and $\times q = \imath_n\colon H_n \to H_{n+1}$ by $\imath_n(k) = qk$ for all $k \in H_n = \mathbb{Z}$ and $n \in \mathbb{N}$. Describe the direct limit $\varinjlim(\mathbb{Z}, \times q)$.

Using the fact that (2.18) is an increasing union, the group operation on H are defined in an obvious way: for $h_1, h_2 \in H$ there exists some $n \in \mathbb{N}$ with $h_1, h_2 \in H_n$ and hence $h_1 h_2, h_1^{-1}$ are defined in H by the using the group operations in H_n.

The topology on $\varinjlim(H_n, \imath_n)$ is defined by the property that each H_n is (homeomorphically embedded as) an open subset of $\varinjlim(H_n, \imath_n)$. More precisely, let the abelian groups H_n and embeddings $\imath_n\colon H_n \to H_{n+1}$ be as in the definition of the direct limit $\varinjlim(H_n, \imath_n)$. Due to the assumed properties of $\imath_n$ we have that H_n can be considered to be an open subgroup of H_{n+1}. This allows us to define the topology on H as in (2.18) as the inductive topology, in which a subset $O \subseteq H$ is open if and only if $O \cap H_n$ is open for all $n \in \mathbb{N}$. In particular, $H_n \subseteq H$ is open for every $n \in \mathbb{N}$.

Proposition 2.36 (Projective and direct limits). *Under the assumptions above, the projective limit $\varprojlim(G_n, \theta_n)$ and the direct limit $\varinjlim(H_n, \imath_n)$ are again locally compact σ-compact metric abelian groups.*

Proof. We first discuss the topological group $\varprojlim(G_n, \theta_n)$. From continuity of $\theta_n\colon G_{n+1} \to G_n$ for $n \in \mathbb{N}$ and the definition of the product topology, it follows that $G = \varprojlim(G_n, \theta_n)$ is a closed subset of $\prod_{n=1}^{\infty} G_n$. As the maps are also homomorphisms, G is in fact a closed subgroup, and hence is a metric abelian topological group. For the local and the σ-compactness, we claim that

[†] For example, one can use the set

$$H_\infty = \{1\} \times H_1 \sqcup \bigsqcup_{n \geqslant 2} \{n\} \times H_n \smallsetminus \imath_{n-1}(H_{n-1}) \subseteq \mathbb{N} \times \bigcup_{n \in \mathbb{N}} H_n.$$

We leave the definition of the maps ϕ_n and the proof of the identity $\phi_{n+1} \circ \imath_n = \phi_n$ for $n \in \mathbb{N}$ to the reader.

the surjective map $\theta_n\colon G_{n+1} \to G_n$ with compact kernel is a proper map. Given $(g_n) \in G = \varprojlim(G_n, \theta_n)$ we then can find a compact neighbourhood K of $g_1 \in G_1$ and obtain the neighbourhood

$$\left(K \times \prod_{n=2}^{\infty} G_n\right) \cap G = \left(\prod_{n=1}^{\infty} K_n\right) \cap G,$$

for $K_1 = K$ and $K_{n+1} = \theta_n^{-1}(K_n)$ for all $n \in \mathbb{N}$. By Tychonoff's theorem, $\prod_{n=1}^{\infty} K_n$ is compact and so G is locally compact. Writing G_1 as a countable union of compact sets and applying the argument above once more, we also see that G is σ-compact.

To prove the claim we simplify notation and suppose that $\theta\colon G \to G'$ is a surjective continuous homomorphism with compact kernel between the abelian groups G and G'. Let $K' \subseteq G'$ be compact. By Proposition B.3 there exists a compact subset $K \subseteq G$ with $K' = \theta(K)$ and hence

$$\theta^{-1}(K) = K + \ker\theta$$

is compact as claimed.

Suppose now that the abelian groups H_n and embeddings $\imath_n\colon H_n \to H_{n+1}$ are as in the definition of the direct limit $\varinjlim(H_n, \imath_n)$. Due to the assumed properties of $\imath_n$ we have that H_n can be considered to be an open subgroup of H_{n+1}. This allows us to define H as in (2.18) equipped with the obvious operations and the inductive topology. Recall that $H_n \subseteq H$ is open for every $n \in \mathbb{N}$. Since each H_n is locally compact, σ-compact, and has second countable topology, the same is true for H, and in particular H is metric. If a sequence in H converges to some $h \in H_n$, then by openness of H_n in H, all but finitely many terms of the sequence must lie in H_n. From this it is easy to conclude that H is also a topological group. □

For the discussion of the Pontryagin dual of projective and direct limits the following two notions and their relation will be useful. We say that a continuous homomorphism $\theta\colon G \to G'$ is a *proper projection* if it is onto and has compact kernel (which as we have shown in the above proof indeed implies properness of the map). Furthermore we say that $\imath\colon H \to H'$ is an *open embedding* if it is an isomorphism between H and an open subgroup $\imath(H)$ of H'.

Lemma 2.37 (Proper projections and open embeddings). *Let G, G' and H, H' be locally compact σ-compact metric abelian groups.*

(1) *If $\theta\colon G \to G'$ is a proper projection, then $\widehat{\theta}\colon \widehat{G'} \to \widehat{G}$ is an open embedding.*
(2) *If $\imath\colon H \to H'$ is an open embedding, then $\widehat{\imath}\colon \widehat{H'} \to \widehat{H}$ is a proper projection.*

Proof. We suppose that θ is a proper projection. Then surjectivity of θ implies injectivity of $\widehat{\theta}$ by Corollary 2.29. Moreover, the last claim in Proposition B.3 shows that $\theta = \overline{\theta} \circ p$, where $p\colon G \to G/\ker\theta$ is the canonical projection map and $\overline{\theta}\colon G/\ker\theta \to G'$ is an isomorphism. Dually, we then have $\widehat{\theta} = \widehat{p} \circ \widehat{\overline{\theta}}$, where $\widehat{\overline{\theta}}$ is an isomorphism and

$$\widehat{p}\colon \widehat{G/\ker\theta} = (\ker\theta)^{\perp} \to \widehat{G}$$

is the canonical embedding by Example 2.31. Furthermore we have

$$\widehat{G}/(\ker\theta)^{\perp} = \widehat{\ker\theta}$$

by Proposition 2.30. By assumption $\ker\theta$ is compact, which implies that $\widehat{\ker\theta}$ is discrete by Proposition 2.25. Therefore, $(\ker\theta)^{\perp}$ is an open subgroup of $\widehat{G}$ and it follows that $\widehat{\theta}$ embeds $\widehat{G'}$ onto the open subgroup $\widehat{\theta}(\widehat{G'}) = (\ker\theta)^{\perp} \subseteq \widehat{G}$ as claimed.

We suppose now that $\imath\colon H \to H'$ is a continuous embedding such that $\imath(H) \subseteq H'$ is an open subgroup and $\imath\colon H \to \imath(H)$ is a group isomorphism. Identifying H with $\imath(H)$, the dual homomorphism to the embedding $\imath\colon H \to H'$ is the canonical projection $\widehat{\imath}\colon \widehat{H'} \to \widehat{H'}/H^{\perp}$ with kernel $H^{\perp} = \widehat{H'/H}$ by Example 2.31. Since H'/H is discrete, Proposition 2.25 shows that $\ker\widehat{\imath} = H^{\perp}$ is compact. □

With these preparations, we can now prove the duality between the two limit constructions.

Proposition 2.38 (Duality of limits). *Let the projective limit* $\varprojlim(G_n, \theta_n)$ *and the direct limit* $\varinjlim(H_n, \imath_n)$ *be as in Proposition* 2.36. *Then*

$$\widehat{\varprojlim(G_n, \theta_n)} = \varinjlim(\widehat{G_n}, \widehat{\theta_n}),$$

where we use the open embedding $\imath_n = \widehat{\theta_n}\colon \widehat{G_n} \to \widehat{G_{n+1}}$ *for all* $n \in \mathbb{N}$ *to define the direct limit. Dually,*

$$\widehat{\varinjlim(H_n, \imath_n)} = \varprojlim(\widehat{H_n}, \widehat{\imath_n}),$$

where we use the proper projection $\theta_n = \widehat{\imath_n}\colon \widehat{H_{n+1}} \to \widehat{H_n}$ *for* $n \in \mathbb{N}$ *to define the projective limit.*

Proof. Lemma 2.37 shows that if $(H_n, \imath_n)$ satisfies the assumptions for the construction of

$$H = \varinjlim(H_n, \imath_n)$$

then $(\widehat{H_n}, \widehat{\imath_n})$ satisfies the assumptions for the construction of

$$G = \varprojlim(\widehat{H_n}, \widehat{\imath_n}).$$

Suppose now $h \in H$ and (t_n) is a sequence in G. Then there exists some $m \in \mathbb{N}$ with $h \in H_m$, and we define

$$\langle h, (t_n) \rangle_{\lim} = \langle h, t_m \rangle.$$

We wish to prove that this is the dual pairing between H and $\widehat{H} \cong G$. Note that

$$\langle h, t_{m+1} \rangle = \langle \imath_m(h), t_{m+1} \rangle = \langle h, \widehat{\imath}_m(t_{m+1}) \rangle = \langle h, t_m \rangle$$

since $\widehat{\imath_n}(t_{n+1}) = t_n$ for all (t_n) in G and $n \in \mathbb{N}$ by construction of the projective limit $G = \varprojlim(\widehat{H_n}, \imath_n)$. Thus the expression $\langle h, (t_n) \rangle_{\lim}$ is independent of the choice of m, which easily implies that

$$H \ni h \longmapsto \langle h, (t_n) \rangle_{\lim} \in \mathbb{S}^1 \tag{2.19}$$

defines a multiplicative homomorphism. Using the fact that H_n is an open subgroup of H and $t_n \in \widehat{H_n}$ we also obtain continuity of the character defined by (2.19) for each $n \geqslant 1$. In other words, we have a well-defined homomorphism $\Phi\colon G \to \widehat{H}$, which sends (t_n) to the character in (2.19). Moreover, if we have $\Phi((t_n)) = 0$ for some $(t_n) \in G$, then $\langle H_n, t_n \rangle = 1$ for all $n \in \mathbb{N}$, and so Φ is injective.

Now let χ be a character on H. Restricting χ to any of the open subgroups H_n for $n \in \mathbb{N}$, we obtain the character $\chi|_{H_n}$ on H_n. Hence there exists a uniquely determined $t_n \in \widehat{H_n}$ with

$$\chi(h) = \langle h, t_n \rangle$$

for all $h \in H_n$. For $h \in H_n$ we also have $h = \imath_n(h) \in H_{n+1}$, and so

$$\langle h, t_n \rangle = \chi(h) = \langle \imath_n(h), t_{n+1} \rangle = \langle h, \widehat{\imath_n}(t_{n+1}) \rangle$$

for all $h \in H_n$. This implies that $t_n = \widehat{\imath}_n(t_{n+1})$ for all $n \in \mathbb{N}$. Therefore (t_n) lies in G, and we have shown that $\Phi\colon G \to \widehat{H}$ is onto and so gives the desired identification.

To see that $\widehat{H}$ and G are also isomorphic as topological groups, we note that $H = \bigcup_{n=1}^{\infty} H_n$ is an open cover and hence any compact set $K \subseteq H$ belongs to some H_m. It follows that $N_{\widehat{H}}(K, \varepsilon)$ corresponds under the isomorphism from $\widehat{H}$ to $G = \varprojlim(\widehat{H_n}, \widehat{\imath_n})$ to the set

$$\{(t_n) \in G \mid t_m \in N_{\widehat{H_m}}(K, \varepsilon)\}.$$

As the latter is an open subset of G (with respect to the restriction of the product topology), we conclude that the map from G to $\widehat{H}$ is continuous. Proposition B.3 now implies that the above isomorphism from G to $\widehat{H}$ as abstract groups is in fact an isomorphism for topological groups.

The dual statement concerning the Pontryagin dual of $\varprojlim(G_n, \theta_n)$ follows from Lemma 2.37, the above, and Pontryagin duality (Theorem 2.20). □

2.4.5 Local Fields*

By definition a *local field* $\mathbb{K}$ is a locally compact σ-compact non-discrete metric field, where all field operations are assumed to be continuous. Our main goal here is to prove the following self duality statement.

Proposition 2.39 (Self-duality). *A local field $\mathbb{K}$ is isomorphic as an additive group to its own Pontryagin dual $\widehat{\mathbb{K}}$. In fact, for any non-trivial character $\chi \in \widehat{\mathbb{K}}$ we can define an isomorphism of topological groups*

$$\mathbb{K} \ni t \longmapsto \chi_t \in \widehat{\mathbb{K}}$$

by $\chi_t \colon \mathbb{K} \ni a \mapsto \chi_t(a) = \chi(at)$.

A useful tool, both for our discussion and for the further study and classification of local fields, is the *induced absolute value* defined by

$$|a|_{\mathbb{K}} = \frac{m_{\mathbb{K}}(aM)}{m_{\mathbb{K}}(M)},$$

where $a \in \mathbb{K}$, $m_{\mathbb{K}}$ is a Haar measure for the group $(\mathbb{K}, +)$, and $M \subseteq \mathbb{K}$ is any Borel subset with positive finite measure.

Lemma 2.40 (Properties of the absolute value). *Let $\mathbb{K}$ be a local field. The absolute value $|\cdot|_{\mathbb{K}} \colon \mathbb{K} \to [0, \infty)$ is positive in the sense that $|a|_{\mathbb{K}} > 0$ for all $a \in \mathbb{K} \smallsetminus \{0\}$, well-defined, multiplicative in the sense that*

$$|a_1 a_1|_{\mathbb{K}} = |a_1|_{\mathbb{K}} |a_2|_{\mathbb{K}}$$

for all $a_1, a_2 \in \mathbb{K}$, continuous, and proper. Moreover, $a_n \to 0$ as $n \to \infty$ if and only if $|a_n|_{\mathbb{K}} \to 0$ as $n \to \infty$.

Proof. For $a = 0$ we have $aM = \{0\}$ for any M as in the definition of $|a|_{\mathbb{K}}$, and the assumption that $\mathbb{K}$ is non-discrete implies that $|0|_{\mathbb{K}} = 0$. That $|a|_{\mathbb{K}}$ is well-defined (that is, independent of the choice of Haar measure $m_{\mathbb{K}}$ and the set M) follows as in our discussion of the modular character in Section 1.2.4. In fact, for any $a \in \mathbb{K} \smallsetminus \{0\}$ defining $\mu_a(B) = m_{\mathbb{K}}(aB)$ for Borel subsets $B \subseteq \mathbb{K}$ gives a Haar measure μ_a on $(\mathbb{K}, +)$ and hence, by uniqueness, μ_a must be a positive multiple of $m_{\mathbb{K}}$. In particular, $|a|_{\mathbb{K}} > 0$ for all $a \in \mathbb{K} \smallsetminus \{0\}$.

To see multiplicativity, suppose without loss of generality that a_1, a_2 are elements of $\mathbb{K} \smallsetminus \{0\}$. Then

$$|a_1a_2|_{\mathbb{K}} = \frac{m_{\mathbb{K}}(a_1a_2M)}{m_{\mathbb{K}}(M)} = \frac{m_{\mathbb{K}}(a_1a_2M)}{m_{\mathbb{K}}(a_2M)}\frac{m_{\mathbb{K}}(a_2M)}{m_{\mathbb{K}}(M)} = |a_1|_{\mathbb{K}}|a_2|_{\mathbb{K}}$$

for any $M \subseteq \mathbb{K}$ as in the definition of $|\cdot|_{\mathbb{K}}$. We will assume in the following that $M \subseteq \mathbb{K}$ is a compact neighbourhood of $0 \in \mathbb{K}$.

For the proof of continuity of $|\cdot|\colon \mathbb{K} \to [0,\infty)$ at 0, we will need the following topological claim for the local field $\mathbb{K}$. For any open neighbourhood U of $0 \in \mathbb{K}$ there exists some neighbourhood V of $0 \in \mathbb{K}$ such that $VM \subseteq U$. Indeed, for any $a \in M$ there exists (by continuity of multiplication) open neighbourhoods V_a of 0 and O_a of a such that $V_aO_a \subseteq U$. By compactness of M we can find a finite subcover

$$M \subseteq O_{a_1} \cup \cdots \cup O_{a_m}$$

and so it follows that $VM \subseteq U$ for

$$V = V_{a_1} \cap \cdots V_{a_m}.$$

To see continuity of $|\cdot|_{\mathbb{K}}$ at 0, we let $\varepsilon > 0$ and choose U to be an open neighbourhood of 0 with measure $m_{\mathbb{K}}(U) < m_{\mathbb{K}}(M)\varepsilon$. Let V be an open neighbourhood of 0 with $VM \subseteq U$ as above. For $a \in V$ we then obtain

$$|a|_{\mathbb{K}} = \frac{m_{\mathbb{K}}(aM)}{m_{\mathbb{K}}(M)} \leqslant \frac{m_{\mathbb{K}}(U)}{m_{\mathbb{K}}(M)} < \varepsilon.$$

As $\varepsilon > 0$ was arbitrary it follows that $|\cdot|_{\mathbb{K}}$ is continuous at $0 \in \mathbb{K}$.

To see continuity of $|\cdot|_{\mathbb{K}}$ at 1 we first prove an analogue of the above topological claim. By local compactness and local finiteness of $m_{\mathbb{K}}$ there exists an open set $O \supseteq M$ with finite Haar measure. We define

$$B_n = \{a \in O \mid \mathsf{d}(a, M) < \tfrac{1}{n}\}$$

for $a \in \mathbb{K}$. Since $O \supseteq B_1 \supseteq B_2 \supseteq \cdots$ is a decreasing sequence, O has finite Haar measure, and

$$M = \bigcap_{n=1}^{\infty} B_n,$$

we have $m_{\mathbb{K}}(M) = \lim_{n\to\infty} m_{\mathbb{K}}(B_n)$. Let $\varepsilon > 0$ and choose $n \geqslant 1$ such that

$$m_{\mathbb{K}}(B_n) < (1+\varepsilon)m_{\mathbb{K}}(M).$$

Just as in the proof of the above topological claim, we can use compactness of M, openness of B_n, and the inclusion $1 \cdot M \subseteq B_n$ to find an open neighbourhood V of $1 \in \mathbb{K}$ with $VM \subseteq B_n$. For $a \in V$ we then obtain

$$|a|_{\mathbb{K}} = \frac{m_{\mathbb{K}}(aM)}{m_{\mathbb{K}}(M)} \leqslant \frac{m_{\mathbb{K}}(B_n)}{m_{\mathbb{K}}(M)} < 1 + \varepsilon.$$

Therefore

$$(1+\varepsilon)^{-1} < |a|_{\mathbb{K}} < 1+\varepsilon$$

for all $a \in V$ with $a^{-1} \in V$, since $|a^{-1}|_{\mathbb{K}} = |a|_{\mathbb{K}}^{-1}$ by the multiplicative property. As $\varepsilon > 0$ was arbitrary and $\{a \in V \mid a^{-1} \in V\}$ is a neighbourhood of 1, we obtain the continuity of $|\cdot|_{\mathbb{K}}$ at $1 \in \mathbb{K}$. Continuity at $a_0 \in \mathbb{K}\smallsetminus\{0\}$ now follows from the identity

$$|a|_{\mathbb{K}} = |aa_0^{-1}a_0|_{\mathbb{K}} = |aa_0^{-1}|_{\mathbb{K}}|a_0|_{\mathbb{K}}$$

for $a \in \mathbb{K}$.

It remains to prove that $|\cdot|_{\mathbb{K}}$ is proper, and that a sequence (a_n) in $\mathbb{K}$ converges to 0 if $(|a_n|_{\mathbb{K}})$ converges to 0 as $n \to \infty$. Let $M \subseteq \mathbb{K}$ be a compact neighbourhood of $0 \in \mathbb{K}$ as above, and suppose in addition that $1 \in M$. Let $U_0 \subseteq M$ be an open neighbourhood of 0 with Haar measure satisfying $m_{\mathbb{K}}(U_0) \leqslant \frac{1}{2}m_{\mathbb{K}}(M)$. Let V_0 be an open neighbourhood of $0 \in \mathbb{K}$ satisfying $V_0 M \subseteq U_0$ as in the topological claim above. For the following we choose and fix some $t \in V_0\smallsetminus\{0\}$. Then $|t|_{\mathbb{K}} \leqslant \frac{1}{2}$ (by the argument proving continuity at 0 above) and so

$$|t^n|_{\mathbb{K}} = |t|_{\mathbb{K}}^n \longrightarrow 0 \tag{2.20}$$

as $n \to \infty$. Moreover, $t = t \cdot 1 \in tM \subseteq U_0 \subseteq M$, and by induction $t^n \in M$ for all $n \geqslant 1$. By compactness of M the sequence (t^n) has a convergent subsequence. However, because of the established positivity and continuity of the absolute value, we conclude from (2.20) that 0 is the only possible limit point of a convergent subsequence. By compactness, we obtain $t^n \to 0 \in \mathbb{K}$ as $n \to \infty$.

For $\ell \in \mathbb{N}$ we now define $P_\ell = t^{-\ell}(M\smallsetminus tM)$ and obtain

$$\mathbb{K} = M \cup \bigcup_{\ell=1}^{\infty} P_\ell.$$

Indeed, for any $a \in \mathbb{K}\smallsetminus M$ we have $t^n a \to 0$ as $n \to \infty$, so there exists a minimal $\ell \geqslant 1$ with $t^\ell a \in M$ (satisfying $t^\ell a \notin tM$), and hence $a \in P_\ell$.

We also define

$$c = \min\{|a|_{\mathbb{K}} \mid a \in \overline{M\smallsetminus tM}\},$$

and note that $c > 0$ since $\overline{M\smallsetminus tM} \subseteq M$ is compact and does not contain 0. It now follows that any $a = t^{-\ell}a_0 \in P_\ell$ with $a_0 \in M\smallsetminus tM$ and $\ell \in \mathbb{N}$ has absolute value

$$|a|_{\mathbb{K}} = |t^{-\ell}a_0|_{\mathbb{K}} = |t|_{\mathbb{K}}^{-\ell}|a_0|_{\mathbb{K}} \geqslant 2^\ell c.$$

As

$$M \cup \bigcup_{\ell=1}^{L} P_\ell = t^{-L}M$$

is compact for every $L \geqslant 1$ and $|a|_{\mathbb{K}} \geqslant 2^{L+1}c$ for all $a \in \mathbb{K} \smallsetminus (t^{-L}M)$, it follows that the absolute value is a proper function.

Now suppose that (a_n) is a sequence in $\mathbb{K}$ with $|a_n|_{\mathbb{K}} \to 0$ as $n \to \infty$. Then $a_n \in M$ for all sufficiently large n since by the above, $|a|_{\mathbb{K}} \geqslant 2c$ if $a \notin M$, and we may again use compactness of M, positivity, and continuity of the absolute value to conclude that $a_n \to 0$ as $n \to \infty$. □

Proof of Proposition 2.39. Let $\chi \in \widehat{\mathbb{K}}$ be any non-trivial character. We define $\chi_t \colon \mathbb{K} \ni a \mapsto \chi_t(a) = \chi(at)$ for any $t \in \mathbb{K}$ as in the proposition. Since $\mathbb{K} \ni a \mapsto at \in \mathbb{K}$ is a continuous homomorphism of the additive group $\mathbb{K}$ it follows that $\chi_t \in \widehat{\mathbb{K}}$ for any $t \in \mathbb{K}$. For $t_1, t_2, a \in \mathbb{K}$ we also have

$$\chi_{t_1+t_2}(a) = \chi\bigl((t_1+t_2)a\bigr) = \chi(t_1a)\chi(t_2a) = \chi_{t_1}(a)\chi_{t_2}(a),$$

and so $\Phi \colon \mathbb{K} \ni t \mapsto \chi_t \in \widehat{\mathbb{K}}$ is a homomorphism. By assumption, χ is a non-trivial character. So let $a_0 \in \mathbb{K}$ be chosen with $\chi(a_0) \neq 1$. If now $t \in \mathbb{K} \smallsetminus \{0\}$ then

$$\chi_t(\tfrac{1}{t}a_0) = \chi(t\tfrac{1}{t}a_0) = \chi(a_0) \neq 1,$$

which shows that χ_t is non-trivial and hence Φ is injective.

To see continuity of Φ, let $M \subseteq \mathbb{K}$ be compact and fix $\varepsilon > 0$. By continuity of χ,

$$U = \{a \in \mathbb{K} \mid |\chi(a) - 1| < \varepsilon\}$$

is an open neighbourhood of $0 \in \mathbb{K}$. By compactness of M there exists an open neighbourhood V of 0 such that $VM \subseteq U$ (just as in the topological claim in the proof of Lemma 2.40). Therefore $t \in V$ gives $|\chi_t(a) - 1| = |\chi(at) - 1| < \varepsilon$ for all $a \in M$. Equivalently, $t \in V$ implies that

$$\chi_t \in N_{\widehat{\mathbb{K}}}(M, \varepsilon),$$

which gives continuity of Φ.

Next we claim that Φ is proper. So let (t_n) be a sequence in $\mathbb{K}$ with the property that $t_n \to \infty$ as $n \to \infty$. By Lemma 2.40, we have

$$a_n = t_n^{-1}a_0 \longrightarrow 0$$

as $n \to \infty$. This shows that (χ_{t_n}) has no convergent subsequence with respect to the compact-open topology. In fact, uniform convergence of $(\chi_{t_{n_k}})$ on the compact set $\{0\} \cup \{a_n \mid n \in \mathbb{N}\}$ to a character χ' would imply that $\lim_{k\to\infty} \chi_{t_{n_k}}(a_{n_k}) = \chi'(0) = 1$, but we have $\chi_{t_n}(a_n) = \chi(a_0)$ for all $n \geqslant 1$, and $\chi(a_0) \neq 1$. This shows that $\chi_{t_n} \to \infty$ as $n \to \infty$, and hence that Φ is proper.

It follows that $\Phi(\mathbb{K}) \subseteq \widehat{\mathbb{K}}$ is a closed subgroup of $\widehat{\mathbb{K}}$. To identify the subgroup $\Phi(\mathbb{K})$ we suppose that $a \in \Phi(\mathbb{K})^{\perp} \subseteq \mathbb{K}$, so that $\langle a, \Phi(t)\rangle = \chi_t(a) = 1$ for all $t \in \mathbb{K}$. If $a \neq 0$ we can set $t = a^{-1}a_0$ and obtain the contradiction $\chi_t(a) = \chi(aa^{-1}a_0) = \chi(a_0) \neq 1$. Hence $\Phi(\mathbb{K})^{\perp} = \{0\}$ and Proposition 2.30 implies that $\Phi(\mathbb{K}) = \widehat{\mathbb{K}}$. By the last claim in Proposition B.3 we now obtain that $\Phi\colon \mathbb{K} \to \widehat{\mathbb{K}}$ is an isomorphism of topological groups, which concludes the proof. □

We note that the above applies to the local fields $\mathbb{R}$ and $\mathbb{C}$. In particular, Proposition 2.39 and Proposition 2.26 together give a complete (but much longer) proof of Exercise 2.6. Let us briefly describe a second class of local fields that are especially important in number theory.

Fix a prime $p \in \mathbb{N}$. The local field $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the so-called *p-adic norm* defined by

$$|a|_p = \begin{cases} 0 & \text{if } a = 0; \\ p^{-k} & \text{if } a = p^k \frac{m}{n} \in \mathbb{Q}^\times \text{ for } k \in \mathbb{Z}, m, n \in \mathbb{Z} \smallsetminus p\mathbb{Z}. \end{cases}$$

The definition of $|\cdot|_p$ implies that $|ab|_p = |a|_p |b|_p$ for all $a, b \in \mathbb{Q}$. Similarly, if $a = p^k \frac{m}{n}$ and $b = p^\ell \frac{r}{s}$ with $k, \ell \in \mathbb{Z}$ and $m, n, r, s \in \mathbb{Z} \smallsetminus p\mathbb{Z}$, then

$$\begin{aligned} |a+b|_p &= \left| p^{\min(k,\ell)} \big(p^{k-\min(k,\ell)} \tfrac{m}{n} + p^{\ell - \min(k,\ell)} \tfrac{r}{s} \big) \right|_p \\ &\leqslant \max(p^{-k}, p^{-\ell}) = \max\big(|a|_p, |b|_p\big) \leqslant |a|_p + |b|_p. \end{aligned}$$

As in the study of norms on vector spaces, it now follows that $|a-b|_p$ defines a metric on $\mathbb{Q}$ which extends to the completion $\mathbb{Q}_p$ of $\mathbb{Q}$. Moreover, the p-adic norm extends from $\mathbb{Q}$ to a continuous function on $\mathbb{Q}_p$ defined by setting

$$|a|_p = \mathsf{d}_p(a, 0)$$

for all $a \in \mathbb{Q}_p$. From these properties it follows that multiplication and addition extend continuously from $\mathbb{Q}$ to $\mathbb{Q}_p$. In fact, by the above discussion we have

$$|a+b|_p \leqslant \max\big(|a|_p, |b|_p\big) \leqslant |a|_p + |b|_p \tag{2.21}$$

and

$$|ab|_p = |a|_p |b|_p \tag{2.22}$$

for all $a, b \in \mathbb{Q}$. This implies that sums and products of Cauchy sequences in $\mathbb{Q}$ are again Cauchy sequences in $\mathbb{Q}$. Hence we obtain the definition of addition and multiplication on $\mathbb{Q}_p$ so that (2.21) and (2.22) also hold for $a, b \in \mathbb{Q}_p$.

To see that $\mathbb{Q}_p$ is a topological field, we claim that any Cauchy sequence (a_n) in $\mathbb{Q} \smallsetminus \{0\}$ with limit $a \in \mathbb{Q}_p \smallsetminus \{0\}$ satisfies that (a_n^{-1}) is again a Cauchy sequence. With $\delta = \frac{1}{2}|a|_p$ it follows that $|a_n|_p \geqslant \delta$ for all $n \geqslant N$ and some $N \geqslant 1$. Therefore

$$|a_m^{-1} - a_n^{-1}|_p \leqslant |a_m^{-1} a_n^{-1} (a_n - a_m)|_p \leqslant \delta^{-2} |a_m - a_n|_p$$

for all $m, n \geqslant N$, which implies the claim since (a_n) is a Cauchy sequence. Thus $a \mapsto a^{-1}$ is continuous on $\mathbb{Q}_p \smallsetminus \{0\}$. Now the limit $b = \lim_{n\to\infty} a_n^{-1} \in \mathbb{Q}_p$

of course satisfies

$$ab = \lim_{n\to\infty} a_n a_n^{-1} = 1.$$

The subgroup $\mathbb{Z}_p$ of $\mathbb{Q}_p$ is defined as the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$, and so $|a|_p \leqslant 1$ for all $a \in \mathbb{Z}_p$. We recall that if $n \in \mathbb{Z} \smallsetminus p\mathbb{Z}$, then there exists some ℓ such that $n\ell \equiv 1$ modulo p. In other words, there exists some $a \in \mathbb{Z}$ with the property that $n\ell = 1 - ap$, which implies that

$$(n\ell)^{-1} = \frac{1}{1-ap} = \sum_{j=0}^{\infty} (ap)^j$$

is a series in $\mathbb{Z}$ that converges in $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ since

$$\Big|\frac{1}{1-ap} - \sum_{j=0}^{J}(ap)^j\Big|_p = \left|\frac{1-(1-ap)\sum_{j=0}^{J}(ap)^j}{1-ap}\right|_p = \left|\frac{(ap)^{J+1}}{1-ap}\right|_p \leqslant p^{-(J+1)}$$

for $J \geqslant 1$. This implies that for $k \geqslant 0$ and $m, n \in \mathbb{Z} \smallsetminus p\mathbb{Z}$ we have

$$p^k \frac{m}{n} = p^k m\ell(n\ell)^{-1} \in \mathbb{Z}_p.$$

Since $|a|_p \in \{0\} \cup p^{\mathbb{Z}}$ for all $a \in \mathbb{Q}$, we obtain that

$$\mathbb{Z}_p = \overline{B}_1(0) = \{a \in \mathbb{Q}_p \mid |a|_p \leqslant 1\} = \{a \in \mathbb{Q}_p \mid |a|_p < p\} = B_p(0)$$

is the closed unit ball in $\mathbb{Q}_p$ and is simultaneously an open ball. Using (2.22) and multiplication by powers of p we deduce that open and closed balls with centre 0 have the form

$$\begin{aligned} p^k\mathbb{Z}_p = \overline{B}_{p^{-k}}(0) &= \left\{a \in \mathbb{Q}_p \mid |a|_p \leqslant p^{-k}\right\} \\ &= B_{p^{1-k}}(0) = \left\{a \in \mathbb{Q}_p \mid |a|_p < p^{1-k}\right\} \end{aligned}$$

for some $k \in \mathbb{Z}$. By (2.21), these sets form additive subgroups of $\mathbb{Q}_p$. Moreover, (2.22) shows that $\mathbb{Z}_p$ is a subring and that the sets $p^k\mathbb{Z}_p \subseteq \mathbb{Z}_p$ are ideals for any $k \geqslant 0$.

For $k \geqslant 1$ division by p^k with remainder in $\mathbb{Z}$ implies that

$$\mathbb{Z} = \bigsqcup_{m=0}^{p^k-1} (m + p^k\mathbb{Z}),$$

where the sets $m + p^k\mathbb{Z}$ for $m \in \{0, \dots, p^k - 1\}$ on the right-hand side are the closed balls of radius p^{-k} around m with respect to $|\cdot|_p$, intersected with $\mathbb{Z}$. Taking the closure in $\mathbb{Q}_p$, we obtain

$$\mathbb{Z}_p = \bigsqcup_{m=0}^{p^k-1} (m + p^k\mathbb{Z}_p). \tag{2.23}$$

As these balls have distance at least p^{-k+1}, the union is disjoint. This shows that $\mathbb{Z}_p$ is totally bounded as a metric space. In particular, as $\mathbb{Z}_p$ is complete by definition, we see that $\mathbb{Z}_p$ is compact and so $\mathbb{Q}_p$ is locally compact and σ-compact.

Exercise 2.41. Show that any element $a \in \mathbb{Z}_p$ may be written as

$$a = \sum_{k=0}^{\infty} a_k p^k$$

with $a_k \in \{0, 1, \dots, p-1\}$ for $k \geqslant 0$, and that any $a \in \mathbb{Q}_p$ may be written as

$$a = \sum_{k=\ell}^{\infty} a_k p^k$$

with $a_k \in \{0, 1, \dots, p-1\}$ for all $k \geqslant \ell$ and some $\ell \in \mathbb{Z}$.

Exercise 2.42. Use (2.23) to show that $\mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/p^k\mathbb{Z} = \mathrm{C}_{p^k}$ for all $k \geqslant 1$ and deduce that we may identify $\mathbb{Z}_p$ with the projective limit $\varprojlim(\mathrm{C}_{p^k}, \theta_k)$ where

$$\begin{aligned} \theta_k \colon \mathrm{C}_{p^{k+1}} &\longrightarrow \mathrm{C}_{p^k} \\ m + p^{k+1}\mathbb{Z} &\longmapsto m + p^k\mathbb{Z} \end{aligned}$$

is the canonical projection map for all $k \geqslant 1$.

Exercise 2.43. Show that $\mathbb{Q}_p$ can be obtained as the inductive limit $\mathbb{Q}_p = \varinjlim(p^{-\ell}\mathbb{Z}_p, \imath_\ell)$, where

$$\begin{aligned} \imath_\ell \colon p^{-\ell}\mathbb{Z}_p &\longrightarrow p^{-\ell-1}\mathbb{Z}_p \\ a &\longmapsto a \end{aligned}$$

is the inclusion map for all $\ell \geqslant 0$.

Exercise 2.44. Find an explicit formula for a non-trivial character on $\mathbb{Q}_p$, and use this to exhibit an explicit isomorphism between $\mathbb{Q}_p$ and $\widehat{\mathbb{Q}_p}$.

Exercise 2.45. (a) Show that the absolute value $|\cdot|_{\mathbb{Q}_p}$ in Lemma 2.40 agrees with the p-adic norm $|\cdot|_p$ on $\mathbb{Q}_p$.
(b) Describe the Haar measure on $\mathbb{Q}_p$.

Exercise 2.46. (a) Show that for a finite set S of primes in $\mathbb{N}$ the ring $\mathbb{Z}[\frac{1}{p} \mid p \in S]$ gives rise by diagonal embedding to a discrete subgroup of $\mathbb{R} \times \prod_{p\in S} \mathbb{Q}_p$.
(b) Show that

$$\widehat{\mathbb{Z}[\tfrac{1}{p} \mid p \in S]} \cong \big(\mathbb{R} \times \textstyle\prod_{p\in S} \mathbb{Q}_p\big)/\mathbb{Z}[\tfrac{1}{p} \mid p \in S].$$

Exercise 2.47. Show that for any non-empty set S of primes there is an injective homomorphism $\mathbb{R} \hookrightarrow \widehat{\mathbb{Z}[\frac{1}{p} \mid p \in S]}$ with dense image.†

† These compact groups are known as *solenoids* as they have a copy of the line 'wrapped' inside them; this terminology seems to have been used first by van Dantzig [17, p. 75].

The case of the characteristic p local field defined by the field of formal Laurent series

$$\mathbb{F}_p(\!(X)\!) = \Big\{ \sum_{k=\ell}^{\infty} a_k X^k \mid \ell \in \mathbb{Z}, a_k \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \text{ for all } k \geqslant \ell \Big\}$$

is quite similar to the above. The norm $|\cdot|\colon \mathbb{F}_p(\!(X)\!) \to \mathbb{R}_{\geqslant 0}$ is defined by

$$\left| \sum_{k=\ell}^{\infty} a_k X^k \right| = \begin{cases} 0 & \text{if } a_k = 0 \text{ for all } k \in \mathbb{Z}, \\ p^{-\ell} & \text{if } \ell \in \mathbb{Z} \text{ and } a_\ell \neq 0. \end{cases}$$

The ring

$$\mathbb{F}_p[\![X]\!] = \Big\{ \sum_{k=0}^{\infty} a_k X^k \mid \ell \in \mathbb{Z}, a_k \in \mathbb{F}_p \text{ for all } k \geqslant 0 \Big\}$$

is an open and compact neighbourhood of 0, and all other open and closed balls about 0 can be obtained by multiplication by X^ℓ for $\ell \in \mathbb{Z}$. We leave the details as an exercise.

Exercise 2.48. Show that $\mathbb{F}_p(\!(X)\!)$ is a locally compact σ-compact metric field if we declare $\mathbb{F}_p(\!(X)\!)$ to be the inductive limit over ℓ of $X^{-\ell}\mathbb{F}_p[\![X]\!]$ and use the inductive topology on the latter group.

We conclude this topic by noting that any local field $\mathbb{K}$ is isomorphic to exactly one of the following three types of field:

- (Archimedean) $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$;
- (Non-Archimedean of zero characteristic) $\mathbb{K} = \mathbb{Q}_p$ or $\mathbb{K}$ is a finite field extension of $\mathbb{Q}_p$ for some prime $p \in \mathbb{N}$;
- (Non-Archimedean of positive characteristic) $\mathbb{K} = \mathbb{F}_p(\!(X)\!)$ or $\mathbb{K}$ is a finite field extension of $\mathbb{F}_p(\!(X)\!)$, for some prime $p \in \mathbb{N}$.

We refer to Weil [67, Sec. I.3] for this classification of local fields.

2.5 Spectral Measures

We already encountered spectral measures implicitly in Bochner's theorem (Theorem 2.8), and explicitly in the description of cyclic representations (Corollary 2.11). In particular, Theorem 2.8 shows the existence and uniqueness of diagonal spectral measures as defined below. Here we study them in greater detail and also consider more general non-diagonal matrix coefficients, whose existence and uniqueness will be shown in Proposition 2.51.

Definition 2.49 (Spectral measures). Let π be a unitary representation of the abelian group G. For any $v \in \mathcal{H}_\pi$ the *(principal* or *diagonal) spectral* *measure* μ_v is the finite measure on $\widehat{G}$ with the property that

$$\left\langle \pi_g v, v \right\rangle = \int_{\widehat{G}} \langle g, t \rangle \, \mathrm{d}\mu_v(t)$$

for all $g \in G$. For $v, w \in \mathcal{H}_\pi$ the *(non-diagonal) spectral measure* $\mu_{v,w}$ is a finite complex-valued measure on $\widehat{G}$ such that

$$\left\langle \pi_g v, w \right\rangle = \int \langle g, t \rangle \, \mathrm{d}\mu_{v,w}(t)$$

for all $g \in G$.

In most of our discussion we will consider only one unitary representation π and will write μ_v and $\mu_{v,w}$ for the spectral measures of $v, w \in \mathcal{H}_\pi$ as in the above definition. If we want to emphasise the unitary representation used in the definition of the spectral measure, we will write μ_v^π or $\mu_{v,w}^\pi$. The main properties of spectral measures are summarized in the next result.

Proposition 2.50 (Geometry of diagonal spectral measures). *Let π be a unitary representation of the abelian group G. The (diagonal) spectral measures μ_v for $v \in \mathcal{H}_\pi$ satisfy the following for all $v, w, v_1, v_2, \dots \in \mathcal{H}_\pi$.*

(1) *We have $\|\mu_v\| = \mu_v(\widehat{G}) = \|v\|^2$.*
(2) *If $w \in \langle v \rangle_\pi$, then $\mu_w \ll \mu_v$. Indeed, if $w \in \langle v \rangle_\pi$ corresponds under a unitary isomorphism as in the cyclic spectral theorem (Corollary 2.11) to $F \in L^2_{\mu_v}(\widehat{G})$, then $\mathrm{d}\mu_w = |F|^2 \, \mathrm{d}\mu_v$.*
(3) *If $\langle v_1 \rangle_\pi \perp \langle v_2 \rangle_\pi$, then $\mu_{v_1+v_2} = \mu_{v_1} + \mu_{v_2}$.*
(4) *If $v = \sum_{k=1}^\infty v_k$ is convergent with $\langle v_k \rangle_\pi \perp \langle v_\ell \rangle_\pi$ for all $k \neq \ell \in \mathbb{N}$, then $\mu_v = \sum_{k=1}^\infty \mu_{v_k}$.*
(5) *If $\mu_{v_1} \perp \mu_{v_2}$, then $\langle v_1 \rangle_\pi \perp \langle v_2 \rangle_\pi$.*
(6) *If $w \in \langle v \rangle_\pi$ and $\mu_v \ll \mu_w$, then $\langle w \rangle_\pi = \langle v \rangle_\pi$.*
(7) *If $\mu_{v_1} \perp \mu_{v_2}$, then we even have $\langle v_1 + v_2 \rangle_\pi = \langle v_1 \rangle_\pi \oplus \langle v_2 \rangle_\pi$.*

We note that we will be able to give a better motivated proof of (6) and (7) using the measurable functional calculus in the next section.

Proof of Proposition 2.50. The equality in (1) already appeared in Corollary 2.11. Applying Corollary 2.11, we have that π restricted to $\langle v \rangle_\pi$ is unitarily isomorphic to $L^2_{\mu_v}(\widehat{G})$, where v corresponds to $\mathbb{1}$. Let $F \in L^2_{\mu_v}(\widehat{G})$ correspond to w. Then

$$\langle \pi_g w, w \rangle = \langle M_g F, F \rangle_{L^2(\widehat{G}, \mu_v)} = \int \langle g, t \rangle |F(t)|^2 \, \mathrm{d}\mu_v(t)$$

for all $g \in G$, showing that $\mathrm{d}\mu_w = |F|^2 \, \mathrm{d}\mu_v$, and hence (2).

Suppose that $\langle v_1 \rangle_\pi \perp \langle v_2 \rangle_\pi$ as in (3). Then

$$\begin{aligned}\int \langle g,t\rangle \,\mathrm{d}\mu_{v_1+v_2}(t) &= \langle \pi_g(v_1+v_2), v_1+v_2\rangle = \langle \pi_g v_1, v_1\rangle + \langle \pi_g v_2, v_2\rangle \\ &= \int \langle g,t\rangle \,\mathrm{d}\mu_{v_1}(t) + \int \langle g,t\rangle \,\mathrm{d}\mu_{v_2}(t) \\ &= \int \langle g,t\rangle \,\mathrm{d}(\mu_{v_1}+\mu_{v_2})(t)\end{aligned}$$

for all $g \in G$, which proves (3) by uniqueness of spectral measures in Bochner's theorem (Theorem 2.8).

Suppose now that $v = \sum_{k=1}^{\infty} v_k$ as in (4) and define $u_N = \sum_{k=1}^{N} v_k$, respectively $w_N = \sum_{k=N+1}^{\infty} v_k$, for some $N \geqslant 1$, so that $v = u_N + w_N$. By (3), we have $\mu_v = \mu_{u_N} + \mu_{w_N}$, and by induction also $\mu_{u_N} = \sum_{k=1}^{N} \mu_{v_k}$. By (1) we have $\|\mu_{w_N}\| = \|w_N\|^2 \to 0$ as $N \to \infty$, which implies that

$$\mu_v = \lim_{N\to\infty} \left(\mu_{u_N} + \mu_{w_N}\right) = \sum_{k=1}^{\infty} \mu_{v_k}.$$

Now suppose that $\mu_{v_1} \perp \mu_{v_2}$ as in (5). We write v_2 as a sum $v_2 = w + \widetilde{w}$ with $w \in \langle v_1\rangle_\pi$ and $\widetilde{w} \in \langle v_1\rangle_\pi^\perp$. Since $\langle w\rangle_\pi \subseteq \langle v_1\rangle_\pi$ and $\langle \widetilde{w}\rangle_\pi \perp \langle v_1\rangle_\pi$, we obtain $\mu_{v_2} = \mu_w + \mu_{\widetilde{w}}$ from (3). By (2) we have $\mu_w \ll \mu_{v_1}$, which together with the assumption $\mu_{v_1} \perp \mu_{v_2}$ implies $\mu_w = 0$ and so also

$$\|w\|^2 = \mu_w(\widehat{G}) = 0$$

by (1). Therefore $v_2 = \widetilde{w} \in \langle v_1\rangle_\pi^\perp$, and (5) follows.

For the proof of (6), we apply the cyclic spectral theorem (Corollary 2.11) to the subspace $\langle v\rangle_\pi$ and assume without loss of generality that $\pi = M$ and

$$v = \mathbb{1} \in L^2(\widehat{G}, \mu)$$

for a finite measure $\mu = \mu_v$. We also recall from Corollary 2.12 that we have $M_*(f) = M_{\check{f}}$ for all $f \in L^1(G)$. It follows that $L^1(G)w \subseteq \langle w\rangle_M$ and, by density of $L^1(G) \subseteq C_0(\widehat{G})$, also $C_0(\widehat{G})w \subseteq \langle w\rangle_M$ for all $w \in L^2_\mu(\widehat{G})$. Suppose now that $w \in L^2_\mu(\widehat{G})$ satisfies $\mu \ll \mu_w$. By (2) we have $\mathrm{d}\mu_w = |w|^2 \,\mathrm{d}\mu$, which implies that $w(t) \neq 0$ for μ-almost every $t \in \widehat{G}$. Finally, recall that $C_0(\widehat{G})$ is dense inside $L^2_{\mu_w}(\widehat{G})$. Now pick some $F \in L^2_\mu(\widehat{G})$ and note that

$$\widetilde{F} = w^{-1} F \in L^2_{\mu_w}(\widehat{G}).$$

Applying the density claim to $\widetilde{F}$, we find a sequence (F_n) in $C_0(\widehat{G})$ that converges to $\widetilde{F}$ in $L^2_{\mu_w}(\widehat{G})$. Together, it follows that $F_n w \in \langle w\rangle_M$ and

$$\begin{aligned}\|F_n w - F\|^2_{L^2_\mu(\widehat{G})} &= \int |F_n w - \underbrace{\widetilde{F} w}_{=F}|^2 \,\mathrm{d}\mu \\ &= \int |F_n - \widetilde{F}| \,\mathrm{d}\mu_w = \|F_n - \widetilde{F}\|^2_{L^2_{\mu_w}(\widetilde{G})} \longrightarrow 0\end{aligned}$$

as $n \to \infty$. Hence $F \in \langle w \rangle_M$ also, and we obtain $\langle w \rangle_M = L^2_\mu(\widehat{G})$ as claimed in (6).

It remains to prove (7). By (5) we have $\langle v_1 \rangle_\pi \perp \langle v_2 \rangle_\pi$. We let $w = v_1 + v_2$ and conclude from (3) that $\mu_w = \mu_{v_1} + \mu_{v_2}$. Since $\mu_{v_1} \perp \mu_{v_2}$ we can decompose $\widehat{G}$ as $\widehat{G} = B_1 \sqcup B_2$ for some measurable sets $B_1, B_2 \subseteq \widehat{G}$ with $\mu_{v_j}(\widehat{G} \smallsetminus B_j) = 0$ for $j = 1, 2$. Using the cyclic spectral theorem (Corollary 2.11), the function $\mathbb{1}_{B_1} \in L^2(\widehat{G}, \mu_w)$ corresponds to some $u_1 \in \langle v_1 + v_2 \rangle_\pi$, which by (2) has spectral measure $\mathrm{d}\mu_{u_1} = \mathbb{1}_{B_1} \mathrm{d}\mu_w = \mathrm{d}\mu_{v_1}$. In particular, $\mu_{u_1} \perp \mu_{v_2}$, which by (5) implies $\langle u_1 \rangle_\pi \perp \langle v_2 \rangle_\pi$ and so $u_1 \in \langle v_1 \rangle_\pi$. Since $\mu_{u_1} = \mu_{v_1}$, property (6) implies $\langle v_1 \rangle_\pi = \langle u_1 \rangle_\pi \subseteq \langle v_1 + v_2 \rangle_\pi$. This also implies $\langle v_2 \rangle_\pi \subseteq \langle v_1 + v_2 \rangle_\pi$, which concludes the proof. □

We now discuss the main properties of the non-diagonal spectral measures. These will be crucial in the next section.

Proposition 2.51 (Non-diagonal spectral measures). *For any unitary representation π of the abelian group G there exists a map*

$$\begin{aligned}\mathcal{H}_\pi \times \mathcal{H}_\pi &\longrightarrow \mathcal{M}^{\mathbb{C}}(\widehat{G}) \\ (v, w) &\longmapsto \mu_{v,w}\end{aligned}$$

sending a pair (v, w) to the spectral measure $\mu_{v,w}$ satisfying

$$\left\langle \pi_g v, w \right\rangle = \int \langle g, t \rangle \,\mathrm{d}\mu_{v,w}(t) \tag{2.24}$$

for all $g \in G$ (as in Definition 2.49) and the following additional properties.

(1) *$\mu_{v,w}$ depends linearly on $v \in \mathcal{H}_\pi$ and semi-linearly on $w \in \mathcal{H}_\pi$.*
(2) *$\mu_{w,v} = \overline{\mu_{v,w}}$ for all $v, w \in \mathcal{H}_\pi$.*
(3) *$\|\mu_{v,w}\| \leqslant \|v\| \|w\|$ for all $v, w \in \mathcal{H}_\pi$.*
(4) *$\langle \pi_*(f) v, w \rangle = \int \check{f} \,\mathrm{d}\mu_{v,w}$ for all $f \in L^1(G)$.*
(5) *If the unitary isomorphism in Corollary 2.11 sends $v, w \in \langle u \rangle_\pi$ to the functions $F_v, F_w \in L^2_{\mu_u}(\widehat{G})$ respectively, then $\mathrm{d}\mu_{v,w} = F_v \overline{F_w} \,\mathrm{d}\mu_u$.*
(6) *If $v = \sum_{k=1}^\infty v_k$, $w = \sum_{k=1}^\infty w_k$, and $v_k, w_k \in \langle u_k \rangle_\pi$, $\langle u_k \rangle_\pi \perp \langle u_\ell \rangle_\pi$ for all $k \neq \ell \in \mathbb{N}$ and some sequence (u_k) in $\mathcal{H}_\pi$, then we have*

$$\mu_{v,w} = \sum_{k=1}^\infty \mu_{v_k, w_k}.$$

Moreover, for a pair $(v, w) \in \mathcal{H}_\pi \times \mathcal{H}_\pi$ *the property* (2.24) *for all* $g \in G$ *(alternatively, property* (4) *for all* $f \in L^1(G)$*) uniquely determines the measure* $\mu_{v,w}$.

Let us highlight a particular case in the following exercise.

Essential Exercise 2.52. Let π be a unitary representation of the abelian group G. Assume the existence and properties of the non-diagonal spectral measures as in Proposition 2.51. Show for $v, w \in \mathcal{H}_\pi$ that $\mu_{v,w} = 0$ if and only if $\langle v \rangle_\pi \perp \langle w \rangle_\pi$.

Proof of Proposition 2.51. Let $v, w \in \mathcal{H}$ and assume that a finite complex-valued measure $\mu_{v,w}$ satisfies (2.24) for all $g \in G$. For any $f \in L^1(G)$ we then have

$$\begin{aligned}\langle \pi_*(f)v, w\rangle &= \int_G f(g)\langle \pi_g v, w\rangle \,\mathrm{d}m(g)\\ &= \int_G f(g)\int_{\widehat{G}} \langle g, t\rangle \,\mathrm{d}\mu_{v,w}(t)\,\mathrm{d}m(g) = \int_{\widehat{G}} \check{f}(t)\,\mathrm{d}\mu_{v,w}(t)\end{aligned}$$

by definition of the convolution operator, property (2.24), Fubini's theorem, and the definition of the Fourier transform. This shows (4) and that $\int_{\widehat{G}} \check{f}\,\mathrm{d}\mu_{v,w}$ is uniquely determined by v, w for all $f \in L^1(G)$. Since $\widecheck{L^1(G)}$ is dense in $C_0(\widehat{G})$ by Corollary 2.5 the identification of finite signed measures on $\widehat{G}$ with linear functionals on $C_0(\widehat{G})$ in the Riesz representation theorem (see [21, Th. 7.54]) implies the uniqueness claim.

Let us next turn to the existence part of the argument. For this we first prove (5). So suppose that $v, w \in \langle u \rangle_\pi$ are sent to $F_v, F_w \in L^2_{\mu_u}(\widehat{G})$ under the unitary isomorphism in Corollary 2.11. Then

$$\langle \pi_g v, w\rangle = \langle M_g F_v, F_w\rangle = \int \langle g, t\rangle\, F_v(t)\overline{F_w(t)}\,\mathrm{d}\mu_u(t)$$

for all $g \in G$, which already implies (5). To obtain the existence we set $u = v$ and decompose $w = w_1 + w_2$ with $w_1 \in \langle v \rangle_\pi$ and $w_2 \in \langle v \rangle_\pi^\perp$, which implies

$$\langle \pi_g v, w\rangle = \langle \pi_g v, w_1 + w_2\rangle = \langle \pi_g v, w_1\rangle$$

for all $g \in G$, and so $\mu_{v,w} = \mu_{v,w_1}$ exists by the above discussion, proving (5). This also implies (3) since

$$\|\mu_{v,w}\| = \int |\mathbb{1}\overline{F_{w_1}}|\,\mathrm{d}\mu_v \leqslant \|\mathbb{1}\|_2 \|F_{w_1}\|_2 = \|v\|\|w_1\| \leqslant \|v\|\|w\|$$

by the Cauchy–Schwarz inequality.

To prove linearity in $v \in \mathcal{H}_\pi$ for a fixed $w \in \mathcal{H}_\pi$ notice that

$$\begin{aligned}\langle\pi_g(\alpha_1 v_1+\alpha_2 v_2),w\rangle &= \alpha_1\langle\pi_g v_1,w\rangle+\alpha_2\langle\pi_g v_2,w\rangle\\ &= \alpha_1\int\langle g,t\rangle\,\mathrm{d}\mu_{v_1,w}(t)+\alpha_2\int\langle g,t\rangle\,\mathrm{d}\mu_{v_2,w}(t)\\ &= \int\langle g,t\rangle\,\mathrm{d}(\alpha_1\mu_{v_1,w}+\alpha_2\mu_{v_2,w})(t)\end{aligned}$$

for any $g\in G$, $v_1,v_2\in\mathcal{H}_\pi$ and $\alpha_1,\alpha_2\in\mathbb{C}$. By uniqueness of the spectral measures, linearity in the first argument follows. Semi-linearity with respect to the second argument follows in the same way, which gives (1).

To prove (2) we fix $v,w\in\mathcal{H}_\pi$. Then

$$\begin{aligned}\int\langle g,t\rangle\,\mathrm{d}\mu_{w,v}(t) &= \langle\pi_g w,v\rangle=\overline{\langle v,\pi_g w\rangle}=\overline{\langle\pi_{-g}v,w\rangle}\\ &= \overline{\int\langle -g,t\rangle\,\mathrm{d}\mu_{v,w}(t)}=\int\langle g,t\rangle\,\mathrm{d}\overline{\mu_{v,w}}(t)\end{aligned}$$

for all $g\in G$, which implies (2) by the uniqueness claim.

It remains to prove (6). Suppose first that $v=v_1+v_2$, $w=w_1+w_2$ with $v_k,w_k\in\langle u_k\rangle$ for $k=1,2$ and some $u_1,u_2\in\mathcal{H}_\pi$ with $\langle u_1\rangle_\pi\perp\langle u_2\rangle_\pi$. Then we have

$$\begin{aligned}\langle\pi_g v,w\rangle &= \langle\pi_g v_1,w_1\rangle+\langle\pi_g v_2,w_2\rangle\\ &= \int\langle g,t\rangle\,\mathrm{d}\mu_{v_1,w_1}(t)+\int\langle g,t\rangle\,\mathrm{d}\mu_{v_2,w_2}(t)\\ &= \int\langle g,t\rangle\,\mathrm{d}(\mu_{v_1,w_1}+\mu_{v_2,w_2})(t)\end{aligned}$$

for all $g\in G$, which implies $\mu_{v,w}=\mu_{v_1,w_1}+\mu_{v_2,w_2}$. This extends by induction to finite sums, and by (3) to the general case (see also the proof of Proposition 2.50(4)). □

A second way to obtain the existence of the non-diagonal spectral measures is to use the *polarization identity* as outlined in the next exercise.

Exercise 2.53. Let π be a unitary representation of the abelian group G. For $v\in\mathcal{H}_\pi$ let μ_v be the diagonal spectral measure of v. Show that

$$\mu_{v,w}=\tfrac{1}{4}\left(\mu_{v+w}-\mu_{v-w}+\mathrm{i}\mu_{v+\mathrm{i}w}-\mathrm{i}\mu_{v-\mathrm{i}w}\right)$$

defines the non-diagonal spectral measure for every $v,w\in\mathcal{H}_\pi$.

The ease with which we were able to encode various properties of unitary representations into properties of diagonal and non-diagonal spectral measures indicates how natural spectral measures are. In the following sections, we will also see how powerful the use of spectral measures can be.

2.5.1 Containment

In this section we use spectral measures to give a complete characterisation of containment for cyclic unitary representations (thus for now avoiding questions concerning multiplicity, which we postpone to Section 2.7). The following generalizes Exercise 2.13.

Proposition 2.54 (Containment). *Let π be a cyclic representation with generator $v \in \mathcal{H}_\pi$ and let ρ be a cyclic representation with generator $w \in \mathcal{H}_\rho$ of the abelian group G. Then $\pi < \rho$ if and only if $\mu_v^\pi \ll \mu_w^\rho$.*

PROOF. We assume first that $\pi < \rho$. Simplifying the notation we may suppose that $\mathcal{H}_\pi = \langle v \rangle_\rho \subseteq \mathcal{H}_\rho = \langle w \rangle_\rho$ with $\pi = \rho|_{\mathcal{H}_\pi}$. By Proposition 2.50(2) this implies $\mu_v \ll \mu_w$ as claimed.

So assume now that $\mu_v^\pi \ll \mu_w^\rho$ and let

$$F = \left(\frac{\mathrm{d}\mu_v^\pi}{\mathrm{d}\mu_w^\rho} \right)^{\frac{1}{2}}.$$

Using Corollary 2.11 we assume that $\mathcal{H}_\rho = L^2(\widehat{G}, \mu_w^\rho)$. Since μ_v^π is a finite measure, we have $F^2 \in L^1(\widehat{G}, \mu_w^\rho)$ and so $F \in \mathcal{H}_\rho = L^2(\widehat{G}, \mu_w^\rho)$. Hence we have

$$\langle M_g F, F \rangle = \int \langle g, t \rangle \, F(t)^2 \, \mathrm{d}\mu_w^\rho(t) = \int \langle g, t \rangle \, \mathrm{d}\mu_v^\pi(t) = \langle \pi_g v, v \rangle$$

for all $g \in G$ by Corollary 2.11. Applying Proposition 1.60 we obtain $\mathcal{H}_\pi < \mathcal{H}_\rho$ as desired. □

2.6 Functional Calculus

As we have used many times, we note that a unitary representation π of G gives rise to a module structure on $\mathcal{H}_\pi$ for the Banach algebra $L^1(G)$. Also notice that for the abelian group G we have already seen the Banach algebra homomorphism

$$\widehat{\ }\colon L^1(G) \longrightarrow C_0(\widehat{G}) \subseteq \mathscr{L}^\infty(\widehat{G}).$$

Using the non-diagonal spectral measures from the previous section, we extend here the scalars of the module $\mathcal{H}_\pi$ to scalars in $\mathscr{L}^\infty(\widehat{G})$.

Proposition 2.55 (Functional calculus for $\mathscr{L}^\infty(\widehat{G})$). *For any unitary representation π of the abelian group G we have a module structure on $\mathcal{H}_\pi$ for the algebra $\mathscr{L}^\infty(\widehat{G})$ that extends the module structure for $L^1(G)$. More formally, for any $F \in \mathscr{L}^\infty(\widehat{G})$ there exists a bounded operator $\pi_{\mathrm{FC}}(F)$ on $\mathcal{H}_\pi$ that depends linearly on F and satisfies*

(1) $\|\pi_{\mathrm{FC}}(F)\|_{\mathrm{op}} \leqslant \|F\|_\infty$, *and if a measurable subset* $B \subseteq \widehat{G}$ *has the property that* $\mu_v(\widehat{G} \smallsetminus B) = 0$ *for all* $v \in \mathcal{H}_\pi$, *then we also have*

$$\|\pi_{\mathrm{FC}}(F)\|_{\mathrm{op}} \leqslant \|F\|_{B,\infty} = \sup\{|F(t)| \mid t \in B\}.$$

(2) $\pi_{\mathrm{FC}}(F)^* = \pi_{\mathrm{FC}}(\overline{F})$,
(3) $\pi_{\mathrm{FC}}(\check{f}) = \pi_*(f)$ *for all* $f \in L^1(G)$, *and*
(4) $\pi_{\mathrm{FC}}(F_1)\pi_{\mathrm{FC}}(F_2) = \pi_{\mathrm{FC}}(F_1F_2)$ *for* $F_1, F_2 \in \mathscr{L}^\infty(\widehat{G})$.
(5) *If* ρ *is a unitary representation of* G *and* $B\colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ *is a bounded and equivariant operator, then we also have* $B \circ \pi_{\mathrm{FC}}(F) = \rho_{\mathrm{FC}}(F) \circ B$ *for* $F \in \mathscr{L}^\infty(\widehat{G})$.
(6) *If* $\mathcal{H}_\pi \cong L^2_\mu(X)$ *for a finite measure* μ *on* $X = \widehat{G} \times \mathbb{N}$ *as in the spectral theorem (Corollary 2.12), then* $\pi_{\mathrm{FC}}(F)$ *corresponds under the isomorphism to the multiplication operator* M_F *on* $L^2_\mu(X)$.

Moreover, for $F \in \mathscr{L}^\infty(\widehat{G})$ *the operator* $\pi_{\mathrm{FC}}(F)$ *is uniquely characterized by the formula*

$$\langle \pi_{\mathrm{FC}}(F)v, w\rangle = \int_{\widehat{G}} F \,\mathrm{d}\mu_{v,w} \tag{2.25}$$

for all $v, w \in \mathcal{H}_\pi$.

Proof. Let $B \subseteq \widehat{G}$ be as in (1) (for example, $B = \widehat{G}$). By Proposition 2.51(5)–(6) or Exercise 2.53, the assumption on B also implies that

$$\mu_{v,w}(\widehat{G} \smallsetminus B) = 0$$

for all $v, w \in \mathcal{H}_\pi$. We are going to define $\pi_{\mathrm{FC}}(F)$ for $F \in \mathscr{L}^\infty(\widehat{G})$ using (2.25) and the Fréchet–Riesz representation theorem. In fact, by Proposition 2.51 the map

$$\mathcal{H}_\pi \times \mathcal{H}_\pi \ni (v, w) \longmapsto \int F \,\mathrm{d}\mu_{v,w} = \int_B F \,\mathrm{d}\mu_{v,w}$$

depends linearly on $v \in \mathcal{H}_\pi$, semi-linearly on w, and satisfies the estimate

$$\left| \int_B F \,\mathrm{d}\mu_{v,w} \right| \leqslant \|F\|_{B,\infty} \|\mu_{v,w}\| \leqslant \|F\|_{B,\infty} \|v\| \|w\|$$

by Proposition 2.51(3). This implies that

$$\mathcal{H}_\pi \ni w \longmapsto \overline{\int F \,\mathrm{d}\mu_{v,w}}$$

is a bounded linear functional, which therefore has the form $w \mapsto \langle w, v_F\rangle$ for some uniquely determined $v_F \in \mathcal{H}_\pi$ with $\|v_F\| \leqslant \|F\|_{B,\infty}\|v\|$. Equivalently, we have $\langle v_F, w\rangle = \int F \,\mathrm{d}\mu_{v,w}$. Using the linearity of $\mu_{v,w}$ in v, we also see that v_F depends linearly on $v \in \mathcal{H}_\pi$ and so uniquely defines a bounded operator

$$\mathcal{H}_\pi \ni v \longmapsto \pi_{\mathrm{FC}}(F)v = v_F \in \mathcal{H}_\pi$$

with $\|\pi_{\mathrm{FC}}(F)\|_{\mathrm{op}} \leqslant \|F\|_{B,\infty}$ as claimed in (1). Note that a function F in $\mathscr{L}^\infty(B)$ can be extended trivially (or in any measurable way) to all of $\widehat{G}$, and also gives rise to the operator $\pi_{\mathrm{FC}}(F)$ independent of the extension.

Clearly $\int F \,\mathrm{d}\mu_{v,w}$ depends, for a given $v, w \in \mathcal{H}_\pi$, linearly on F, which implies that $\pi_{\mathrm{FC}}(F)$ depends linearly on $F \in \mathscr{L}^\infty(\widehat{G})$.

Given $F \in \mathscr{L}^\infty(\widehat{G})$, we have

$$\begin{aligned}\left\langle \pi_{\mathrm{FC}}(F)^* v, w\right\rangle &= \langle v, \pi_{\mathrm{FC}}(F)w\rangle = \overline{\langle \pi_{\mathrm{FC}}(F)w, v\rangle} \\ &= \int \overline{F}\,\mathrm{d}\overline{\mu_{w,v}} = \int \overline{F}\,\mathrm{d}\mu_{v,w} = \left\langle \pi_{\mathrm{FC}}(\overline{F})v, w\right\rangle\end{aligned}$$

by Proposition 2.51(2) for any $v, w \in \mathcal{H}_\pi$, which implies the conjugation formula $\pi_{\mathrm{FC}}(F)^* = \pi_{\mathrm{FC}}(\overline{F})$ as claimed in (2).

For $F = \check{f}$ with $f \in L^1(G)$ we have

$$\left\langle \pi_{\mathrm{FC}}(\check{f})v, w\right\rangle = \int \check{f}\,\mathrm{d}\mu_{v,w} = \langle \pi_*(f)v, w\rangle$$

for all $v, w \in \mathcal{H}_\pi$, by definition of the spectral measure $\mu_{v,w}$ in Proposition 2.51. Therefore $\pi_{\mathrm{FC}}(\check{f}) = \pi_*(f)$ as claimed in (3).

Next we prove (6), and assume first that $\mathcal{H}_\pi = \langle u\rangle_\pi \cong L^2_\mu(\widehat{G})$ as in the cyclic spectral theorem (Corollary 2.11). By Proposition 2.51(5) we have for $v, w \in \langle u\rangle_\pi$ corresponding to $F_v, F_w \in L^2_\mu(\widehat{G})$ that $\mathrm{d}\mu_{v,w} = F_v\overline{F_w}\,\mathrm{d}\mu_u$, and so

$$\langle \pi_{\mathrm{FC}}(F)v, w\rangle_{\mathcal{H}_\pi} = \int F F_v \overline{F_w}\,\mathrm{d}\mu_u = \langle M_F F_v, F_w\rangle_{L^2_\mu(\widehat{G})}$$

for all $v, w \in \langle u\rangle_\pi$ and all $F \in \mathscr{L}^\infty(\widehat{G})$. This proves the claim for cyclic representations. However, by linearity and continuity of all operators involved, this extends to the general case.

To prove (4), we now simply apply the spectral theorem in Corollary 2.12 and property (6) proven above. Under the unitary and equivariant isomorphism $\mathcal{H}_\pi \cong L^2_\mu(X)$ for $X = \widehat{G}\times\mathbb{N}$ we have that $\pi_{\mathrm{FC}}(F_1), \pi_{\mathrm{FC}}(F_2), \pi_{\mathrm{FC}}(F_1F_2)$ correspond to $M_{F_1}, M_{F_2}, M_{F_1F_2}$ respectively, satisfying $M_{F_1}M_{F_2} = M_{F_1F_2}$.

It remains to prove (5). So suppose that ρ is a unitary representation, and let $v \in \mathcal{H}_\pi$ and $w \in \mathcal{H}_\rho$. Then

$$\int \langle g, t\rangle \,\mathrm{d}\mu^\rho_{Bv,w}(t) = \langle \rho_g Bv, w\rangle = \langle B\pi_g v, w\rangle = \langle \pi_g v, B^* w\rangle = \int \langle g, t\rangle\,\mathrm{d}\mu^\pi_{v,B^*w}(t)$$

for all $g \in G$, which implies that $\mu^\rho_{Bv,w} = \mu^\pi_{v,B^*w}$ by uniqueness of spectral measures. We note that $\mu^\rho_{Bv,w}$ is a spectral measure defined by ρ, and μ^π_{v,B^*w} is a spectral measure defined by π. For $F \in \mathscr{L}^\infty(\widehat{G})$ we then have

$$\begin{aligned}\langle B\pi_{\mathrm{FC}}(F)v,w\rangle &= \left\langle \pi_{\mathrm{FC}}(F)v,B^*w\right\rangle \\ &= \int F\,\mathrm{d}\mu^{\pi}_{v,B^*w} \\ &= \int F\,\mathrm{d}\mu^{\rho}_{Bv,w} \\ &= \langle \rho_{\mathrm{FC}}(F)Bv,w\rangle\,.\end{aligned}$$

As this holds for all $v \in \mathcal{H}_\pi$ and $w \in \mathcal{H}_\rho$, property (5), and hence the proposition, follow. □

The following exercises should help the reader to become more familiar with the concepts introduced here.

Essential Exercise 2.56 (Functional calculus and invariance). Let π be a unitary representation of the abelian group G. Show that a closed subspace $\mathcal{V} \subseteq \mathcal{H}_\pi$ is invariant under π if and only if it is invariant under $\pi_{\mathrm{FC}}(\mathscr{L}^\infty(\widehat{G}))$.

Essential Exercise 2.57 (Functional calculus and spectral measures). Let π be a unitary representation of the abelian group G. Show that

$$\mathrm{d}\mu_{\pi_{\mathrm{FC}}(F)v} = |F|^2\,\mathrm{d}\mu_v,\ \mathrm{d}\mu_{\pi_{\mathrm{FC}}(F)v,w} = F\,\mathrm{d}\mu_{v,w}$$

for $v,w \in \mathcal{H}_\pi$ and $F \in \mathscr{L}^\infty(\widehat{G})$.

Exercise 2.58. Let π be a unitary representation of the abelian group G. Show that

$$\pi_g\pi_{\mathrm{FC}}(\mathbb{1}_{\{t\}}) = \langle g,t\rangle\pi_{\mathrm{FC}}(\mathbb{1}_{\{t\}})$$

for all elements g in G and t in $\widehat{G}$.

As mentioned above, the functional calculus gives a different approach to the proof of the last two parts of Proposition 2.50.

Proof of Proposition 2.50(6) and (7). Let π be a unitary representation of the abelian group G, and let $w \in \langle v\rangle_\pi$ such that $\mu_v \ll \mu_w$ as in (6). We apply the cyclic spectral theorem and suppose $v = \mathbb{1} \in L^2(\widehat{G},\mu)$ for $\mu = \mu_v$. Then $w \in L^2(\widehat{G},\mu)$ and $\mathrm{d}\mu_w = |w|^2\,\mathrm{d}\mu$ (by Proposition 2.50(2)) imply with our assumption that $w(t) \neq 0$ for μ-almost every $t \in \widehat{G}$. For $n \in \mathbb{N}$ we now define

$$F_n(t) = \begin{cases} w(t)^{-1} & \text{if } |w(t)| \geqslant \frac{1}{n}, \\ 0 & \text{if not.}\end{cases}$$

Using the measurable functional calculus in Proposition 2.55 (see also Exercise 2.56 and its hint on p. 477), we obtain $\pi_{\mathrm{FC}}(F_n)w \in \langle w\rangle_\pi$ and

$$\pi_{\mathrm{FC}}(F_n)w = \mathbb{1}_{\{t\in\widehat{G}\,|\,|w(t)|\geqslant 1/n\}}$$

which converges to $v = \mathbb{1}_{\{t|w(t)\neq 0\}} = \mathbb{1}$ in $L^2(\widehat{G},\mu)$ as $n\to\infty$ (by dominated convergence). Therefore $v \in \langle w\rangle_\pi$ as claimed.

Suppose now as in (7) that $\mu_{v_1} \perp \mu_{v_2}$. By Proposition 2.50(5), we have

$$\langle v_1\rangle_\pi \perp \langle v_2\rangle_\pi$$

and so $\langle v_1+v_2\rangle_\pi \subseteq \langle v_1\rangle_\pi \oplus \langle v_2\rangle_\pi$. By Proposition 2.50(3) we also have

$$\mu_{v_1+v_2} = \mu_{v_1} + \mu_{v_2}$$

and we can find measurable sets B_1, B_2 with $\widehat{G} = B_1 \sqcup B_2$ with $\mu_{v_j}(\widehat{G}\smallsetminus B_j) = 0$ for $j = 1, 2$. We define

$$w_j = \pi_{\mathrm{FC}}(\mathbb{1}_{B_j})(v_1+v_2) \in \langle v_1+v_2\rangle_\pi$$

for $j = 1, 2$. By Exercise 2.57 (see also its hint on p. 477), we have

$$\mathrm{d}\mu_{w_j} = |\mathbb{1}_{B_j}|^2\,\mathrm{d}\mu_{v_1+v_2} = \mathbb{1}_{B_j}\,\mathrm{d}\mu_{v_1} + \mathbb{1}_{B_j}\,\mathrm{d}\mu_{v_2} = \mathrm{d}\mu_{v_j}$$

for $j = 1, 2$. By Proposition 2.50(5) we obtain $w_1 \in \langle v_2\rangle_\pi^\perp$ and $w_2 \in \langle v_1\rangle_\pi^\perp$. Together with $v_1 + v_2 = \pi_{\mathrm{FC}}(\mathbb{1}_{B_1} + \mathbb{1}_{B_2})(v_1+v_2) = w_1 + w_2$ we obtain that $v_1 = w_1 \in \langle v_1+v_2\rangle_\pi$ and $v_2 = w_2 \in \langle v_1+v_2\rangle_\pi$, which proves (7). □

2.6.1 Projection-Valued Measures

As a special case of the functional calculus above, we obtain for a unitary representation π of the abelian group G a *projection-valued measure* defined by

$$\mathcal{B}(\widehat{G}) \ni B \longmapsto \mathit{\Pi}_B = \pi_{\mathrm{FC}}(\mathbb{1}_B),$$

which gives for every Borel subset $B \subseteq \widehat{G}$ an orthogonal projection $\mathit{\Pi}_B$ onto the subspace of $\mathcal{H}_\pi$ corresponding to '*the generalized sum of eigenspaces for all characters in* B' as we will explain now.

The projection-valued measure allows us to reconstruct the functional calculus and, in particular, to reconstruct the unitary representation. In fact if $F \in \mathscr{L}^\infty(\widehat{G})$ and $F_0 = \sum_{j=1}^n c_j \mathbb{1}_{B_j}$ is an approximation of F by a simple function, then

$$\Big\|\pi_{\mathrm{FC}}(F) - \sum_{j=1}^n c_j \mathit{\Pi}_{B_j}\Big\|_{\mathrm{op}} = \|\pi_{\mathrm{FC}}(F-F_0)\|_{\mathrm{op}} \leqslant \|F-F_0\|_\infty$$

shows that $\pi_{\mathrm{FC}}(F)$ can be approximated by the finite sums

$$\sum_{j=1}^{n} c_j \Pi_{B_j} = \pi_{\text{FC}}(F_0).$$

We make the definition

$$\int_{\widehat{G}} F_0 \, \mathrm{d}\Pi = \sum_{j=1}^{n} c_j \Pi_{B_j}.$$

With this interpretation we can now define

$$\int_{\widehat{G}} F \, \mathrm{d}\Pi = \pi_{\text{FC}}(F) = \lim_{F_0 : \|F_0 - F\|_\infty \to 0} \pi_{\text{FC}}(F_0),$$

where convergence holds in the uniform topology.

Also notice that for $g \in G$ we may define a function $F_g \in \mathscr{L}^\infty(\widehat{G})$ by setting $F_g(t) = \langle g, t \rangle$. Since

$$\big\langle \pi_{\text{FC}}(F_g) v, w \big\rangle = \int_{\widehat{G}} \langle g, t \rangle \, \mathrm{d}\mu_{v,w}(t) = \big\langle \pi_g v, w \big\rangle$$

for all $v, w \in \mathcal{H}_\pi$, this has the property that

$$\pi_{\text{FC}}(F_g) = \pi_g = \int_{\widehat{G}} \langle g, \cdot \rangle \, \mathrm{d}\Pi. \tag{2.26}$$

As mentioned above, this suggests that for any measurable $B \subseteq \widehat{G}$ we may think of $\operatorname{Im} \Pi_B = \Pi_B \mathcal{H}_\pi$ as the generalized sum of the eigenspaces for characters[†] χ_t with $t \in B$. The generalization concerns the integral and the fact that, strictly speaking, there may not be a single eigenvector, but only approximate eigenvectors. Similarly, $\pi_{\text{FC}}(F)$ can now be interpreted as the operator that multiplies (generalized) eigenvectors in $\mathcal{H}_\pi$ for eigenvalue $t \in \widehat{G}$ by $F(t)$. Informally, we may also write (2.26) in the form

$$\pi_g v = \int_{\widehat{G}} \langle g, t \rangle \Pi_t v \, \mathrm{d}m(t)$$

and think of $\Pi_t v$ as the projection of $v \in \mathcal{H}_\pi$ to the eigenspace in $\mathcal{H}_\pi$ corresponding to the character χ_t. However, as the latter may be trivial for all $t \in \widehat{G}$ this formally makes no sense, and this is the reason we prefer the notation of (2.26). This provides a useful viewpoint, but does not provide additional formal properties. We refer to [21, Sec. 12.7] for a more thorough discussion of spectral-valued measures.

[†] Whenever we have a collection of operators that commute with each other, an eigenvalue is really a function on the collection of operators, and here — in the context of unitary representations — a unitary character on the group.

Exercise 2.59. If B_1 and B_2 are disjoint measurable sets, show that Π_{B_1} and Π_{B_2} are orthogonal projections.

Essential Exercise 2.60. Let π be a unitary representation of the abelian group G. Let $B_1, B_2, \dots$ be a sequence of pairwise disjoint measurable subsets of $\widehat{G}$ and define $B = \bigsqcup_{n=1}^{\infty} B_n$. Show that

$$\Pi_B = \pi_{\text{FC}}(\mathbb{1}_B) = \sum_{n=1}^{\infty} \pi_{\text{FC}}(\mathbb{1}_{B_n}) = \sum_{n=1}^{\infty} \Pi_{B_n},$$

where the convergence holds in the strong operator topology. Find an example where the convergence of the series fails in the uniform topology.

2.7 Spectral Theory and Multiplicity

In this section we will revisit the spectral theorem for the abelian group G, and derive with some more work a version that presents all the information regarding multiplicity.

2.7.1 Maximal Spectral Type

Proposition 2.61 (Maximal spectral type). *Let π be a unitary representation of the abelian group G. Then there exists a vector $v_{\max} \in \mathcal{H}_\pi$ with spectral measure $\mu_{\max} = \mu_{v_{\max}}$ with the property that $\mu_v \ll \mu_{\max}$ for any $v \in \mathcal{H}_\pi$. The measure equivalence class of $\mu_{\max}$ is uniquely characterized by this property, and is called the* maximal spectral type. *Moreover, given some $v_0 \in \mathcal{H}_\pi$ the vector $v_{\max}$ can be chosen so that $v_0 \in \langle v_{\max} \rangle_\pi$.*

Proof. We apply the spectral theorem in the form of Corollary 2.12, and find orthonormal vectors $v_1, v_2, \dots$ of $\mathcal{H}_\pi$ such that

$$\mathcal{H}_\pi = \bigoplus_{n \geqslant 1} \langle v_n \rangle_\pi \cong \bigoplus_{n \geqslant 1} L^2_{\mu_n}(\widehat{G})$$

with $\mu_n = \mu_{v_n}$ for all $n \geqslant 1$. We now define $v_{\max} = \sum_{n \geqslant 1} \frac{1}{n} v_n$ and obtain from Proposition 2.50(4) that the spectral measure $\mu_{\max} = \mu_{v_{\max}}$ is given by

$$\mu_{\max} = \sum_{n \geqslant 1} \frac{1}{n^2} \mu_n. \tag{2.27}$$

For $u \in \mathcal{H}_\pi$ we then have $u = \sum_{n \geqslant 1} u_n$ with $u_n \in \langle v_n \rangle_\pi$ for $n \geqslant 1$, which also implies that $\mu_{u_n} \ll \mu_n \ll \mu_{\max}$ for all $n \geqslant 1$. It follows from Proposition 2.50(4) and the definition of absolute continuity that

$$\mu_u = \sum_{n\geqslant 1} \mu_{u_n} \ll \mu_{\max}.$$

As $u \in \mathcal{H}_\pi$ was arbitrary, it follows that $\mu_{\max}$ is indeed a maximal spectral measure for π.

We now prove the claimed uniqueness of the measure class of $\mu_{\max}$. Suppose that $v \in \mathcal{H}_\pi$ is another vector with the property that $\mu_u \ll \mu_v$ for all $u \in \mathcal{H}_\pi$. Setting $u = v_{\max}$ gives $\mu_{\max} = \mu_{v_{\max}} \ll \mu_v$. Setting $u = v$ in the discussion above gives $\mu_v \ll \mu_{\max}$, and hence $\mu_{\max}$ and μ_v lie in the same measure class.

Now let $v_0 \in \mathcal{H}_\pi$ be arbitrary, and write $\mathcal{V} = \langle v_0\rangle_\pi^\perp$, so that we may write $\mathcal{H}_\pi = \langle v_0\rangle_\pi \oplus \mathcal{V}$. We apply the above to find a vector $w \in \mathcal{V}$ of maximal spectral type for the representation $\pi|_{\mathcal{V}}$. Next apply the Lebesgue decomposition theorem (see [21, Sec. 3.1.3]) to μ_w and μ_{v_0} to write μ_w as a sum $\mu_w = \mu_{\text{abs}} + \mu_{\text{sing}}$, where $\mu_{\text{abs}} \ll \mu_{v_0}$ and $\mu_{\text{sing}} = \mu_w|_B \perp \mu_{v_0}$ for some measurable subset $B \subseteq \widehat{G}$. By the cyclic spectral theorem (Corollary 2.11) there exists some $w_{\text{sing}} \in \langle w\rangle_\pi \subseteq \mathcal{V}$ corresponding to $\mathbb{1}_B \in L^2_{\mu_w}(\widehat{G})$, which by Proposition 2.50(2) has spectral measure $\mathrm{d}\mu_{w_{\text{sing}}} = \mathbb{1}_B \,\mathrm{d}\mu_w = \mathrm{d}\mu_{\text{sing}} \perp \mu_{v_0}$. (Equivalently, we may set $w_{\text{sing}} = \pi_{\text{FC}}(\mathbb{1}_B)w$ and apply Exercises 2.56 and 2.57.) We define $v_{\max} = v_0 + w_{\text{sing}}$ and note that

$$\mu_{v_{\max}} = \mu_{v_0} + \mu_{\text{sing}}$$

by Proposition 2.50(3) and $\langle w\rangle_\pi \subseteq \mathcal{V} = \langle v_0\rangle_\pi^\perp$. Proposition 2.50(7) also implies that $\langle v_{\max}\rangle_\pi = \langle v_0\rangle_\pi \oplus \langle w_{\text{sing}}\rangle_\pi$, and in particular $\langle v_{\max}\rangle_\pi$ contains v_0. For any $u = u_0 + u_1 \in \mathcal{H}_\pi = \langle v_0\rangle_\pi \oplus \mathcal{V}$ with $u_0 \in \langle v_0\rangle_\pi$ and $u_1 \in \mathcal{V}$ we again have, by various parts of Proposition 2.50, that

$$\mu_u = \mu_{u_0} + \mu_{u_1} \ll \mu_{v_0} + \mu_w = \mu_{v_0} + \mu_{\text{abs}} + \mu_{\text{sing}} \ll \mu_{v_0} + \mu_{\text{sing}} = \mu_{v_{\max}}.$$

This concludes the proof. □

Corollary 2.62 (Spectral theorem with descending measures). *Let π be a unitary representation of the abelian group G. Then there exists a (possibly finite) sequence of vectors $u_1, u_2, \dots$ in $\mathcal{H}_\pi$ such that $\mathcal{H}_\pi = \bigoplus_{n\geqslant 1}\langle u_n\rangle_\pi$ and $\mu_{\max} = \mu_{u_1} \gg \mu_{u_2} \gg \cdots$.*

Proof. Let $v_1, v_2, \dots$ be a basis of $\mathcal{H}_\pi$. Applying Proposition 2.61 gives a vector $u_1 = v_{\max}$ whose spectral measure represents the maximal spectral type and whose cyclic representation contains v_1.

Now project the vector v_2 to $\langle u_1\rangle_\pi^\perp$ and apply Proposition 2.61 above again to find a vector u_2 with the property that $v_2 \in \langle u_1\rangle_\pi \oplus \langle u_2\rangle_\pi$ and such that μ_{u_2} represents the maximal spectral type of $\pi|_{\langle u_1\rangle_\pi^\perp}$.

Proceeding like this we can make sure that each of the basis vectors v_n in $\mathcal{H}_\pi$ belongs to $V_n = \langle u_1\rangle_\pi \oplus \langle u_2\rangle_\pi \oplus \cdots \oplus \langle u_n\rangle_\pi$ and that the maximal spectral type on the orthogonal complement $V_n^\perp$ is realized by u_{n+1} for $n \in \mathbb{N}$. This gives the corollary. □

Weakening the notion of maximal spectral measure, we also obtain the following topological object related to a unitary representation. For this, note that the support of two measures in the same measure class agree, which makes the following independent of the choice of the maximal measure.

Definition 2.63 (Support of π). Let π be a unitary representation of the abelian group G. Then the support $\operatorname{supp}(\pi) \subseteq \widehat{G}$ of π is defined as the support of the maximal spectral measure from Proposition 2.61.

Exercise 2.64. Let π be a unitary representation of an abelian group. Show that any two vectors in $\mathcal{H}_\pi$ of maximal spectral type (that is, vectors whose spectral measures are maximal spectral measures) have isomorphic cyclic representations.

2.7.2 Spectral Multiplicity*

Theorem 2.65 (Spectral theorem with multiplicities). *Let π be a unitary representation of the abelian group G. Then there exists a collection*

$$(\mu_1, \mu_2, \dots, \mu_\infty) = (\mu_n)_{n \in \mathbb{N} \cup \{\infty\}}$$

of finite measures on $\widehat{G}$ that satisfy the following properties and so completely describe the unitary representation π.

(1) *For $m \neq n \in \mathbb{N} \cup \{\infty\}$ the measures $\mu_m \perp \mu_n$ are singular. In fact, if $\mu_{\max}$ be a maximal spectral measure as in Proposition* 2.61 *then there exists a measurable partition $\{P_n \mid n \in \mathbb{N} \cup \{\infty\}\}$ of $\widehat{G}$ with the property that $\mu_n = \mu_{\max}|_{P_n}$ for all $n \in \mathbb{N} \cup \{\infty\}$.*
(2) *The representation π is unitarily isomorphic to the multiplication representation on $\bigoplus_{n \in \mathbb{N}} \bigl(L^2_{\mu_n}(P_n)\bigr)^n \oplus \bigl(L^2_{\mu_\infty}(P_\infty)\bigr)^\infty$, where $\bigl(L^2_{\mu_\infty}(P_\infty)\bigr)^\infty$ denotes the Hilbert space direct sum of countably many copies of $L^2_{\mu_\infty}(P_\infty)$.*
(3) *Even though the isomorphism in* (2) *is not canonical, the subspace $\mathcal{H}_\pi^{(n)}$ of $\mathcal{H}_\pi$ corresponding to $\bigl(L^2_{\mu_n}(P_n)\bigr)^n$ for $n \in \mathbb{N} \cup \{\infty\}$ is independent of the choice of the isomorphism.*
(4) *The measure class of μ_n for $n \in \mathbb{N} \cup \{\infty\}$ is uniquely determined by* (1) *and* (2).

Proof of (1) and (2). Let $u_1, u_2, \dots$ be as in Corollary 2.62. We claim that it is possible to replace $u_2, u_3, \dots$ by another sequence $w_2, w_3, \dots$ without changing their respective cyclic subspaces so that $\frac{\mathrm{d}\mu_{w_n}}{\mathrm{d}\mu_{\max}} \in \{0, 1\}$ almost everywhere with respect to $\mu_{\max} = \mu_{u_1}$. To see this, fix $n \geqslant 2$, let $F_n = \frac{\mathrm{d}\mu_{u_n}}{\mathrm{d}\mu_{\max}}$ and

$$B_n = \{t \in \widehat{G} \mid F_n(t) > 0\}.$$

Then the function

$$\widetilde{F_n}(t) = \begin{cases} F_n^{-\frac{1}{2}}(t) & t \in B_n, \\ 0 & t \in \widehat{G} \smallsetminus B_n \end{cases}$$

satisfies $\widetilde{F_n} \in L^2_{\mu_{u_n}}(\widehat{G})$ since

$$\int_{\widehat{G}} \widetilde{F_n}^2 \,\mathrm{d}\mu_{u_n} = \int_{B_n} F_n^{-1} \,\mathrm{d}\mu_{u_n} = \int_{B_n} F_n^{-1} F_n \,\mathrm{d}\mu_{\max} = \mu_{\max}(B_n) < \infty.$$

We see from Proposition 2.50(2) that the vector $w_n \in \langle u_n \rangle_\pi$ corresponding to $\widetilde{F_n} \in L^2_{\mu_{u_n}}(\widehat{G})$ has spectral measure

$$\mu_{w_n} = |\widetilde{F_n}|^2 \,\mathrm{d}\mu_{u_n} = |\widetilde{F_n}|^2 F_n \,\mathrm{d}\mu_{\max} = \mu_{\max}|_{B_n},$$

since $|\widetilde{F_n}|^2 F_n = \mathbb{1}_{B_n}$. Moreover, μ_{w_n} defines the same measure class as μ_{u_n} (since $\widetilde{F_n} > 0$ μ_{u_n}-almost everywhere). Hence by Proposition 2.50(6) we see that $\langle w_n \rangle_\pi = \langle u_n \rangle_\pi$. This proves the claim.

So we suppose now that the vectors $u_1, u_2, \dots$ are as in Corollary 2.62 with $\mu_{\max} = \mu_{u_1}$, and in addition that $\frac{\mathrm{d}\mu_{u_n}}{\mathrm{d}\mu_{\max}} = \mathbb{1}_{B_n}$ for some measurable set $B_n \subseteq \widehat{G}$ for all $n \geqslant 2$. Since $\mu_{\max} = \mu_{u_1} \gg \mu_{u_2} \gg \cdots$ and

$$\mu_{u_n}\big(\widehat{G} \smallsetminus B_n\big) = 0$$

for all $n \geqslant 2$, we have

$$0 = \mu_{u_m}\big(\widehat{G} \smallsetminus B_n\big) = \mu_{\max}(B_m \smallsetminus B_n)$$

by definition of absolute continuity for all $m \geqslant n$. We now replace B_2 by $B_1 \cap B_2$, B_3 by $B_1 \cap B_2 \cap B_3$ and so on, without changing their defining properties. Hence we may also assume that these subsets comprise a descending chain $\widehat{G} = B_1 \supseteq B_2 \supseteq \cdots$ of measurable subsets with the property that $\mu_{u_n} = \mu_{\max}|_{B_n}$ for all $n \in \mathbb{N}$. We now define $\mu_n = \mu_{\max}|_{P_n}$ where $P_n = B_n \smallsetminus B_{n+1}$ for all $n \in \mathbb{N}$, and $\mu_\infty = \mu|_{P_\infty}$ where $P_\infty = \bigcap_{n \geqslant 1} B_n$. This gives the unitary and equivariant isomorphism

$$L^2_{\mu_{u_n}}(\widehat{G}) = L^2_{\mu_{u_n}}(B_n) \cong \bigoplus_{k \in \{n, \dots, \infty\}} L^2_{\mu_k}(P_k),$$

where the projection map from the left to the factors on the right is simply multiplication by the characteristic function of P_k for $n \leqslant k \leqslant \infty$. Together with Corollary 2.62 this gives

$$\mathcal{H}_\pi \cong \bigoplus_{n \geqslant 1} L^2_{\mu_{u_n}}(\widehat{G}) \cong \bigoplus_{n \geqslant 1} \Big(\bigoplus_{k \in \{n, \dots, \infty\}} L^2_{\mu_k}(P_k) \Big) \cong \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} \Big(L^2_{\mu_n}(P_n) \Big)^n.$$

The first isomorphism is equivariant by Corollary 2.62, the second is realized by multiplication operators and so commutes with the multiplication representation, and the final isomorphism is simply a permutation of the invariant factors. This proves (1) and (2). □

2.7.3 The Multiplicity Subspaces*

Next we will prove that the multiplicity n subspace $\mathcal{H}_\pi^{(n)}$ in Theorem 2.65(3) is canonical for every $n \in \mathbb{N} \cup \{\infty\}$. The canonical description of $\mathcal{H}_\pi^{(n)}$ uses spectral measures, and will be easy to prove using our knowledge of them together with some measure-theoretic constructions.

For every $n \in \mathbb{N}$ the subspace $\mathcal{H}_\pi^{(\geqslant n)}$ is given by

$$\mathcal{H}_\pi^{(\geqslant n)} = \{v \in \mathcal{H}_\pi \mid v \text{ has spectral multiplicity at least } n\},$$

where we say that $v \in \mathcal{H}_\pi$ has *spectral multiplicity at least* n if there exist n vectors

$$v_1 = v, v_2, \dots, v_n \in \mathcal{H}_\pi$$

such that $\langle v_k \rangle_\pi \perp \langle v_\ell \rangle_\pi$ for $1 \leqslant k < \ell \leqslant n$ and $\mu_{v_k}^\pi = \mu_v^\pi$ for $k \leqslant n$. We also define

$$\mathcal{H}_\pi^{(n)} = \mathcal{H}_\pi^{(\geqslant n)} \cap \left(\mathcal{H}_\pi^{(\geqslant n+1)}\right)^\perp \tag{2.28}$$

as the relative orthogonal complement of $\mathcal{H}_\pi^{(\geqslant n+1)}$ within $\mathcal{H}_\pi^{(\geqslant n)}$. Similarly, for $n = \infty$ the *infinite multiplicity subspace* $\mathcal{H}_\pi^{(\infty)}$ is given by

$$\mathcal{H}_\pi^{(\infty)} = \bigcap_{n \in \mathbb{N}} \mathcal{H}_\pi^{(\geqslant n)}. \tag{2.29}$$

We note that the above definition of $\mathcal{H}_\pi^{(n)}$ for $n \in \mathbb{N} \cup \{\infty\}$ is canonical, since spectral measures are uniquely determined by a given vector and the unitary representation.

Before starting with the argument for part (3) of the theorem, we wish to set up some helpful further notation. In fact, it will be convenient to write $\mathcal{V}_\infty = \ell^2(\mathbb{N})$, $\mathcal{V}_n = \mathbb{C}^n$ for $n \in \mathbb{N}$, and in either case $\|\cdot\|_\mathcal{V}$ for its standard Hilbert space norm and $\langle \cdot, \cdot \rangle_\mathcal{V}$ for its inner product. For any $n \in \mathbb{N} \cup \{\infty\}$ we may and will think of an element $v = (v_j)_j \in \left(L^2_{\mu_n}(P_n)\right)^n$ as a $\mathcal{V}_n$-valued function

$$P_n \ni t \longmapsto (v_j(t))_j \in \mathcal{V}_n$$

satisfying $\|v\|^2 = \sum_j \|v_j\|^2 = \int_{P_n} \|(v_j(t))_j\|^2_\mathcal{V} \,\mathrm{d}\mu_n(t)$ (by a simple form of Fubini's theorem). We denote the space of measurable, square-integrable, $\mathcal{V}_n$-valued functions by

$$L^2_{\mu_n}(P_n,\mathcal{V}_n)=\left\{v\colon P_n\to\mathcal{V}_n\mid \|v\|^2=\int_{P_n}\|v(t)\|^2_{\mathcal{V}}\,\mathrm{d}\mu_n(t)<\infty\right\}.$$

Using this notation, parts (1) and (2) of the theorem, and the fact that the definition of $\mathcal{H}_\pi^{(n)}$ in (2.28) and (2.29) is canonical, we may and will assume that

$$\mathcal{H}_\pi=\bigoplus_{n\in\mathbb{N}\cup\{\infty\}}L^2_{\mu_n}(P_n,\mathcal{V}_n)\tag{2.30}$$

and that the representation is given by the multiplication representation. Using the identity $\mu_n=\mu_{\max}|_{P_n}$ for the measurable partition

$$\{P_n\mid n\in\mathbb{N}\cup\{\infty\}\}$$

of $\widehat{G}$, we may even identify an element $v\in\mathcal{H}_\pi$ with a function on $\widehat{G}$, which satisfies $v|_{P_n}\in L^2_{\mu_n}(P_n,\mathcal{V}_n)$ for all $n\in\mathbb{N}\cup\{\infty\}$ and for which

$$\widehat{G}\ni t\longmapsto\|v(t)\|_{\mathcal{V}}$$

is square-integrable with respect to $\mu_{\max}$. For $n\in\mathbb{N}\cup\{\infty\}$ and a measurable subset $B\subseteq P_n$ we will also write $L^2_{\mu_n}(B,\mathcal{V}_n)$ for the subspace of $L^2_{\mu_n}(P_n,\mathcal{V}_n)$ consisting of functions that vanish on $P_n\smallsetminus B$.

Using these identifications it remains to prove that $\mathcal{H}_\pi^{(n)}=L^2_{\mu_n}(P_n,\mathcal{V}_n)$ for $n\in\mathbb{N}\cup\{\infty\}$. For this we need the following measurable construction.

Lemma 2.66 (Measurable selection of orthonormal basis). *Let $n\in\mathbb{N}$ and $n\leqslant m\in\mathbb{N}\cup\{\infty\}$. Then there exists a measurable map sending a vector $v_1\in\{v\in\mathcal{V}_m\mid\|v\|=1\}$ to a list $(v_1,v_2,\dots,v_n)$ consisting of n orthonormal entries.*

Proof. Given v_1 we may apply the Gram–Schmidt orthonormalization procedure to the list of vectors $v_1,e_1,e_2,\dots$. As usual, we discard at each step the vector if its projection onto the orthogonal complement of the span of the previous vectors vanishes. This gives a definition of the required orthonormal basis in a measurable way. In fact, on each of the measurable subsets $\langle e_1\rangle_{\mathbb{C}}$, $\langle e_1,e_2\rangle_{\mathbb{C}}\smallsetminus\langle e_1\rangle_{\mathbb{C}},\dots$ of $\mathcal{V}_n$ with the same discarding steps, the new vectors depend continuously on v_1. For instance, for the second vector v_2 we have $v_2=e_2$ if $v_1\in\langle e_1\rangle_{\mathbb{C}}$ and

$$v_2=\frac{1}{\|v_1-\langle v_1,e_1\rangle e_1\|}\big(v_1-\langle v_1,e_1\rangle e_1\big)$$

if $v_1\notin\langle e_1\rangle_{\mathbb{C}}$. □

Exercise 2.67. Give a more careful general definition of v_k for $k\in\{2,\dots,n\}$ in Lemma 2.66.

Proof of Theorem 2.65(3) using spectral measures. As mentioned above, the definition of $\mathcal{H}_\pi^{(n)}$ for $n \in \mathbb{N} \cup \{\infty\}$ given in (2.28)–(2.29) using spectral measures is canonical (as spectral measures are uniquely determined by a given vector and the unitary representation). This and parts (1) and (2) allow us to assume that π is the multiplication representation on the Hilbert space in (2.30). It remains to prove that

$$\mathcal{H}_\pi^{(\geqslant n)} = \bigoplus_{m \geqslant n} L^2_{\mu_m}(P_m, \mathcal{V}_m)$$

for $n \in \mathbb{N}$, where m is allowed to be ∞. This will give $\mathcal{H}_\pi^{(n)} = L^2_{\mu_n}(P_n, \mathcal{V}_n)$ for $n \in \mathbb{N}$ and $\mathcal{H}_\pi^{(\infty)} = L^2_{\mu_\infty}(P_\infty, \mathcal{V}_\infty)$.

For this, we note that the definition of spectral measures shows that

$$\mathrm{d}\mu_v^\pi(t) = \sum_{n \in \mathbb{N} \cup \{\infty\}} \|v(t)\|^2_{\mathcal{V}} \,\mathrm{d}\mu_n(t) = \|v(t)\|^2_{\mathcal{V}} \,\mathrm{d}\mu_{\max} \tag{2.31}$$

and, more generally,

$$\mathrm{d}\mu_{v,w}^\pi(t) = \sum_{n \in \mathbb{N} \cup \{\infty\}} \langle v(t), w(t)\rangle_{\mathcal{V}} \,\mathrm{d}\mu_n(t) = \langle v(t), w(t)\rangle_{\mathcal{V}} \,\mathrm{d}\mu_{\max} \tag{2.32}$$

for $t \in \widehat{G}$ and all $v, w \in \mathcal{H}_\pi$ (also see Proposition 2.51).

We suppose first that $v \in \bigoplus_{m \geqslant n} L^2_{\mu_m}(P_m, \mathcal{V}_m)$. Using (2.31) we see that the spectral measure is given by $F^2 \,\mathrm{d}\mu_n$, where $F(t) = \|v(t)\|_{\mathcal{V}}$ for all $t \in \bigcup_{m \geqslant n} P_m$. We also define $B = \{t \in \bigcup_{m \geqslant n} P_m \mid F(t) \neq 0\}$. For $t \in B$ we apply Lemma 2.66 to $\widetilde{v}_1(t) = F^{-1} v(t)$ to obtain the orthonormal vectors $\widetilde{v}_1(t), \widetilde{v}_2(t), \ldots, \widetilde{v}_n(t)$. We extend these functions trivially to all of $\widehat{G}$. By measurability of the construction in Lemma 2.66, we now obtain the vectors

$$v_k = F \widetilde{v}_k \in L^2_{\mu_n}(P_n, \mathcal{V}_n)$$

for $k \in \{2, \ldots, n\}$. We also define $v_1 = v$. This ensures that

$$\|v_k(t)\|_{\mathcal{V}} = F(t) = \|v(t)\|_{\mathcal{V}}$$

for all $k \geqslant 1$ and $t \in \widehat{G}$, which implies that these n vectors have the same spectral measures as v by (2.31). Moreover, we also have $\langle v_k(t), v_\ell(t)\rangle_{\mathcal{V}} = 0$ by construction for $k \neq \ell$ and all $t \in P_n$. Hence $\langle v_k\rangle_\pi \perp \langle v_\ell\rangle_\pi$ by (2.32) and Exercise 2.52 (see also the hint on p. 477), which proves the requirement for v to have spectral multiplicity at least n.

Suppose now that $v \in \mathcal{H}_\pi^{(\geqslant n)} \subseteq \bigoplus_{m \in \mathbb{N} \cup \{\infty\}} L^2_{\mu_m}(P_m, \mathcal{V}_m)$ has the property that there exist n vectors $v_1 = v, v_2, \ldots, v_n$ with orthogonal cyclic representations and spectral measures equal to μ_v. As in the previous step, we can define $B = \{t \in \widehat{G} \mid \|v(t)\|_{\mathcal{V}} \neq 0\}$. We note that $\langle v_k(t), v_\ell(t)\rangle_{\mathcal{V}} = 0$ for almost every $t \in \widehat{G}$ (with respect to $\mu_{\max}$) for $1 \leqslant k < \ell \leqslant n$. Indeed, Exercise 2.52 shows that we have $\mu^{\pi}_{v_k,v_\ell} = 0$, which, together with (2.32), gives $\langle v_k(t), v_\ell(t)\rangle_{\mathcal{V}} = 0$ for almost every $t \in \widehat{G}$. This shows that the n vectors $v_1(t), v_2(t), \dots, v_n(t)$ are, for almost every $t \in B$, orthogonal. For $t \in P_m$ and $m < n$ the vectors $v_k(t)$ take values in $\mathcal{V}_m = \mathbb{C}^m$, which shows that $\mu_{\max}(B \cap P_m) = 0$. Hence we have the opposite inclusion

$$\mathcal{H}_\pi^{(\geqslant n)} \subseteq \bigoplus_{m \geqslant n} L^2_{\mu_m}(P_m, \mathcal{V}_m).$$

Together with the first part of the proof, this gives equality, and so that

$$\mathcal{H}_\pi^{(n)} = L^2_{\mu_n}(P_n, \mathcal{V}_n)$$

for $n \in \mathbb{N} \cup \{\infty\}$. □

Proof of (4) in Theorem 2.65. By the above, we know that $\mathcal{H}_\pi^{(n)} \subseteq \mathcal{H}_\pi$ is a canonical invariant subspace that is isomorphic to the multiplication representation on $\bigl(L^2_{\mu_n}(P_n)\bigr)^n$. However, by the properties of spectral measures in Proposition 2.61 (see also Proposition 2.50(2) and (4)) this implies that μ_n is a maximal spectral measure of $\pi|_{\mathcal{H}_\pi^{(n)}}$, which in particular implies that its measure class is uniquely and canonically determined by π. □

We note that the phrase *countable Lebesgue spectrum* is used, for example, in dynamical systems to refer to the case of a measure-preserving invertible transformation (corresponding to the case $G = \mathbb{Z}$ in Proposition 1.3) where $\mu_{\max} = \mu_\infty$ represents the measure class of the Lebesgue measure on $\widehat{\mathbb{Z}} = \mathbb{T}$. Other possibilities are described using the terminology *pure point* (or *discrete*) *spectrum*, *continuous spectrum* or *mixed spectrum*, but these do not reflect any multiplicity information. We refer to Cornfeld, Fomin and Sinaĭ [12] for more on how spectral methods are used in the study of measure-preserving dynamical systems.

2.8 The Centralizer of the Representation*

At times it is useful to understand the centralizer of a unitary representation. Let π be a unitary representation of the abelian group G, and $\mu_{\max}$ a maximal spectral measure associated to π as in Proposition 2.61. Applying Theorem 2.65, we find a partition $\{P_n \mid n \in \mathbb{N} \cup \{\infty\}\}$ and measures $\mu_n = \mu_{\max}|_{P_n}$ so that

$$\mathcal{H}_\pi \cong \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} L^2_{\mu_n}(P_n, \mathcal{V}_n), \tag{2.33}$$

where we again write $\mathcal{V}_n = \mathbb{C}^n$ for $n \in \mathbb{N}$, $\mathcal{V}_\infty = \ell^2(\mathbb{N})$, $L^2_{\mu_n}(P_n, \mathcal{V}_n)$ for the space of square-integrable $\mathcal{V}_n$-valued functions on P_n for $n \in \mathbb{N} \cup \{\infty\}$, and the representation on the right is the multiplication representation M.

Proposition 2.68 (The centralizer of π). *With the above assumptions the centralizer of π,*

$$C(\pi) = \{B \in \mathrm{B}(\mathcal{H}_\pi) \mid B \textit{ is equivariant}\}$$

corresponds under the isomorphism 2.33 *to the set of all operators*

$$T = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} T_n$$

with the following properties:

- *For $n \in \mathbb{N}$ the operator T_n can be identified with an n-by-n matrix having entries in the ring $L^\infty_{\mu_n}(P_n)$. More formally, $T_n \in \mathrm{Mat}_{n,n}\big(L^\infty_{\mu_n}(P_n)\big)$ maps $v \in L^2_{\mu_n}(P_n, \mathcal{V}_n)$ to $T_n v \in L^2_{\mu_n}(P_n, \mathcal{V}_n)$ defined by*

$$(T_n v)(t) = T_n(t)v(t) \in \mathcal{V}_n = \mathbb{C}^n$$

  for $t \in P_n$.
- *The operator T_∞ can be defined by a measurable map $T_\infty \colon P_\infty \to \mathrm{B}(\mathcal{V}_\infty)$ such that*

$$(T_\infty v)(t) = T_\infty(t)v(t)$$

  for $t \in P_\infty$ and $v \in L^2_{\mu_\infty}(P_\infty, \mathcal{V}_\infty)$.
- *We have $\|T\| = \sup_{n \in \mathbb{N} \cup \{\infty\}} \operatorname{ess\,sup}_{P_n} \|T_n(\cdot)\|_{\mathrm{op}} < \infty$.*

We note that measurability of the map $T_\infty \colon P_\infty \to \mathrm{B}(\mathcal{V}_\infty)$ is defined by measurability of the inner products

$$P_\infty \ni t \longmapsto \langle T_\infty(t)v, w\rangle_{\mathcal{V}}$$

for all pairs $v, w \in \mathcal{V}_\infty$ (also see Exercise 2.69).

Exercise 2.69. Show that the Borel σ-algebra $\mathcal{B}_\tau$ generated by open sets on $\mathrm{B}(\mathcal{V}_\infty)$ with respect to the strong operator topology τ and the Borel σ-algebra $\mathcal{B}_{\tau_w}$ generated by open sets on $\mathrm{B}(\mathcal{V}_\infty)$ with respect to the weak operator topology τ_w coincide. Also show that the above notion of measurability agrees with measurability of the map $t \mapsto T_\infty(t)$ with respect to the σ-algebra $\mathcal{B}_\tau = \mathcal{B}_{\tau_w}$.

Proof of Proposition 2.68. To simplify notation, we assume that the isomorphism in (2.33) is an identity. Suppose that T is in the centralizer of π.

Linearity over $\mathscr{L}^\infty(\widehat{G})$. For $F \in \mathscr{L}^\infty(\widehat{G})$ and

$$v \in \mathcal{H}_\pi = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} L^2_{\mu_n}(P_n, \mathcal{V}_n)$$

we have, by the properties of the measurable functional calculus in Proposition 2.55(5) and (6), that T also commutes with $\pi_{\mathrm{FC}}(F) = M_F$, so that

$$T(Fv) = T(M_F v) = M_F T(v) = FT(v). \tag{2.34}$$

Put in algebraic terms, we see that T is linear with respect to the $\mathscr{L}^\infty(\widehat{G})$-module structure of $\mathcal{H}_\pi$. Below we will upgrade this to the statement that T itself has a pointwise definition, as in the proposition.

Applying this for the characteristic functions of the elements of the partition $\{P_n \mid n \in \mathbb{N} \cup \{\infty\}\}$ we obtain

$$T(L^2_{\mu_n}(P_n, \mathcal{V}_n)) = T(M_{\mathbb{1}_{P_n}} \mathcal{H}_\pi) = M_{\mathbb{1}_{P_n}} T(\mathcal{H}_\pi) \subseteq L^2_{\mu_n}(P_n, \mathcal{V}_n)$$

for $n \in \mathbb{N} \cup \{\infty\}$. We will describe the restriction of T to $L^2_{\mu_n}(P_n, \mathcal{V}_n)$ for every $n \in \mathbb{N} \cup \{\infty\}$.

Matrix coefficients lie in $L^\infty_{\mu_n}(P_n)$. We fix $n \in \mathbb{N} \cup \{\infty\}$ and two vectors $v, w \in \mathcal{V}_n$ that will be considered as constant functions on P_n belonging to $L^2_{\mu_n}(P_n, \mathcal{V}_n)$. We will now describe the inner product $\langle (Tv)(t), w \rangle_{\mathcal{V}}$ as a function of $t \in P_n$. In fact, we claim that

$$\Big(P_n \ni t \longmapsto F_{v,w}(t) = \langle (Tv)(t), w \rangle_{\mathcal{V}}\Big) \in L^\infty_{\mu_n}(P_n)$$

and, moreover,

$$\|F_{v,w}\|_\infty \leqslant \|T\|_{\mathrm{op}} \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}}. \tag{2.35}$$

For this, we define

$$B = \{t \in P_n \mid |F_{v,w}(t)| > \|T\|_{\mathrm{op}} \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}}\}, \tag{2.36}$$

set $f = \mathbb{1}_B$ and $f_2 = \arg(F_{v,w})\mathbb{1}_B$. Then $f_1, f_2 \in L^\infty_{\mu_n}(P_n) \cap L^2_{\mu_n}(P_n)$ with $\|f_1\|_2 = \|f_2\|_2 = \mu(B)^{\frac{1}{2}}$. Together with (2.34), we deduce that the integral in

$$\begin{aligned} \langle T f_1 v, f_2 w \rangle_{L^2_{\mu_n}(P_n, \mathcal{V}_n)} &= \langle f_1 T v, f_2 w \rangle_{L^2_{\mu_n}(P_n, \mathcal{V}_n)} \\ &= \int_B f_1(t) \overline{f_2(t)} \underbrace{\langle (Tv)(t), w \rangle_{\mathcal{V}}}_{= F_{v,w}} \,\mathrm{d}\mu_n(t) \\ &= \int_B |F_{v,w}| \,\mathrm{d}\mu_n \end{aligned}$$

exists for $v, w \in \mathcal{V}_n$ and is bounded by

$$\begin{aligned}\|T\|_{\mathrm{op}}\|f_1 v\|_2\|f_2 w\|_2 &\leqslant \|T\|_{\mathrm{op}}\|f_1\|_2\|v\|_{\mathcal{V}}\|f_2\|_2\|w\|_{\mathcal{V}}\\ &\leqslant \|T\|_{\mathrm{op}}\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}}\mu(B).\end{aligned}$$

Combining the last two facts and recalling the definition of B in (2.36), this implies that $\mu(B)=0$, and hence (2.35).

THE CASE OF $n\in\mathbb{N}$. We now fix some $n\in\mathbb{N}$ and define

$$T_n(t)=\big((Te_1)(t),(Te_2)(t),\dots,(Te_n)(t)\big)\in\mathrm{Mat}_{n,n}(\mathbb{C})$$

for $t\in P_n$ or, keeping $t\in P_n$ implicit,

$$T_n=(Te_1,Te_2,\dots,Te_n),$$

where we again write e_k for the kth basis vector of $\mathcal{V}_n$ considered as a constant function in $L^2_{\mu_n}(P_n,\mathcal{V}_n)$. Writing Te_k as a column vector belonging to $L^2_{\mu_n}(P_n,\mathcal{V}_n)$, we obtain from the above that $T_n\in\mathrm{Mat}_{n,n}\big(L^\infty_{\mu_n}(P_n)\big)$. Indeed, for $j,k\in\{1,\dots,n\}$ the (j,k)th matrix entry is given by

$$T_{n,j,k}=\langle Te_k,e_j\rangle_{\mathcal{V}}\in L^\infty_{\mu_n}(P_n).$$

It follows that we can use T_n to define a bounded operator on $L^2_{\mu_n}(P_n,\mathcal{V}_n)$ by

$$L^2_{\mu_n}(P_n,\mathcal{V}_n)\ni\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}\longmapsto T_n\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}=\begin{pmatrix}T_{n,1,1}v_1+\dots+T_{n,1,n}v_n\\ \vdots\\ T_{n,n,1}v_1+\dots+T_{n,n,n}v_n\end{pmatrix}.$$

To see that the operator T restricted to $L^2_{\mu_n}(P_n,\mathcal{V}_n)$ agrees with the operator defined by T_n, we suppose that $v=\sum_k v_ke_k\in L^2_{\mu_n}(P_n,\mathcal{V}_n)$. If we assume in addition that $v_k\in L^\infty_{\mu_n}(P_n)$ for $k=1,\dots,n$, then we can use (2.34) to obtain

$$Tv=\sum_k v_kT(e_k)=\sum_{j,k}v_k\underbrace{\langle T(e_k),e_j\rangle_{\mathcal{V}}}_{T_{n,j,k}}e_j=T_nv.$$

Since $L^\infty_{\mu_n}(P_n)$ is a dense subspace of $L^2_{\mu_n}(P_n)$, we deduce that T restricted to $L^2_{\mu_n}(P_n,\mathcal{V}_n)$ coincides with the operator defined above by the matrix T_n.

THE CASE $n=\infty$. Next we study the case $n=\infty$. Just as in the finite-dimensional case, we will use T to define operators $T_\infty(t)\in\mathrm{B}(\mathcal{V}_\infty)$ for almost every $t\in P_\infty$, use these to define an operator T_∞ on $L^2_{\mu_\infty}(P_\infty,\mathcal{V}_\infty)$, and show that T restricted to $L^2_{\mu_\infty}(P_\infty,\mathcal{V}_\infty)$ is equal to T_∞. For this, we let

$$\mathcal{W}=\{w_1,w_2,\dots\}$$

be a dense subset of $\mathcal{V}_\infty$ that is also a subspace over the field $\mathbb{Q}[\mathrm{i}]$. Fixing indices $j,k\in\mathbb{N}$, considering $w_j,w_k\in L^2_{\mu_\infty}(P_\infty,\mathcal{V}_\infty)$ again as constant functions on P_∞, and using (2.35) we see that $\langle (Tw_j)(t),w_k\rangle_{\mathcal{V}}\in L^\infty_{\mu_\infty}(P_\infty)$ is bounded by $\|T\|_{\mathrm{op}}\|w_j\|_{\mathcal{V}}\|w_k\|_{\mathcal{V}}$ for almost every $t\in\mathcal{P}_\infty$. Moreover, we also have that $\langle (Tw_j)(t),w_k\rangle_{\mathcal{V}}$ depends linearly on the vectors w_j,w_k as elements of the vector space $\mathcal{W}$ over $\mathbb{Q}[\mathrm{i}]$, almost surely. Collecting these countably many null sets, applying density of $\mathcal{W}$ in $\mathcal{V}_\infty$, and applying the Riesz representation theorem on $\mathcal{V}_\infty$, it follows that for almost every $t\in P_\infty$ we have

$$\langle (Tw_j)(t),w_k\rangle_{\mathcal{V}}=\langle T_\infty(t)w_j,w_k\rangle_{\mathcal{V}}$$

for some bounded operator $T_\infty(t)\in\mathrm{B}(\mathcal{V}_\infty)$ satisfying $\|T_\infty(t)\|_{\mathrm{op}}\leqslant\|T\|_{\mathrm{op}}$. Measurability of $P_\infty\ni t\mapsto T_\infty(t)$ (that is, of the map $t\mapsto\langle T_\infty(t)v,w\rangle_{\mathcal{V}}$ for all $v,w\in\mathcal{V}_\infty$) follows once again from density of $\mathcal{W}$.

We let $e_1,e_2,\ldots$ be the standard orthonormal basis of $\mathcal{V}_\infty$ and may assume that $e_k\in\mathcal{W}$ for $k\in\mathbb{N}$. Similarly to the finite-dimensional case we can now use $T_\infty(t)$ for $t\in P_\infty$ to define a bounded operator T_∞ on $L^2_{\mu_\infty}(P_\infty,\mathcal{V}_\infty)$ by sending $\sum_k v_ke_k\in L^2_{\mu_\infty}(P_\infty,\mathcal{V}_\infty)$ to

$$\Bigl(T_\infty\Bigl(\sum_k v_ke_k\Bigr)\Bigr)(t)=\sum_{j,k}\langle T_\infty(t)e_k,e_j\rangle_{\mathcal{V}}v_k(t)e_j \tag{2.37}$$

for all $t\in P_\infty$. Indeed, we first note that each summand on the right is measurable by construction and that

$$\begin{aligned}\sum_j\Bigl|\Bigl\langle\sum_k T_\infty(t)v_k(t)e_k,e_j\Bigr\rangle_{\mathcal{V}}\Bigr|^2&=\Bigl\|T_\infty(t)\sum_k v_k(t)e_k\Bigr\|^2_{\mathcal{V}}\\&\leqslant\|T\|_{\mathrm{op}}\Bigl\|\sum_k v_k(t)e_k\Bigr\|^2_{\mathcal{V}}\end{aligned}$$

for almost every $t\in P_\infty$. Moreover,

$$\int_{P_\infty}\Bigl\|\sum_k v_k(t)e_k\Bigr\|^2_{\mathcal{V}}\,\mathrm{d}\mu_\infty(t)<\infty$$

by assumption on $\sum_k v_ke_k$. This shows that (2.37) gives a well-defined operator with norm $\|T_\infty\|_{\mathrm{op}}\leqslant\|T\|_{\mathrm{op}}$.

It remains to see that the operator T_∞ coincides with T when it is restricted to $L^2_{\mu_\infty}(P_\infty,\mathcal{V}_\infty)$. The construction of T_∞ implies that

$$\bigl\langle (Te_k)(t),e_j\bigr\rangle_{\mathcal{V}}=\bigl\langle T_\infty(t)e_k,e_j\bigr\rangle_{\mathcal{V}}$$

for almost every $t\in P_\infty$ and all $j,k\in\mathbb{N}$. This implies that

$$(Te_k)(t)=T_\infty(t)e_k$$

for all $k \in \mathbb{N}$. For $N \in \mathbb{N}$ and $f_1, \dots, f_N \in L^\infty_{\mu_\infty}(P_\infty)$ we now have by (2.34) that

$$T\Big(\sum_{k\leqslant N} f_k e_k\Big) = \sum_{k\leqslant N} f_k T e_k = \sum_{k\leqslant N} f_k T_\infty e_k = T_\infty\Big(\sum_{k\leqslant N} f_k e_k\Big).$$

Varying $N \in \mathbb{N}$ and the functions in $L^\infty_{\mu_\infty}(P_\infty)$, this gives a dense subset of $L^2_{\mu_\infty}(P_\infty, \mathcal{V}_\infty)$. It follows that T_∞ is indeed the restriction of T to $L^2_{\mu_\infty}(P_\infty, \mathcal{V}_\infty)$.

The operator norms. We note that we could also have used the above argument for $n \in \mathbb{N}$, so that we also have $\|T_n(t)\|_{\mathrm{op}} \leqslant \|T\|_{\mathrm{op}}$ for almost every $t \in P_n$ and $n \in \mathbb{N}$. In particular, we have

$$\sup_{n\in\mathbb{N}\cup\{\infty\}} \operatorname*{ess\,sup}_{t\in P_n} \|T_n(t)\|_{\mathrm{op}} \leqslant \|T\|_{\mathrm{op}}.$$

To see the opposite inequality, let S denote the supremum on the left and let $v, w \in \mathcal{H}_\pi$. Then

$$\begin{aligned}|\langle Tv, w\rangle| &\leqslant \sum_{n\in\mathbb{N}\cup\{\infty\}} \int_{P_n} |\langle T_n(t)v(t), w(t)\rangle_{\mathcal{V}}| \,\mathrm{d}\mu_{\max}(t)\\ &\leqslant \int_{\widehat{G}} S\|v(t)\|_{\mathcal{V}}\|w(t)\|_{\mathcal{V}} \,\mathrm{d}\mu_{\max}(t) \leqslant S\|v\|\|w\|\end{aligned}$$

by the description of $\mathcal{H}_\pi$, the pointwise definition of T obtained above, the Cauchy–Schwarz inequality on $\mathcal{V}_n$ for $n \in \mathbb{N}\cup\{\infty\}$, and finally the Cauchy–Schwarz inequality on $L^2_{\mu_{\max}}(\widehat{G})$. However, this implies that $\|T\|_{\mathrm{op}} \leqslant S$.

The converse. The converse statement that any $T = \bigoplus_{n\in\mathbb{N}\cup\{\infty\}} T_n$ as in the proposition defines a bounded operator we leave as an exercise (see Exercise 2.70). As the unitary representation corresponds under the isomorphism to the multiplication representation, it follows that a so-defined operator T belongs to the centralizer of π. This gives the proposition. □

Exercise 2.70. Show that any operator $T = \bigoplus_{n\in\mathbb{N}\cup\{\infty\}} T_n$ as in Proposition 2.68 defines a bounded equivariant operator by the following steps.
(a) Let $n \in \mathbb{N}$, let (X, μ) be a finite measure space and let T lie in $\mathrm{Mat}_{n,n}\big(L^\infty_\mu(X)\big)$. Show that T induces a bounded operator on $L^2_\mu\big(X, \mathcal{V}_n\big)$ satisfying $(Tv)(t) = T(t)v(t)$ for $v \in L^2_\mu\big(X, \mathcal{V}_n\big)$ and almost every $t \in X$ (where we use matrix multiplication on the right), and that $\|T\|_{\mathrm{op}} = \operatorname{ess\,sup}_{t\in X} \|T(t)\|_{\mathrm{op}}$.
(b) Let $n = \infty$ and let (X, μ) be a finite measure space. Show that a measurable map

$$T\colon X \ni t \longmapsto \mathrm{B}(\mathcal{V}_\infty)$$

with $\operatorname{ess\,sup}_{t\in X} \|T(t)\|_{\mathrm{op}} < \infty$ induces a bounded operator T on $L^2_\mu\big(X, \mathcal{V}_\infty\big)$ satisfying $(Tv)(t) = T(t)v(t)$ for $v \in L^2_\mu\big(X, \mathcal{V}_\infty\big)$ and almost every $t \in \widehat{G}$, and that T satisfies $\|T\|_{\mathrm{op}} = \operatorname{ess\,sup}_{t\in X} \|T(t)\|_{\mathrm{op}}$.

(c) Let μ be a finite measure on $\widehat{G}$, and let $\{P_n \mid n \in \mathbb{N} \cup \{\infty\}\}$ be a countable measurable partition of $\widehat{G}$. Suppose T_n is defined as in (a) or (b), using $\mu_n = \mu|_{P_n}$ for $n \in \mathbb{N} \cup \{\infty\}$. Suppose, moreover, that

$$\sup_{n \in \mathbb{N} \cup \{\infty\}} \operatorname*{ess\,sup}_{t \in P_n} \|T(t)\|_{\mathrm{op}} < \infty.$$

Show that this implies that $T = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} T_n$ is a bounded operator on

$$\bigoplus_{n \in \mathbb{N} \cup \{\infty\}} L^2_{\mu_n}(P_n, \mathcal{V}_n)$$

that is equivariant for the canonical multiplication representation of G.

2.9 Summary and Outlook

The rather complete understanding of unitary representations of abelian groups in terms of:

- spectral measures, which completely encode matrix coefficients;
- the measurable functional calculus, which allows us to isolate parts of the spectrum at will;
- the spectral theorem with complete multiplicity data; and
- their centralizer

obtained in this chapter is rewarding and important in itself. However, it will also be the key for understanding the unitary dual and unitary representations of other groups. Most notably, this applies to metabelian groups (that is, semi-direct products of two abelian groups) as discussed in Chapter 3. Moreover, the abelian theory (together with Section 3.1) will also be important for understanding certain aspects of unitary representations of semi-simple groups like $\mathrm{SL}_3(\mathbb{R})$ as we will see in Chapter 7.

Pontryagin duality and the Plancherel formula are useful tools for constructing and understanding concrete abelian groups. Moreover, local fields as introduced in Section 2.4.5 are of fundamental importance to modern number theory.(3)

The reader may continue with Chapter 3 or 5, returning to Chapters 3 and 4 when needed.