# Chapter 9
# Unitary Representations of SL(2)

In this chapter we will again focus our attention on $\mathrm{SL}_2(\mathbb{R})$ and its unitary representations. For this, we recall some of our previous discussions:

- We already classified all finite-dimensional representations of $\mathrm{SL}_2(\mathbb{R})$ in Section 6.1. This result, and even more so the method of proof, will be of interest here.
- In Section 8.3 we recalled the geometric significance of $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SU}_{1,1}(\mathbb{R})$ by connecting it to the hyperbolic plane $\mathbb{H} \cong \mathbb{D}$.
- In Section 8.4 we already found our first two types of non-trivial irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$, namely the discrete series and the mock discrete series representations.
- In Section 8.5 we studied the regular representation of $\mathrm{SL}_2(\mathbb{R})$.
- This allowed us to characterize temperedness for $\mathrm{SL}_2(\mathbb{R})$ in terms of integrability and decay of matrix coefficients in Section 8.6.

We will extend these results here, to obtain a complete description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$. Moreover, we will decompose natural unitary representations into irreducible representations.[(18)] In particular, we will study the Koopman representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{H})$, and see that the 'hyperbolic Fourier transform' is intimately related to the principal series representations of $\mathrm{SL}_2(\mathbb{R})$. Finally, the complementary series representation will allow a better understanding of (the lack of) spectral gap, decay rates, and integrability exponents for $\mathrm{SL}_2(\mathbb{R})$.

## 9.1 The Universal Enveloping Algebra and Smooth Vectors

We recall that the Casimir operator for $\mathrm{SU}_2(\mathbb{R})$ in Proposition 7.21 and Corollary 7.22 commutes with $\mathrm{SU}_2(\mathbb{R})$. Because of the connection between $\mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ developed in Section 6.1.2, it stands to reason that $\mathrm{SL}_2(\mathbb{R})$ should also possess a Casimir operator. However, the Casimir operator of $\mathrm{SL}_2(\mathbb{R})$

will have ‘mixed signature’, and will not arise from an application of Proposition 7.21.

After developing the necessary abstract machinery in this section, it will also be relatively straightforward to define the raising and lowering operators for any unitary representation of $\mathrm{SL}_2(\mathbb{R})$, which will lead to the description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ in Section 9.2.

### 9.1.1 The Universal Enveloping Algebra

We briefly introduce the algebra $\mathfrak{E}$ containing the Lie algebra $\mathfrak{g}$ of a Lie group $G$ as well as higher-order terms like the Casimir elements. We refer to Knapp [44, Ch. 3] for a more careful introduction to this concept.

**Definition 9.1 (The algebra $\mathfrak{E}$).** The *universal enveloping algebra* $\mathfrak{E}$ of a Lie algebra $\mathfrak{g}$ is the linear hull of all formal multi-linear associative non-commuting products $\mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n$ for $n \in \mathbb{N}_0$ and $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathfrak{g}$ modulo the ideal generated by the expressions $\mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a} - [\mathbf{a}, \mathbf{b}]$ for $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$. By also allowing the empty product $\mathbb{1}_{\mathfrak{E}}$ (corresponding to $n = 0$) the algebra $\mathfrak{E}$ is also unital. We will write $\mathbf{b}^{\circ n} = \mathbf{b} \circ \mathbf{b} \circ \cdots \circ \mathbf{b}$ for powers of $\mathbf{b} \in \mathfrak{E}$ in the algebra $\mathfrak{E}$ for all $n \in \mathbb{N}$, and define $\mathbf{b}^{\circ 0}$ to be $\mathbb{1}_{\mathfrak{E}}$ for all $\mathbf{b} \in \mathfrak{E}$.

At first sight this definition might look very much like abstract nonsense. However, as we will see, we should think of $\mathfrak{E}$ as the algebra of all partial differential operators that can be obtained by composition from the first order differential operators that correspond to elements of $\mathfrak{g}$.

Because of its definition, $\mathfrak{E}$ is not a graded algebra (that is, there is no good definition of homogeneous degree in $\mathfrak{E}$), since the generators of the ideal in its definition have terms of different degree. However, it can be written as an increasing union

$$\mathfrak{E} = \bigcup_{d=0}^{\infty} \mathfrak{E}_{\leqslant d},$$

where $\mathfrak{E}_{\leqslant 0} = \mathbb{C}\mathbb{1}_{\mathfrak{E}}$, $\mathfrak{E}_{\leqslant 1} = \mathfrak{E}_{\leqslant 0} + \mathfrak{g}$, and $\mathfrak{E}_{\leqslant d}$ is the subspace of $\mathfrak{E}$ generated by all products $\mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n$ with $n \leqslant d$ and $\mathbf{b}_1, \dots, \mathbf{b}_n \in \mathfrak{g}$. We say that $\mathbf{e} \in \mathfrak{E}$ has degree $d \in \mathbb{N}$ if $\mathbf{e} \in \mathfrak{E}_{\leqslant d} \smallsetminus \mathfrak{E}_{\leqslant d-1}$, and has degree 0 if $\mathbf{e} \in \mathfrak{E}_{\leqslant 0}$.

We note that the algebra $\mathfrak{E}$ is *enveloping* in the sense that it contains the Lie algebra $\mathfrak{g}$, and that it is *universal* in the sense that any Lie algebra representation of $\mathfrak{g}$ (sending Lie brackets to Lie brackets as in (6.7)) can be extended to an algebra representation of $\mathfrak{E}$ (sending products to products). This functorial property holds essentially by definition of $\mathfrak{E}$.

However, let us start by extending the adjoint representation to $\mathfrak{E}$. We define $\mathrm{Ad}_g(\mathbb{1}_{\mathfrak{E}}) = \mathbb{1}_{\mathfrak{E}}$,

$$\mathrm{Ad}_g(\mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n) = (\mathrm{Ad}_g \mathbf{b}_1) \circ \cdots \circ (\mathrm{Ad}_g \mathbf{b}_n)$$

for all $g \in G$, $n \in \mathbb{N}$, and $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathfrak{g}$ and extend $\mathrm{Ad}_g$ linearly to all of $\mathfrak{E}$. For this we need to point out that

$$\begin{aligned}\mathrm{Ad}_g(\mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a} - [\mathbf{a}, \mathbf{b}]) &= (\mathrm{Ad}_g \mathbf{a}) \circ (\mathrm{Ad}_g \mathbf{b}) - (\mathrm{Ad}_g \mathbf{b}) \circ (\mathrm{Ad}_g \mathbf{a}) \\ &\qquad - \mathrm{Ad}_g([\mathbf{a}, \mathbf{b}]) \\ &= (\mathrm{Ad}_g \mathbf{a}) \circ (\mathrm{Ad}_g \mathbf{b}) - (\mathrm{Ad}_g \mathbf{b}) \circ (\mathrm{Ad}_g \mathbf{a}) \\ &\qquad - [\mathrm{Ad}_g(\mathbf{a}), \mathrm{Ad}_g(\mathbf{b})]\end{aligned}$$

for $g \in G$ and $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$. Hence the adjoint action of $G$ on the algebra of formal products sends the ideal appearing in Definition 9.1 to itself, and we obtain a well-defined representation of $G$ on $\mathfrak{E}$ and, by restriction. also on $\mathfrak{E}_{\leqslant d}$ for all $d \in \mathbb{N}_0$.

By the discussion in Section 6.1.3, we may also take the derivative of the adjoint representation of $G$ on $\mathfrak{E}$ (or, said more carefully, on $\mathfrak{E}_{\leqslant d}$ for all $d \in \mathbb{N}_0$) to obtain a representation of $\mathfrak{g}$ on $\mathfrak{E}$. We will again call this representation the adjoint representation, and denote it by

$$\begin{aligned}\mathrm{ad} \colon \mathfrak{g} &\longrightarrow \mathrm{End}(\mathfrak{E}) \\ \mathbf{c} &\longmapsto \mathrm{ad}_{\mathbf{c}}\,.\end{aligned}$$

In fact, for $n \in \mathbb{N}$, $\mathbf{b}_1, \ldots, \mathbf{b}_n, \mathbf{c} \in \mathfrak{g}$ the adjoint representation satisfies

$$\begin{aligned}\mathrm{ad}_{\mathbf{c}}(\mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n) &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big(\mathrm{Ad}_{\exp(t\mathbf{c})} \mathbf{b}_1\big) \circ \cdots \circ \big(\mathrm{Ad}_{\exp(t\mathbf{c})} \mathbf{b}_n\big) \\ &= (\mathrm{ad}_{\mathbf{c}} \mathbf{b}_1) \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n + \cdots + \mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ (\mathrm{ad}_{\mathbf{c}} \mathbf{b}_n). \qquad (9.1)\end{aligned}$$

This generalized product rule follows, for example, by restricting to $\mathfrak{E}_{\leqslant d}$ for some $d \geqslant n$, applying the approximation formula

$$\mathrm{Ad}_{\exp(t\mathbf{c})}(\mathbf{b}_j) = \mathbf{b}_j + t\,\mathrm{ad}_{\mathbf{c}}(\mathbf{b}_j) + \mathrm{O}(t^2)$$

for $t \to 0$ and $j \in \{1, \ldots, n\}$, using multi-linearity to expand the product above, and letting $t$ go to zero.

We note that substituting the relation

$$\mathrm{ad}_{\mathbf{c}}(\mathbf{b}_j) = [\mathbf{c}, \mathbf{b}_j] = \mathbf{c} \circ \mathbf{b}_j - \mathbf{b}_j \circ \mathbf{c}$$

for $j = 1, \ldots, n$ into (9.1) gives the telescoping sum

$$\begin{aligned}\mathrm{ad}_{\mathbf{c}}(\mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n) &= (\mathbf{c} \circ \mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n - \mathbf{b}_1 \circ \mathbf{c} \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n) \\ &\quad + (\mathbf{b}_1 \circ \mathbf{c} \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n - \mathbf{b}_1 \circ \mathbf{b}_2 \circ \mathbf{c} \circ \cdots \circ \mathbf{b}_n) \\ &\quad + \cdots \\ &\quad + (\mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{c} \circ \mathbf{b}_n - \mathbf{b}_1 \circ \mathbf{b}_2 \circ \cdots \circ \mathbf{b}_n \circ \mathbf{c}) \\ &= \mathbf{c} \circ \mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n - \mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n \circ \mathbf{c},\end{aligned}$$

which by linearity shows that

$$\mathrm{ad}_{\mathbf{c}}(\mathbf{e}) = \mathbf{c} \circ \mathbf{e} - \mathbf{e} \circ \mathbf{c} \tag{9.2}$$

for $\mathbf{c} \in \mathfrak{g}$ and $\mathbf{e} \in \mathfrak{E}$.

To get a better feeling for $\mathfrak{E}$, we describe how to obtain a basis for it. For this, suppose that $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathfrak{g}$ form a basis of the Lie algebra $\mathfrak{g}$. Then, by the Poincaré–Birkhoff–Witt Theorem (see Knapp [44, Th. 3.8]) the products

$$\mathbf{b}_1^{\circ k_1} \circ \cdots \circ \mathbf{b}_n^{\circ k_n} \tag{9.3}$$

for $(k_1, \ldots, k_n) \in \mathbb{N}_0^n$ form a basis of $\mathfrak{E}$. We will not need or prove this in detail, but wish to indicate briefly why this should be true.

Firstly, because of the assumed multi-linearity of the products appearing in Definition 9.1, it is clear that $\mathfrak{E}$ is the linear hull of $\mathbb{1}_{\mathfrak{E}}$, $\mathbf{b}_1,\ldots,\mathbf{b}_n$, and all non-commuting products of $\mathbf{b}_1,\ldots,\mathbf{b}_n$ in any order and multiplicity. However, using the ideal appearing in Definition 9.1 we may swap two basis elements in such a product, possibly at the cost of adding a term of lower degree. This allows us to order the basis elements in a product so that they are of the form in (9.3). Using this and an induction on the degree, we arrive at the statement that $\mathfrak{E}$ is the linear hull of products as in (9.3) for $(k_1, \ldots, k_n) \in \mathbb{N}_0^n$. In a way the ideal appearing in Definition 9.1 precisely allows us to do this re-ordering of arbitrary products of basis elements to transform them to the shape of (9.3), but does not allow anything else. This suggests the second half of the Poincaré–Birkhoff–Witt Theorem, namely the statement that the products in (9.3) are all linearly independent within $\mathfrak{E}$.

### 9.1.2 The Casimir Element for $\mathfrak{sl}_2(\mathbb{R})$

As it is our goal in this chapter to describe the unitary dual $\widehat{\mathrm{SL}_2(\mathbb{R})}$, and the notion of universal enveloping algebra is meant to be a tool for that goal, it is only natural that we want to study the universal enveloping algebra $\mathfrak{E}$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. This will, in particular, reveal that $\mathfrak{E}$ has some interesting central elements.

Hence we let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. We will use the basis elements $\mathbf{a}, \mathbf{e}, \mathbf{f}$ as in (6.2). These form an $\mathfrak{sl}_2$-triple as they satisfy the relations $[\mathbf{a}, \mathbf{e}] = 2\mathbf{e}$, $[\mathbf{a}, \mathbf{f}] = -2\mathbf{f}$, and $[\mathbf{e}, \mathbf{f}] = \mathbf{a}$ by (6.3). In addition to these, we will also use the Lie algebra elements

$$\mathbf{d} = \mathbf{e} + \mathbf{f} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R})$$

and

$$\mathbf{k} = -\mathbf{e} + \mathbf{f} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}).$$

We will define the central element $\Omega$ of the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{sl}_2(\mathbb{R})$ directly below. However, we first wish to explain an abstract argument (relying on the Poincaré–Birkhoff–Witt Theorem and the theory of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$ in Section 6.1) showing that such a central element of degree two has to exist.

For this, note first that by the Poincaré–Birkhoff–Witt Theorem $\mathfrak{E}_{\leqslant 2}$ has the 10 elements

$$\mathbb{1}_{\mathfrak{E}}, \mathbf{a}, \mathbf{e}, \mathbf{f}, \mathbf{a}^{\circ 2}, \mathbf{a} \circ \mathbf{e}, \mathbf{a} \circ \mathbf{f}, \mathbf{e}^{\circ 2}, \mathbf{e} \circ \mathbf{f}, \mathbf{f}^{\circ 2}$$

as a basis. To understand $\mathfrak{E}_{\leqslant 2}$ as a finite-dimensional representation for the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, we have to find its highest weight vectors. Clearly $\mathbb{1}_{\mathfrak{E}}$ is a highest weight vector corresponding to weight 0 and spanning a trivial sub-representation. Next we note that $\mathbf{e}$ is a highest weight vector (since we know that $\mathrm{ad}_{\mathbf{a}}(\mathbf{e}) = 2\mathbf{e}$ and $\mathrm{ad}_{\mathbf{e}}(\mathbf{e}) = 0$). Also note that $\mathbf{e}$ generates the sub-representation $\mathfrak{g} \subseteq \mathfrak{E}_{\leqslant 2}$. Moreover, $\mathbf{e}^{\circ 2}$ is another highest weight vector, since

$$\mathrm{ad}_{\mathbf{a}}(\mathbf{e} \circ \mathbf{e}) = \mathrm{ad}_{\mathbf{a}}(\mathbf{e}) \circ \mathbf{e} + \mathbf{e} \circ \mathrm{ad}_{\mathbf{a}}(\mathbf{e}) = 4\mathbf{e} \circ \mathbf{e}$$

and

$$\mathrm{ad}_{\mathbf{e}}(\mathbf{e} \circ \mathbf{e}) = \mathrm{ad}_{\mathbf{e}}(\mathbf{e}) \circ \mathbf{e} + \mathbf{e} \circ \mathrm{ad}_{\mathbf{e}}(\mathbf{e}) = 0.$$

Since $\mathbf{e}^{\circ 2}$ has weight 4, it generates a 5-dimensional irreducible subspace $\mathcal{V}_5$ inside $\mathfrak{E}_{\leqslant 2}$. Together we see that the sub-representations $\mathbb{C}\mathbb{1}_{\mathfrak{E}}$, $\mathfrak{g}$, and $\mathcal{V}_5$ found so far have dimension 1, 3, and 5 respectively. As $\mathfrak{E}_{\leqslant 2}$ has dimension 10 we see that there exists a one-dimensional invariant complement $\mathbb{C}\Omega$ to

$$\mathbb{C}\mathbb{1}_{\mathfrak{E}} \oplus \mathfrak{g} \oplus \mathcal{V}_5 = \mathfrak{E}_{\leqslant 1} \oplus \mathcal{V}_5,$$

on which $\mathfrak{sl}_2(\mathbb{R})$ acts trivially. This shows the existence of an element $\Omega$ of degree 2 with $\mathrm{ad}_{\mathbf{b}}(\Omega) = 0$ for all $\mathbf{b} \in \mathfrak{sl}_2(\mathbb{R})$. By (9.2) we may also write this as

$$\mathbf{b} \circ \Omega = \Omega \circ \mathbf{b}$$

for all $\mathbf{b} \in \mathfrak{sl}_2(\mathbb{R})$. Since $\mathfrak{E}$ is generated by $\mathbb{1}_{\mathfrak{E}}$ and $\mathfrak{sl}_2(\mathbb{R})$ as an algebra, we find that $\Omega$ belongs to the centre of $\mathfrak{E}$.

In order to be logically independent of the Poincaré–Birkhoff–Witt Theorem and, more importantly, to be able to do concrete calculations with $\Omega$, we now give a direct definition of $\Omega$.

**Lemma 9.2 (The Casimir element for $\mathfrak{sl}_2(\mathbb{R})$).** *Let* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ *and let* $\mathbf{a}$, $\mathbf{e}$, $\mathbf{f}$ *be the* $\mathfrak{sl}_2$*-triple from* (6.2). *Define* $\mathbf{d} = \mathbf{e} + \mathbf{f}$ *and* $\mathbf{k} = -\mathbf{e} + \mathbf{f}$. *Then*

$$\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{d}^{\circ 2} - \mathbf{k}^{\circ 2} = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + 2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}$$

*is a degree two element in the centre of* $\mathfrak{E}$.

Proof. We use the definitions of $\mathbf{d}$ and $\mathbf{k}$ to see that

$$\begin{aligned}
&\mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + (\mathbf{e}+\mathbf{f})^{\circ 2} - (-\mathbf{e}+\mathbf{f})^{\circ 2} \\
&\quad = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{e}^{\circ 2} + \mathbf{e}\circ\mathbf{f} + \mathbf{f}\circ\mathbf{e} + \mathbf{f}^{\circ 2} - \left(\mathbf{e}^{\circ 2} - \mathbf{e}\circ\mathbf{f} - \mathbf{f}\circ\mathbf{e} + \mathbf{f}^{\circ 2}\right) \\
&\quad = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + 2\mathbf{e}\circ\mathbf{f} + 2\mathbf{f}\circ\mathbf{e}.
\end{aligned}$$

Hence the two expressions in the lemma define the same element $\Omega$. Next we note that $\mathrm{ad}_{\mathbf{b}}(\mathbb{1}_{\mathfrak{E}}) = 0$ for all $\mathbf{b} \in \mathfrak{g}$. Hence we obtain, by the product rule in (9.1), that

$$\begin{aligned}
\mathrm{ad}_{\mathbf{a}}(\Omega) &= \mathrm{ad}_{\mathbf{a}}\left(\mathbf{a}^{\circ 2} + 2\mathbf{e}\circ\mathbf{f} + 2\mathbf{f}\circ\mathbf{e}\right) \\
&= 2(\mathrm{ad}_{\mathbf{a}}\,\mathbf{e})\circ\mathbf{f} + 2\mathbf{e}\circ(\mathrm{ad}_{\mathbf{a}}\,\mathbf{f}) + 2(\mathrm{ad}_{\mathbf{a}}\,\mathbf{f})\circ\mathbf{e} + 2\mathbf{f}\circ\mathrm{ad}_{\mathbf{a}}(\mathbf{e}) \\
&= 2(2\mathbf{e})\circ\mathbf{f} + 2\mathbf{e}\circ(-2\mathbf{f}) + 2(-2\mathbf{f})\circ\mathbf{e} + 2\mathbf{f}\circ(2\mathbf{e}) = 0
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{ad}_{\mathbf{e}}(\Omega) &= \mathrm{ad}_{\mathbf{e}}\left(\mathbf{a}^{\circ 2} + 2\mathbf{e}\circ\mathbf{f} + 2\mathbf{f}\circ\mathbf{e}\right) \\
&= \mathrm{ad}_{\mathbf{e}}(\mathbf{a})\circ\mathbf{a} + \mathbf{a}\circ\mathrm{ad}_{\mathbf{e}}(\mathbf{a}) + 2\mathbf{e}\circ(\mathrm{ad}_{\mathbf{e}}\,\mathbf{f}) + 2(\mathrm{ad}_{\mathbf{e}}\,\mathbf{f})\circ\mathbf{e} \\
&= -2\mathbf{e}\circ\mathbf{a} - 2\mathbf{a}\circ\mathbf{e} + 2\mathbf{e}\circ\mathbf{a} + 2\mathbf{a}\circ\mathbf{e} = 0.
\end{aligned}$$

The calculation $\mathrm{ad}_{\mathbf{f}}(\Omega) = 0$ is similar, but also follows from the properties of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$ in Section 6.1.4 applied to ad on $\mathfrak{E}_{\leqslant 2}$: Since $\mathrm{ad}_{\mathbf{a}}(\Omega) = \mathrm{ad}_{\mathbf{e}}(\Omega) = 0$ it follows that $\Omega \in \mathfrak{E}_{\leqslant 2}$ is a highest weight vector with weight zero, and so generates the trivial representation of $\mathfrak{sl}_2(\mathbb{R})$.

Applying (9.2) and using the fact that $\mathfrak{E}$ is generated by $\mathbb{1}_{\mathfrak{E}}$ and $\mathfrak{sl}_2(\mathbb{R})$ as an algebra, this implies (as discussed just before the lemma) that $\Omega$ belongs to the centre of $\mathfrak{E}$.

It remains to prove that $\Omega$ has degree two. Assume the opposite, so that $\Omega \in \mathfrak{E}_{\leqslant 1}$. Since $\mathfrak{E}_{\leqslant 1} = \mathbb{1}_{\mathfrak{E}} \oplus \mathfrak{g}$ and $\mathfrak{g}$ has no centre, this would imply that $\Omega = \alpha\mathbb{1}_{\mathfrak{E}}$ for some $\alpha \in \mathbb{C}$. To derive a contradiction from this, we apply the universal property of $\mathfrak{E}$: Any Lie algebra representation $\rho$ of $\mathfrak{sl}_2(\mathbb{R})$ on a real vector space $\mathcal{V}$ can be extended to an algebra representation of $\mathfrak{E}$. Indeed, we may simply define $\rho$ by $\rho(\mathbb{1}_{\mathfrak{E}}) = I$, composition as in

$$\rho(\mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n) = \rho(\mathbf{b}_1)\cdots\rho(\mathbf{b}_n)$$

for $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathfrak{g}$ and $n \in \mathbb{N}$, and linear extension. For this, note that the ideal appearing in Definition 9.1 is sent to 0 due to the definition of a Lie algebra homomorphism (see (6.7)).

We first apply this universal property to the trivial representation $\rho$ on $\mathbb{C}$. Here $\rho(\mathbf{b}) = 0$ for all $\mathbf{b} \in \mathfrak{g}$, which implies that $\rho(\Omega) = \rho(\mathbb{1}_{\mathfrak{E}}) = 1$. Hence we see that $\alpha = 1$.

Next we apply the universal property to the standard representation $\rho$ on $\mathcal{V} = \mathbb{C}^2$ and the vector

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In this case

$$\rho(\Omega)v = v + \mathbf{a}^2 v + \mathbf{d}^2 v - \mathbf{k}^2 v,$$

where $\mathbf{a}^2 = I$, $\mathbf{d}^2 = I$, $\mathbf{k}^2 = -I$ are just the matrix squares. This gives $\rho(\Omega)v = 4v$ and hence with $\alpha = 4$ a contradiction to the previous calculation. It follows that the central element $\Omega \in \mathfrak{E}_{\leqslant 2}$ is not a multiple of $\mathbb{1}_{\mathfrak{E}}$, does not belong to $\mathfrak{E}_{\leqslant 1}$, and hence has degree two as claimed. □

The reader who hopes to find more central elements in $\mathfrak{E}_{\leqslant d}$ for larger values of $d \in \mathbb{N}$, is invited to solve the following exercise.

**Essential Exercise 9.3.** Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, and assume the Poincaré–Birkhoff–Witt Theorem for its universal enveloping algebra.
(a) For every $d \in \mathbb{N}$, calculate the weight of $\mathbf{e}^{\circ d} \in \mathfrak{E}_{\leqslant d}$. Show that all other eigenvectors of $\mathfrak{E}_{\leqslant d}$ have smaller weight. Conclude that $\mathfrak{E}_{\leqslant d}$ contains an irreducible invariant subspace $\mathcal{V}_{2d+1}$ of dimension $(2d+1)$ that is not contained in $\mathfrak{E}_{\leqslant d-1}$.
(b) Calculate the dimension of $\mathfrak{E}_{\leqslant 3}$ and $\mathfrak{E}_{\leqslant 4}$, analyze the representation appearing, and show that the centre of $\mathfrak{E}$ intersected with $\mathfrak{E}_{\leqslant 4}$ is the linear hull of $\mathbb{1}, \Omega, \Omega^{\circ 2}$.
(c) Repeat (b) for all $\mathfrak{E}_{\leqslant d}$ with $d \in \mathbb{N}$ to see that the centre of $\mathfrak{E}$ is the linear hull of $\mathbb{1}_{\mathfrak{E}}$ and $\Omega^{\circ n}$ for $n \in \mathbb{N}$.

### 9.1.3 Higher-order Differential Operators

We return to the general case and now show, as promised, that $\mathfrak{E}$ can be thought of as the algebra of differential operators arising by composition from $\mathfrak{g}$.

**Proposition 9.4 (Differential operators coming from $\mathfrak{E}$).** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\pi$ be a unitary representation of $G$. Then the representation of $\mathfrak{g}$ via $\pi_\partial$ on smooth vectors extends to a representation of the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{g}$ on smooth vectors in such a way that*

$$\pi_\partial(\mathbf{b}_1 \circ \cdots \circ \mathbf{b}_n) = \pi_\partial(\mathbf{b}_1) \cdots \pi_\partial(\mathbf{b}_n) \tag{9.4}$$

*for all $n \in \mathbb{N}$ and $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathfrak{g}$. Furthermore,*

$$\pi_g \pi_\partial(\mathbf{e}) \pi_{g^{-1}} = \pi_\partial\big(\mathrm{Ad}_g(\mathbf{e})\big) \tag{9.5}$$

*for all* $\mathbf{e} \in \mathfrak{E}$ *and* $g \in G$.

PROOF. Let $v \in \mathcal{H}_\pi$ be smooth. Recall that by Lemma 7.3 the vector $\pi_\partial(\mathbf{b})v$ depends linearly on $\mathbf{b} \in \mathfrak{g}$. This extends to multi-linear dependence of $\pi_\partial(\mathbf{b}_1)\cdots\pi_\partial(\mathbf{b}_n)v$ on $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathfrak{g}$. Therefore $\pi_\partial$ extends from $\mathfrak{g}$ to the algebra of formal multi-linear non-commuting products of elements of $\mathfrak{g}$ as appearing in Definition 9.1.

However, to see that $\pi_\partial$ extends to $\mathfrak{E}$ we have to show that $\pi_\partial$ sends the ideal appearing in Definition 9.1 to zero, or equivalently that

$$\pi_\partial\big([\mathbf{a}, \mathbf{b}]\big)v = \big(\pi_\partial(\mathbf{a})\pi_\partial(\mathbf{b}) - \pi_\partial(\mathbf{b})\pi_\partial(\mathbf{a})\big)v \tag{9.6}$$

for all $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$ and all smooth $v \in \mathcal{H}_\pi$. To see this, we fix a vector $w \in \mathcal{H}_\pi$ and look at the matrix coefficient $\varphi = \varphi^\pi_{w,v}$. Let us use left-translation by elements of $G$ to define a vector field $\lambda_\partial(\mathbf{m})$ on $G$ for every $\mathbf{m} \in \mathfrak{g}$, as in Proposition 7.7. Smoothness of $v$ and Lemma 7.18 show that

$$\begin{aligned}\lambda_\partial(\mathbf{m})\varphi(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\varphi^\pi_{w,v}\big(\exp(-t\mathbf{m})g\big) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big\langle \pi_{\exp(-t\mathbf{m})}\pi_g w, v\big\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big\langle \pi_g w, \pi_{\exp(t\mathbf{m})}v\big\rangle \\ &= \big\langle \pi_g w, \pi_\partial(\mathbf{m})v\big\rangle = \varphi^\pi_{w,\pi_\partial(\mathbf{m})v}(g)\end{aligned}$$

exists for all $\mathbf{m} \in \mathfrak{g}$ and $g \in G$, which, when iterated, shows that $\varphi \in C^\infty(G)$. However, for smooth functions on $G$, the formula

$$\lambda_\partial([\mathbf{a}, \mathbf{b}])\varphi = \lambda_\partial(\mathbf{a})\lambda_\partial(\mathbf{b})\varphi - \lambda_\partial(\mathbf{b})\lambda_\partial(\mathbf{a})\varphi \tag{9.7}$$

is the definition of the Lie bracket for general Lie groups (see Exercise 9.5). At $g = e$, together with the above, this becomes

$$\begin{aligned}\big\langle w, \pi_\partial\big([\mathbf{a}, \mathbf{b}]\big)v\big\rangle &= \lambda_\partial([\mathbf{a}, \mathbf{b}])\varphi^\pi_{w,v}(e) \\ &= \big(\lambda_\partial(\mathbf{a})\lambda_\partial(\mathbf{b})\varphi^\pi_{w,v} - \lambda_\partial(\mathbf{b})\lambda_\partial(\mathbf{a})\varphi^\pi_{w,v}\big)(e) \\ &= \big(\lambda_\partial(\mathbf{a})\varphi^\pi_{w,\pi_\partial(\mathbf{b})v} - \lambda_\partial(\mathbf{b})\varphi^\pi_{w,\pi_\partial(\mathbf{a})v}\big)(e) \\ &= \langle w, \pi_\partial(\mathbf{a})\pi_\partial(\mathbf{b})v\rangle - \langle w, \pi_\partial(\mathbf{b})\pi_\partial(\mathbf{a})v\rangle\,.\end{aligned}$$

As this holds for all $w \in \mathcal{H}_\pi$, we obtain (9.6).

For $g \in G$ and $\mathbf{e} \in \mathfrak{g}$, the identity in (9.5) is simply the chain rule in Proposition 7.6. Moreover, if $\mathbf{e} = \mathbf{a}_1 \circ \mathbf{a}_2 \circ \cdots \circ \mathbf{a}_n \in \mathfrak{E}$ then we also have

$$\begin{aligned}\pi_g\pi_\partial(\mathbf{e})\pi_{g^{-1}} &= \pi_g\pi_\partial(\mathbf{a}_1)\pi_\partial(\mathbf{a}_2)\cdots\pi_\partial(\mathbf{a}_n)\pi_{g^{-1}} \\ &= \pi_g\pi_\partial(\mathbf{a}_1)\pi_{g^{-1}}\pi_g\pi_\partial(\mathbf{a}_2)\pi_{g^{-1}}\cdots\pi_g\pi_\partial(\mathbf{a}_n)\pi_{g^{-1}} \\ &= \pi_\partial\big(\mathrm{Ad}_g(\mathbf{a}_1)\big)\pi_\partial\big(\mathrm{Ad}_g(\mathbf{a}_2)\big)\cdots\pi_\partial\big(\mathrm{Ad}_g(\mathbf{a}_n)\big) \\ &= \pi_\partial\big(\mathrm{Ad}_g(\mathbf{a}_1)\circ\cdots\circ\mathrm{Ad}_g(\mathbf{a}_n)\big) = \pi_\partial\big(\mathrm{Ad}_g(\mathbf{e})\big)\end{aligned}$$

which, together with linearity, proves (9.5) for $\mathbf{e} \in \mathfrak{E}$. □

**Essential Exercise 9.5 (Lie brackets for linear groups).** Prove (9.7) for closed linear groups, where the Lie bracket is defined by (6.1) using matrix products.

**Corollary 9.6 (Adjoint representation).** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and universal enveloping algebra $\mathfrak{E}$. Then there exists a linear anti-homomorphism*

$$^*\colon\ \mathfrak{E} \longrightarrow \mathfrak{E}$$

*satisfying $\mathbf{a}^* = -\mathbf{a}$ and $(\mathbf{e}\circ\mathbf{f})^* = \mathbf{f}^*\circ\mathbf{e}^*$ for all $\mathbf{a} \in \mathfrak{g}$, and $\mathbf{e}, \mathbf{f} \in \mathfrak{E}$. Moreover, we have*

$$\langle \pi_\partial(\mathbf{e})u, v\rangle = \langle u, \pi_\partial(\mathbf{e}^*)v\rangle$$

*for all $\mathbf{e} \in \mathfrak{E}$ whenever $\pi$ is a unitary representation of $G$ and $u, v \in \mathcal{H}_\pi$ are smooth.*

We will refer to $*$ on $\mathfrak{E}$ as the *formal adjoint*.

Proof of Corollary 9.6. We define $^*\colon \mathfrak{g} \ni \mathbf{a} \mapsto -\mathbf{a} \in \mathfrak{g}$ and extend $^*$ to a linear map from $\mathfrak{E}$ to $\mathfrak{E}$ by requiring

$$\big(\mathbf{e}_1 \circ \mathbf{e}_2\big)^* = \mathbf{e}_2^* \circ \mathbf{e}_1^* \tag{9.8}$$

for all $\mathbf{e}_1, \mathbf{e}_2 \in \mathfrak{E}$.

To see that $^*$ is well-defined, we verify that the relations appearing in Definition 9.5 are preserved. For $\mathbf{a}, \mathbf{b} \in \mathfrak{g}$ we have $\mathbf{a}^* = -\mathbf{a}$, $\mathbf{b}^* = -\mathbf{b}$, and that $[\mathbf{a}, \mathbf{b}] \in \mathfrak{g}$ satisfies $[\mathbf{a}, \mathbf{b}]^* = -[\mathbf{a}, \mathbf{b}]$. Together with (9.8), we obtain that

$$\begin{aligned}\big(\mathbf{a}\circ\mathbf{b} - \mathbf{b}\circ\mathbf{a} - [\mathbf{a},\mathbf{b}]\big)^* &= (-\mathbf{b})\circ(-\mathbf{a}) - (-\mathbf{a})\circ(-\mathbf{b}) + [\mathbf{a},\mathbf{b}]\\ &= \mathbf{b}\circ\mathbf{a} - \mathbf{a}\circ\mathbf{b} - [\mathbf{b},\mathbf{a}]\end{aligned}$$

indeed is once more in the ideal appearing in the definition of $\mathfrak{E}$.

Now let $u, v \in \mathcal{H}_\pi$ be smooth and $\mathbf{a} \in \mathfrak{g}$. Then

$$\begin{aligned}\langle \pi_\partial(\mathbf{a})u, v\rangle &= \lim_{t\to 0}\big\langle \tfrac{1}{t}\big(\pi_{\exp(t\mathbf{a})}u - u\big), v\big\rangle\\ &= \lim_{t\to 0}\tfrac{1}{t}\big(\big\langle u, \pi_{\exp(-t\mathbf{a})}v\big\rangle - \langle u, v\rangle\big)\\ &= \lim_{t\to 0}\big\langle u, \tfrac{1}{t}\big(\pi_{\exp(-t\mathbf{a})}v - v\big)\big\rangle\\ &= \langle u, -\pi_\partial(\mathbf{a})v\rangle = \langle u, \pi_\partial(\mathbf{a}^*)v\rangle\,.\end{aligned}$$

Moreover, if $\mathbf{b} \in \mathfrak{g}$ then

$$\langle \pi_\partial(\mathbf{a}\circ\mathbf{b})u, v\rangle = \langle \pi_\partial(\mathbf{b})u, \pi_\partial(\mathbf{a}^*)v\rangle = \langle u, \pi_\partial(\mathbf{b}^*\circ\mathbf{a}^*)v\rangle = \big\langle u, \pi_\partial\big((\mathbf{a}\circ\mathbf{b})^*\big)v\big\rangle\,.$$

The corollary follows by iterating this, and combining it with sesqui-linearity of the inner product. □

### 9.1.4 Central Elements of the Universal Enveloping Algebra

We have seen in Section 9.1.2 that the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{sl}_2(\mathbb{R})$ has a central element, to which we will apply the following general results.

**Essential Exercise 9.7.** Let $G$ be a connected closed linear group with Lie algebra $\mathfrak{g}$ and universal enveloping algebra $\mathfrak{E}$. Suppose that $\Omega \in \mathfrak{E}$ is central in the sense that $\mathbf{a} \circ \Omega = \Omega \circ \mathbf{a}$ for all $\mathbf{a} \in \mathfrak{g}$ (or, equivalently, $\mathbf{a} \in \mathfrak{E}$). Show that $\mathrm{Ad}_g(\Omega) = \Omega$ for all $g \in G$.

**Proposition 9.8 (Operators coming from the centre of $\mathfrak{E}$).** *Let $G$ be a closed linear group with Lie algebra $\mathfrak{g}$, and let $\Omega$ be a central element of the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{g}$ satisfying $\mathrm{Ad}_g(\Omega) = \Omega$ for all $g \in G$. Then, for any unitary representation $\pi$ of $G$, the closure of $\pi_\partial(\Omega)$ (defined on smooth vectors) is a well-defined closed equivariant operator. If $\Omega^* = \Omega$, then the closure is a self-adjoint operator. If $\pi$ is irreducible, then the closure is multiplication by a scalar $\alpha_{\Omega,\pi} \in \mathbb{C}$ (respectively $\alpha_{\Omega,\pi} \in \mathbb{R}$ if $\Omega^* = \Omega$ also).*

Proof. Let $\Omega \in \mathfrak{E}$ be central and let $\pi$ be a unitary representation as in the proposition. We define $T_\pi$ as the closure of $\pi_\partial(\Omega)$ acting on smooth vectors. More formally, we have that $v \in D_{T_\pi}$ belongs to the domain of $T_\pi$, and $T_\pi v = w$ is the image, if there exists a sequence $(v_n)$ in $\mathcal{H}_\pi$ of smooth vectors with

$$\big(v_n, \pi_\partial(\Omega) v_n\big) \in \mathrm{Graph}\big(\pi_\partial(\Omega)\big)$$

converging to $(v, w)$ as $n \to \infty$. To see that this defines a well-defined operator, we need to show that

$$(v, w) = \lim_{n\to\infty} \big(v_n, \pi_\partial(\Omega) v_n\big) \in \overline{\mathrm{Graph}\big(\pi_\partial(\Omega)\big)}$$

with $v = 0$ implies that $w = 0$. To see this, suppose $u \in \mathcal{H}_\pi$ is smooth, then

$$\langle w, u\rangle = \lim_{n\to\infty} \langle \pi_\partial(\Omega) v_n, u\rangle = \lim_{n\to\infty} \langle v_n, \pi_\partial(\Omega^*) u\rangle\,,$$

where we applied Corollary 9.6. Since $\lim_{n\to\infty} v_n = 0$ by assumption, we obtain $\langle w, u\rangle = 0$ for all smooth $u \in \mathcal{H}_\pi$. Since smooth vectors are dense by Proposition 7.7, it follows that $w = 0$. Hence $T_\pi$ is a well-defined closed operator.

Suppose now that $v \in D_{T_\pi}$ so that, by definition,

$$(v, T_\pi v) = \lim_{n\to\infty} (v_n, \pi_\partial(\Omega) v_n) \in \mathrm{Graph}(T_\pi),$$

and fix some $g \in G$. Then

$$\pi_g \pi_\partial(\Omega) v_n = \pi_\partial\big(\underbrace{\mathrm{Ad}_g(\Omega)}_{=\Omega}\big)\pi_g v = \pi_\partial(\Omega)\pi_g v_n$$

by Corollary 9.6 and our assumptions on $\Omega$. Now $v_n \to v$ and $\pi_\partial(\Omega)v_n \to T_\pi v$ as $n \to \infty$, so $\pi_g v_n \to \pi_g v$ and

$$\pi_\partial(\Omega)\pi_g v_n = \pi_g \pi_\partial(\Omega) v_n \longrightarrow \pi_g T_\pi v$$

as $n \to \infty$. Therefore

$$\big(\pi_g v, \pi_g T_\pi v\big) = \lim_{n\to\infty} \big(\pi_g v_n, \pi_\partial(\Omega)\pi_g v_n\big),$$

which shows that $T_\pi \pi_g v = \pi_g T_\pi v$ for all $g \in G$ and $v \in D_{T_\pi}$. In other words, $T_\pi$ is a well-defined closed equivariant operator.

We note that if $\pi$ is irreducible then Schur's lemma in the form of Corollary 1.38 implies that $T_\pi$ is simply multiplication by some $\alpha_{\pi,\Omega} \in \mathbb{C}$. If, in addition, $\Omega^* = \Omega$ and $v \in \mathcal{H}_\pi$ is a smooth unit vector, then

$$\alpha_{\pi,\Omega} = \alpha_{\pi,\Omega} \langle v, v\rangle = \langle \pi_\partial(\Omega)v, v\rangle = \langle v, \pi_\partial(\Omega^*)v\rangle = \overline{\alpha_{\pi,\Omega}} \langle v, v\rangle = \overline{\alpha_{\pi,\Omega}}$$

shows that $\alpha_{\pi,\Omega} \in \mathbb{R}$. We note that this case suffices (for instance) for the classification of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$ in the next section.

For completeness we now drop the assumption of irreducibility but still suppose that $\Omega^* = \Omega$ , and will show that $T_\pi$ is in this case a closed self-adjoint operator. For this we define the operator

$$B = P_1 P_{\mathrm{Graph}(T_\pi)} \imath_1,$$

where

$$\imath_1 \colon \mathcal{H}_\pi \ni v \longmapsto (v, 0) \in \mathcal{H}_\pi^2$$

is the embedding of $\mathcal{H}_\pi$ into the first factor, $P_{\mathrm{Graph}(T_\pi)}$ is the orthogonal projection onto the closed subspace $\mathrm{Graph}(T_\pi) \subseteq \mathcal{H}_\pi^2$ and

$$P_1 \colon \mathcal{H}_\pi^2 \ni (v_1, v_2) \longmapsto v_1 \in \mathcal{H}_\pi$$

is the projection onto the first factor.

Since the adjoint of $\imath_1$ is $P_1$, it follows that $B \colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ is self-adjoint. Note that if $P_{\mathrm{Graph}(T_\pi)}(v, 0) = (\widetilde{v}, T_\pi \widetilde{v})$ for some $v \in \mathcal{H}_\pi$, then $Bv = \widetilde{v} \in D_{T_\pi}$. From this and $(\widetilde{v}, T_\pi\widetilde{v}) - (v, 0) \in \mathrm{Graph}(T_\pi)^\perp$, it follows that

$$\langle Bv, v\rangle = \langle \widetilde{v}, v\rangle = \langle (\widetilde{v}, T_\pi\widetilde{v}), (v, 0)\rangle = \langle (\widetilde{v}, T_\pi\widetilde{v}), (\widetilde{v}, T_\pi\widetilde{v})\rangle = \|(\widetilde{v}, T_\pi\widetilde{v})\|^2_{\mathcal{H}_\pi^2}$$

belongs to $[0, \|v\|^2]$. Since $B$ is self-adjoint, this proves that the spectrum of $B$ is contained in $[0, 1]$. Moreover, suppose now that $Bv = \widetilde{v} = 0$. Recalling that by definition $(\widetilde{v}, T_\pi\widetilde{v}) = (0, 0)$ is the orthogonal projection of $(v, 0)$ onto $\mathrm{Graph}(T_\pi)$, density of the domain of $D_{T_\pi} \supseteq D_{\pi_\partial(\Omega)}$ now forces $v = 0$. This implies that $B$ is injective.

Recall that $T_\pi$ is equivariant by the first part of the proof. Hence $\mathrm{Graph}(T_\pi)$ is invariant, and $B$ is also equivariant. We use the measurable functional calculus for $B$ to define

$$\mathcal{V}_n = \big(\mathbb{1}_{(\frac{1}{n+1},\frac{1}{n}]}\big)_{\mathrm{FC}}\mathcal{H}_\pi.$$

Hence

$$\mathcal{H}_\pi = \big(\mathbb{1}_{(0,1]}\big)_{\mathrm{FC}}\mathcal{H}_\pi = \bigoplus_{n\in\mathbb{N}} \mathcal{V}_n \tag{9.9}$$

is a decomposition into closed invariant subspaces. Indeed, the measurable functional calculus also shows that all operators that commute with $B$ (for example, $\pi_g$ for any $g \in G$) also commute with any operators constructed by the measurable functional calculus. Since the spectrum of $B$ is contained in $[0,1]$, but the kernel of $B$ is trivial, this gives (9.9).

Let $\pi_n = \pi|_{\mathcal{V}_n}$. Then $B|_{\mathcal{V}_n}\colon \mathcal{V}_n \to \mathcal{V}_n$ has a bounded inverse defined on all of $\mathcal{V}_n$, which shows that for any $\widetilde{v} \in \mathcal{V}_n$ we have that

$$(\widetilde{v}, T_\pi\widetilde{v}) = P_{\mathrm{Graph}(T_\pi)}(B^{-1}\widetilde{v}, 0) \in \mathrm{Graph}(T_\pi)$$

and

$$\|(\widetilde{v}, T_\pi\widetilde{v})\|_{\mathcal{H}_\pi^2} \leqslant \|B^{-1}\widetilde{v}\| \leqslant (n+1)\widetilde{v}. \tag{9.10}$$

Therefore, $\mathcal{V}_n \subseteq D_{T_\pi}$ for all $n \in \mathbb{N}$.

Let $v = \sum_{n\in\mathbb{N}} v_n \in \mathcal{H}_\pi$ be smooth with $v_n = P_{\mathcal{V}_n} v \in \mathcal{V}_n$ for all $n \in \mathbb{N}$, where $P_{\mathcal{V}_n}\colon \mathcal{H}_\pi \to \mathcal{V}_n$ is the (equivariant) orthogonal projection. Then the definition of smoothness implies that $v_n$ is smooth for $\pi_n$ for all $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}$ we have that $v_n \in \mathcal{V}_n$ is smooth for $\pi_n$ if and only if $v_n$ is smooth for $\pi$. Using the definition

$$\mathrm{Graph}(T_\pi) = \overline{\mathrm{Graph}\big(\pi_\partial(\Omega)\big)},$$

this implies that $T_\pi|_{\mathcal{V}_n} = T_{\pi_n}$. Using (9.10), $\Omega^* = \Omega$, and Corollary 9.6, we obtain that $T_{\pi_n}$ is a bounded self-adjoint operator.

We now define the subspace

$$D = \left\{ \sum_{n=1}^{\infty} v_n \;\middle|\; v_n \in \mathcal{V}_n \text{ for } n \in \mathbb{N}, \sum_{n=1}^{\infty} \|v_n\|^2 < \infty, \text{ and} \sum_{n=1}^{\infty} \|T_{\pi_n} v_n\|^2 < \infty \right\}.$$

For $\sum_{n=1}^{\infty} v_n \in D$ and any $N \in \mathbb{N}$ we have $\sum_{n=1}^{N} v_n \in D_{T_\pi}$ and

$$\left( \sum_{n=1}^{N} v_n, \sum_{n=1}^{N} T_{\pi_n} v_n \right) \in \mathrm{Graph}(T_\pi).$$

Letting $N \to \infty$, and using the fact that $\mathrm{Graph}(T_\pi)$ is closed, we obtain that $D \subseteq D_{T_\pi}$.

Assume now that, on the other hand, $v \in \mathrm{Graph}(T_\pi)$. We may write

$$v = \sum_{n=1}^{\infty} v_n$$

for some $v_n \in \mathcal{V}_n$ for $n \in \mathbb{N}$, with $\sum_{n=1}^{\infty} \|v_n\|^2 < \infty$. Projecting $(v, T_\pi v)$ onto $\mathcal{V}_n \oplus \mathcal{V}_n \subseteq \mathcal{H}_\pi \oplus \mathcal{H}_\pi$ and noting again that the projection of a smooth vector in $\mathcal{H}_\pi$ to $\mathcal{V}_n$ is a smooth vector in $\mathcal{V}_n$, it follows that

$$T_\pi v = \sum_{n=1}^{\infty} T_{\pi_n} v_n$$

and hence $\sum_{n=1}^{\infty} \|T_{\pi_n} v_n\|^2 < \infty$. This shows that $D = D_{T_\pi}$. Since the operator $T_{\pi_n} \colon \mathcal{V}_n \to \mathcal{V}_n$ are bounded and self-adjoint, it is now a standard exercise (see [24, Ch. 3] and Exercise 9.9) to show that $T_\pi$ is a closed self-adjoint operator. □

**Exercise 9.9.** Let $\mathcal{H}_n$ be a Hilbert space and $T_n \colon \mathcal{H}_n \to \mathcal{H}_n$ a self-adjoint bounded operator for all $n \in \mathbb{N}$. Then

$$T\left(\sum_{n=1}^{\infty} v_n\right) = \sum_{n=1}^{\infty} T_n v_n$$

for all $\sum_{n=1}^{\infty} v_n \in \bigoplus_{n \geqslant 1} \mathcal{H}_n$ with $\sum_{n=1}^{\infty} \|T_n v_n\|^2 < \infty$ defines a self-adjoint operator.

### 9.1.5 Complexification of the Universal Enveloping Algebra

For some of our discussions, the following extension of Definition 9.1 will be important.

**Definition 9.10 (The complexification $\mathfrak{E}_\mathbb{C}$).** Let $\mathfrak{g}$ be a Lie algebra with universal enveloping algebra $\mathfrak{E}$. We define the *complexification of* $\mathfrak{E}$ by

$$\mathfrak{E}_\mathbb{C} = \mathfrak{E} \otimes_\mathbb{R} \mathbb{C}.$$

For $\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1, \mathbf{a}_2 + \mathrm{i}\mathbf{b}_2 \in \mathfrak{E}_\mathbb{C}$ with $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2 \in \mathfrak{E}$ and $g \in G$, we also define multiplication by

$$(\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1) \circ (\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2) = \mathbf{a}_1 \circ \mathbf{a}_2 - \mathbf{b}_1 \circ \mathbf{b}_2 + \mathrm{i}(\mathbf{a}_1 \circ \mathbf{b}_2 + \mathbf{b}_1 \circ \mathbf{a}_2),$$

the adjoint operator by

$$\mathrm{Ad}_g(\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1) = \mathrm{Ad}_g(\mathbf{a}_1) + \mathrm{i}\,\mathrm{Ad}_g(\mathbf{b}_2),$$

conjugation by

$$\overline{\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1} = \mathbf{a}_1 - \mathrm{i}\mathbf{b}_1,$$

and the formal adjoint by $(\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1)^* = \mathbf{a}_1^* - \mathrm{i}\mathbf{b}_1^*$.

We verify that multiplication is actually bilinear over $\mathbb{C}$ on $\mathfrak{E}_{\mathbb{C}}$. To see this, note first that multiplication is clearly linear over $\mathbb{R}$. Moreover,

$$\begin{aligned}\big(\mathrm{i}(\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1)\big) \circ (\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2) &= (-\mathbf{b}_1 + \mathrm{i}\mathbf{a}_1) \circ (\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2)\\ &= -\mathbf{b}_1 \circ \mathbf{a}_2 - \mathbf{a}_1 \circ \mathbf{b}_2 + \mathrm{i}(-\mathbf{b}_1 \circ \mathbf{b}_2 + \mathbf{a}_1 \circ \mathbf{a}_2)\\ &= \mathrm{i}\big(\mathbf{a}_1 \circ \mathbf{a}_2 - \mathbf{b}_1 \circ \mathbf{b}_2 + \mathrm{i}(\mathbf{a}_1 \circ \mathbf{b}_2 + \mathbf{b}_1 \circ \mathbf{a}_2)\big)\\ &= \mathrm{i}\big((\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1) \circ (\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2)\big).\end{aligned}$$

Together with linearity over $\mathbb{R}$ in the first argument, linearity over $\mathbb{C}$ in the first argument follows. The proof the second argument is identical. Finally, the proof of conjugate-linearity of conjugation and the formal adjoint on $\mathfrak{E}_{\mathbb{C}}$ are similar.

**Proposition 9.11 (Differentials operators coming from $\mathfrak{E}_{\mathbb{C}}$).** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, and let $\pi$ be a unitary representation of $G$. Then the representation of $\mathfrak{g}$ via $\pi_\partial$ on smooth vectors extends to a complex representation of the complexification $\mathfrak{E}_{\mathbb{C}}$ of the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{g}$ on smooth vectors in $\mathcal{H}_\pi$ satisfying the properties of Proposition* 9.4 *and Corollary* 9.6*.*

Proof. For $\mathbf{a}, \mathbf{b} \in \mathfrak{E}$, we define $\pi_\partial(\mathbf{a} + \mathrm{i}\mathbf{b}) = \pi_\partial(\mathbf{a}) + \mathrm{i}\pi_\partial(\mathbf{b})$. This gives an extension of the above representation to $\mathfrak{E}_{\mathbb{C}}$, satisfying linearity over $\mathbb{C}$. Moreover, if $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2 \in \mathfrak{E}$, then, by definition of $\circ$ on $\mathfrak{E}_{\mathbb{C}}$,

$$\begin{aligned}\pi_\partial\big((\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1) \circ (\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2)\big) &= \pi_\partial\big(\mathbf{a}_1 \circ \mathbf{a}_2 - \mathbf{b}_1 \circ \mathbf{b}_2\big) + \mathrm{i}\pi_\partial\big(\mathbf{a}_1 \circ \mathbf{b}_2 + \mathbf{b}_1 \circ \mathbf{a}_2\big)\\ &= \pi_\partial(\mathbf{a}_1)\pi_\partial(\mathbf{a}_2) - \pi_\partial(\mathbf{b}_1)\pi_\partial(\mathbf{b}_2)\\ &\qquad + \mathrm{i}\big(\pi_\partial(\mathbf{a}_1)\pi_\partial(\mathbf{b}_2) - \pi_\partial(\mathbf{b}_1)\pi_\partial(\mathbf{a}_2)\big)\\ &= \pi_\partial(\mathbf{a}_1 + \mathrm{i}\mathbf{b}_1)\pi_\partial(\mathbf{a}_2 + \mathrm{i}\mathbf{b}_2),\end{aligned}$$

so (9.4) also holds for $\mathfrak{E}_{\mathbb{C}}$.

The extension of (9.5) follows simply by linearity of both sides over $\mathbb{C}$. Finally, the complex version of Corollary 9.6 follows from its real counterpart and sesqui-linearity of the inner product on $\mathcal{H}_\pi$. □

To conclude, we wish to emphasize that the operators $\pi_\partial(\mathbf{e})$ for

$$\mathbf{e} \in \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \subseteq \mathfrak{E}_{\mathbb{C}}$$

(or $\mathbf{e} \in \mathfrak{E}_{\mathbb{C}}$) do not in general directly correspond to first (or higher order) partial derivatives, but instead to complex linear combinations of these.

We also note that the complexification $\mathfrak{E}_{\mathbb{C}}$ of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{R})$ reveals that our definition of $\Omega$ as in Lemma 9.2 formally matches the definition of the Casimir operator in Corollary 7.22. Indeed, there we used the basis vectors

$$\mathbf{b}_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \mathrm{i}\mathbf{a}, \mathbf{b}_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\mathbf{d}, \mathbf{b}_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{k}$$

of $\mathfrak{su}_2(\mathbb{R}) \subseteq \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, and hence

$$\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{d}^{\circ 2} - \mathbf{k}^{\circ 2} = \mathbb{1}_{\mathfrak{E}} - \mathbf{b}_1^{\circ 2} - \mathbf{b}_2^{\circ 2} - \mathbf{b}_3^{\circ 2}$$

within the complex universal enveloping algebra $\mathfrak{E}_{\mathbb{C}}$.

## 9.2 Raising, Lowering, and the Dual of $\mathbf{SL_2}(\mathbb{R})$

We now specialize and extend the material from the previous section in the case of $\mathrm{SL}_2(\mathbb{R})$. Our final goal of the section is a description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$, although the concrete construction of the principal and complementary series representations will be postponed to later sections.

### 9.2.1 The Casimir Operator

It follows from the discussion in Section 9.1.2 that Proposition 9.8 applies to unitary representations of $\mathrm{SL}_2(\mathbb{R})$ as in Corollary 9.12 below (see also Exercise 9.7). Indeed, the Casimir element $\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{d}^{\circ 2} - \mathbf{k}^{\circ 2}$ of $\mathfrak{sl}_2(\mathbb{R})$ satisfies $\Omega^* = \Omega$ by its definition in Lemma 9.2.

**Corollary 9.12 (Casimir operator).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ and let $\Omega \in \mathfrak{E}$ be the Casimir element in the centre of the enveloping algebra $\mathfrak{E}$ of $\mathfrak{sl}_2(\mathbb{R})$. Then the closure $T_\pi$ of $\pi_\partial(\Omega)$ is a self-adjoint equivariant operator. If $\pi$ is in addition irreducible, then $T_\pi = \alpha_\pi I$ for some $\alpha_\pi \in \mathbb{R}$.*

### 9.2.2 The Raising and Lowering Operators

We define[†] the elements $\mathbf{r}^+ = \frac{1}{2}(\mathbf{a} - \mathrm{i}\mathbf{d})$ and $\mathbf{r}^- = \frac{1}{2}(\mathbf{a} + \mathrm{i}\mathbf{d})$ of $\mathfrak{sl}_2(\mathbb{C})$, and note that $\overline{\mathbf{r}^+} = \mathbf{r}^-$. We calculate

$$\mathrm{ad}_{\mathbf{k}}(\mathbf{a}) = [-\mathbf{e} + \mathbf{f}, \mathbf{a}] = [\mathbf{a}, \mathbf{e}] - [\mathbf{a}, \mathbf{f}] = 2\mathbf{e} + 2\mathbf{f} = 2\mathbf{d}$$

and

$$\mathrm{ad}_{\mathbf{k}}(\mathbf{d}) = [-\mathbf{e} + \mathbf{f}, \mathbf{e} + \mathbf{f}] = -[\mathbf{e}, \mathbf{f}] + [\mathbf{f}, \mathbf{e}] = -2[\mathbf{e}, \mathbf{f}] = -2\mathbf{a},$$

[†] It is an unfortunate coincidence that $\mathbf{d} \in \mathfrak{sl}_2(\mathbb{R})$ multiplied by $\mathrm{i} \in \mathbb{C}$, giving $\mathrm{i}\mathbf{d}$, is notationally close to a familiar 'identity' notation, id.

which gives

$$\mathrm{ad}_{\mathbf{k}}(\mathbf{r}^+) = \mathbf{d} - \mathrm{i}(-\mathbf{a}) = \mathrm{i}(\mathbf{a} - \mathrm{i}\mathbf{d}) = 2\mathrm{i}\mathbf{r}^+$$

and, by conjugation,

$$\mathrm{ad}_{\mathbf{k}}(\mathbf{r}^-) = -2\mathrm{i}\mathbf{r}^-$$

also. With $\mathrm{Ad}_{\exp(\theta\mathbf{k})} = \exp(\mathrm{ad}_{\theta\mathbf{k}})$ for $\theta \in \mathbb{R}$, this also implies

$$\begin{cases} \mathrm{Ad}_{k_\theta}(\mathbf{r}^+) = \mathrm{e}^{2\mathrm{i}\theta}\mathbf{r}^+ \\ \mathrm{Ad}_{k_\theta}(\mathbf{r}^-) = \mathrm{e}^{-2\mathrm{i}\theta}\mathbf{r}^- \end{cases} \tag{9.11}$$

for all $k_\theta \in K$.

For a unitary representation $\pi$ of $\mathrm{SL}_2(\mathbb{R})$, we will call $\pi_\partial(\mathbf{r}^+)$ the *raising operator* and $\pi_\partial(\mathbf{r}^-)$ the *lowering operator*.

**Proposition 9.13 (Raising and lowering operators).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. For any smooth $K$-eigenvector $v_n \in \mathcal{H}_\pi$ of weight $n \in \mathbb{Z}$, the vector $\pi_\partial(\mathbf{r}^\pm)v_n$ has weight $n \pm 2$ (but either or both might be zero).*

Proof. Let $k_\theta \in K$ and let $v_n \in \mathcal{H}_\pi$ be a smooth vector of $K$-weight $n$. Then the chain rule in Proposition 9.4 and (9.11) imply that

$$\begin{aligned} \pi_{k_\theta}\pi_\partial(\mathbf{r}^+)v_n &= \pi_\partial\big(\mathrm{Ad}_{k_\theta}(\mathbf{r}^+)\big)\pi_{k_\theta}v_n \\ &= \pi_\partial\big(\mathrm{e}^{2\mathrm{i}\theta}\mathbf{r}^+\big)\mathrm{e}^{\mathrm{i}n\theta}v_n \\ &= \mathrm{e}^{\mathrm{i}(n+2)\theta}\pi_\partial(\mathbf{r}^+)v_n, \end{aligned}$$

and the calculation for $\pi_\partial(\mathbf{r}^-)v_n$ is identical. □

We also wish to relate the elements $\mathbf{r}^\pm$ to the Casimir operator in Lemma 9.2. By definition of $\mathbf{r}^+$ and $\mathbf{r}^-$, we have

$$\begin{aligned} \mathbf{a} &= \mathbf{r}^+ + \mathbf{r}^-, \\ \mathbf{d} &= \mathrm{i}(\mathbf{r}^+ - \mathbf{r}^-), \end{aligned}$$

and

$$\begin{aligned} [\mathbf{r}^+, \mathbf{r}^-] &= \tfrac{1}{4}[\mathbf{a} - \mathrm{i}\mathbf{d}, \mathbf{a} + \mathrm{i}\mathbf{d}] \\ &= \tfrac{1}{4}\left(\mathrm{i}[\mathbf{a}, \mathbf{d}] - \mathrm{i}[\mathbf{d}, \mathbf{a}]\right) \\ &= \tfrac{\mathrm{i}}{2}[\mathbf{a}, \mathbf{d}] = -\mathrm{i}\mathbf{k} \end{aligned} \tag{9.12}$$

since

$$[\mathbf{a}, \mathbf{d}] = [\mathbf{a}, \mathbf{e} + \mathbf{f}] = 2\mathbf{e} - 2\mathbf{f} = -2\mathbf{k}.$$

Using the definition of $\Omega$ in Lemma 9.2, we now obtain

$$
\begin{aligned}
\Omega &= \mathbb{1}_{\mathfrak{E}} + (\mathbf{r}^+ + \mathbf{r}^-) \circ (\mathbf{r}^+ + \mathbf{r}^-) - (\mathbf{r}^+ - \mathbf{r}^-) \circ (\mathbf{r}^+ - \mathbf{r}^-) - \mathbf{k}^{\circ 2} \\
&= \mathbb{1}_{\mathfrak{E}} + (\mathbf{r}^+ \circ \mathbf{r}^+ + \mathbf{r}^+ \circ \mathbf{r}^- + \mathbf{r}^- \circ \mathbf{r}^+ + \mathbf{r}^- \circ \mathbf{r}^-) \\
&\qquad - (\mathbf{r}^+ \circ \mathbf{r}^+ - \mathbf{r}^+ \circ \mathbf{r}^- - \mathbf{r}^- \circ \mathbf{r}^+ + \mathbf{r}^- \circ \mathbf{r}^-) - \mathbf{k}^{\circ 2} \\
&= \mathbb{1}_{\mathfrak{E}} + 2\mathbf{r}^+ \circ \mathbf{r}^- + 2\mathbf{r}^- \circ \mathbf{r}^+ - \mathbf{k}^{\circ 2}.
\end{aligned}
$$

We can also write this in the form

$$\Omega = \mathbb{1}_{\mathfrak{E}} + 4\mathbf{r}^+ \circ \mathbf{r}^- - 2[\mathbf{r}^+, \mathbf{r}^-] - \mathbf{k}^{\circ 2}.$$

Using (9.12), this gives

$$\Omega = 4\mathbf{r}^+ \circ \mathbf{r}^- + \underbrace{\mathbb{1}_{\mathfrak{E}} + 2\mathrm{i}\mathbf{k} - \mathbf{k}^{\circ 2}}_{(\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})^{\circ 2}}.$$

Therefore

$$\Omega = 4\mathbf{r}^+ \circ \mathbf{r}^- + (\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})^{\circ 2} = 4\mathbf{r}^- \circ \mathbf{r}^+ + (\mathbb{1}_{\mathfrak{E}} - \mathrm{i}\mathbf{k})^{\circ 2}, \tag{9.13}$$

where the second formula follows from the first by conjugation.

**Corollary 9.14 (Norm of raised and lowered vectors).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ so that the closure of $\pi_\partial(\Omega)$ is multiplication by $\alpha_\pi \in \mathbb{R}$. Then for any smooth $K$-eigenvector $v_n \in \mathcal{H}_\pi$ of weight $n \in \mathbb{Z}$, we have*

$$
\begin{cases}
\|\pi_\partial(\mathbf{r}^+)v_n\|^2 & = \frac{1}{4}\big((n+1)^2 - \alpha_\pi\big)\|v_n\|^2 \\
\|\pi_\partial(\mathbf{r}^-)v_n\|^2 & = \frac{1}{4}\big((n-1)^2 - \alpha_\pi\big)\|v_n\|^2.
\end{cases}
$$

To make these two formulas more memorable, we note that in both cases $n \pm 1$ is precisely the weight that lies in between the weight $n$ of $v_n$ and the weight $n \pm 2$ of the vector $\pi_\partial(\mathbf{r}^\pm)v_n$.

Proof of Corollary 9.14. We note that $(\mathbf{r}^+)^* = -\overline{\mathbf{r}^+} = -\mathbf{r}^-$ by Corollary 9.6 and Definition 9.10. Using Proposition 9.11 and the identity (9.13) in the form

$$\mathbf{r}^- \circ \mathbf{r}^+ = \tfrac{1}{4}\left(\Omega - (\mathbb{1}_{\mathfrak{E}} - \mathrm{i}\mathbf{k})^{\circ 2}\right)$$

gives

$$
\begin{aligned}
\|\pi_\partial(\mathbf{r}^+)v_n\|^2 &= \left\langle \pi_\partial(\mathbf{r}^+)v_n, \pi_\partial(\mathbf{r}^+)v_n \right\rangle \\
&= -\left\langle \pi_\partial(\mathbf{r}^- \circ \mathbf{r}^+)v_v, v_n \right\rangle \\
&= -\tfrac{1}{4}\left\langle \pi_\partial(\Omega - (\mathbb{1}_{\mathfrak{E}} - \mathrm{i}\mathbf{k})^{\circ 2})v_n, v_n \right\rangle.
\end{aligned}
$$

Since

$$\pi_\partial(\mathbb{1}_{\mathfrak{E}} - \mathrm{i}\mathbf{k})v_n = v_n - \mathrm{i}\pi_\partial(\mathbf{k})v_n = v_n + nv_n = (n+1)v_n,$$

we obtain

$$\|\pi_\partial(\mathbf{r}^+)v_n\|^2 = -\tfrac{1}{4}\left(\alpha_\pi - (n+1)^2\right)\|v_n\|^2.$$

Similarly,

$$\begin{aligned}\|\pi_\partial(\mathbf{r}^-)v_n\|^2 &= \left\langle \pi_\partial(\mathbf{r}^-)v_n, \pi_\partial(\mathbf{r}^-)v_n\right\rangle \\ &= -\left\langle \pi_\partial(\mathbf{r}^+\circ\mathbf{r}^-)v_v, v_n\right\rangle \\ &= -\tfrac{1}{4}\left\langle \pi_\partial(\Omega - (\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})^{\circ 2})v_n, v_n\right\rangle \\ &= -\tfrac{1}{4}\left(\alpha_\pi - (n-1)^2\right)\|v_n\|^2,\end{aligned}$$

completing the proof. □

We note that the raising and lowering operators will be useful for proving irreducibility of the principal and complementary series representations in the next sections. In fact we already used these operators implicitly in the proofs of irreducibility of the discrete and mock discrete series representations of $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SU}_{1,1}(\mathbb{R})$ in Section 8.4. More importantly, we will use the more abstract framework above involving the Casimir operator to classify all irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$. In fact Corollary 9.14 already implies important restrictions on the Casimir eigenvalue $\alpha_\pi$: If $v_n$ as in the corollary is non-zero, then we must have

$$\alpha_\pi \leqslant (n \pm 1)^2. \tag{9.14}$$

Applying our above discussion to the irreducible unitary representation of Section 8.4, we obtain their Casimir eigenvalue.

**Corollary 9.15 (Casimir for discrete and mock discrete series).** *For every integer $n \geqslant 2$, we have*

$$\alpha_{\delta^{n,+}} = \alpha_{\delta^{n,-}} = (n-1)^2$$

*for the discrete series representations $\delta^{n,\pm}$. Similarly, we have*

$$\alpha_{\delta^{1,+}} = \alpha_{\delta^{1,-}} = 0$$

*for the two mock discrete series representations $\delta^{1,\pm}$.*

Proof. Let $n \geqslant 2$ and write $e_0$ for the constant function in $A_n(\mathbb{D})$ which has $K$-weight $n$ by Lemma 8.22. The proof of irreducibility in Theorem 8.23 shows that $e_0$ is smooth (see also Lemma 9.17). By Proposition 9.13 we know that $\delta_\partial^{n,+}(\mathbf{r}^-)e_0$ has weight $n-2$, and by Corollary 9.14 we have

$$\|\delta_\partial^{n,+}(\mathbf{r}^-)e_0\| = \tfrac{1}{4}\left((n-1)^2 - \alpha_{\delta^{n,+}}\right)\|e_0\|.$$

However, by Theorem 8.23, there is no vector of weight $n-2$. Hence $\delta_\partial^{n,+}(\mathbf{r}^-)e_0$ must be trivial, and so

$$\alpha_{\delta^{n,+}} = (n-1)^2,$$

as claimed. The argument for the mock discrete series representation $\delta^{1,+}$ uses Theorem 8.30, but is otherwise identical.

This implies the same formulas for the contragredient representations $\delta^{n,-}$ for $n \geqslant 2$ and for $\delta^{1,-}$. □

**Exercise 9.16.** Explain the last step in the proof of Corollary 9.15 more carefully.

### 9.2.3 Smooth *K*-finite Vectors

For a unitary representation $\pi$ of $\mathrm{SL}_2(\mathbb{R})$ and a vector $v \in \mathcal{H}_\pi$, we say that $v$ is $K$-finite if $\dim\langle \pi(K)v\rangle < \infty$ (or, equivalently, if $v$ is a finite sum of $K$-eigenfunctions).

Because of the discussions above, the following lemma is useful for the study of general unitary representations of $\mathrm{SL}_2(\mathbb{R})$.

**Lemma 9.17 (Smooth $K$-finite vectors).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. Then the subspace of smooth $K$-finite vectors is dense in $\mathcal{H}_\pi$. Moreover, if for some $n \in \mathbb{Z}$ the space of $K$-eigenvectors in $\mathcal{H}_\pi$ of weight $n$ is finite-dimensional, then every $K$-eigenvector of weight $n$ is smooth.*

Proof. Let $v \in \mathcal{H}_\pi$ and let $\psi \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$. By Proposition 7.7 we know that $\pi_*(\psi)v$ is a smooth vector. Now let $n \in \mathbb{Z}$ and let $\chi_n(k_\theta) = \mathrm{e}^{\mathrm{i}n\theta}$ for $k_\theta$ in $K$ be the $n$th character on $K$. Recall that $\bigl(\pi|_K\bigr)_*(\overline{\chi_n})w$ is a $K$-eigenvector of weight $n$ for all $w \in \mathcal{H}_\pi$. Combining these two with $w = \pi_*(\psi)v$, we obtain using Fubini's theorem and the substitution $h = k_\theta g$ that

$$\begin{aligned}\bigl(\pi|_K\bigr)_*(\overline{\chi_n})\pi_*(\psi)v &= \int_K \overline{\chi_n}(k_\theta)\pi_{k_\theta}\int_G \psi(g)\pi_g v \,\mathrm{d}m(g)\,\mathrm{d}m_K(k_\theta)\\ &= \int_K\int_G \overline{\chi_n}(k_\theta)\psi(g)\underbrace{\pi_{k_\theta g}v}_{=\pi_h v}\,\mathrm{d}m(g)\,\mathrm{d}m_K(k_\theta)\\ &= \int_G \underbrace{\int_K \overline{\chi_n}(k_\theta)\psi(k_\theta^{-1}h)\,\mathrm{d}m_K(k_\theta))}_{\psi_n(h)}\pi_h(v)\,\mathrm{d}m(h)\end{aligned}$$

where

$$\psi_n(h) = \int_K \overline{\chi_n}(k_\theta)\psi(k_\theta^{-1}h)\,\mathrm{d}m_K(k_\theta)$$

for $h \in \mathrm{SL}_2(\mathbb{R})$. We note that $\psi_n$ has compact support, with

$$\operatorname{supp}\psi_n \subseteq K\operatorname{supp}\psi.$$

Moreover, $\psi_n$ is also smooth, which follows for instance by considering the derivatives

$$\rho_\partial(\mathbf{m})\psi_n = \lim_{s\to 0}\frac{1}{s}\bigl(\rho_{\exp(\mathbf{m})}\psi_n - \psi_n\bigr)$$

for the right-regular representation and proving that

$$\rho_\delta(\mathbf{m})\psi_n = \int_K \overline{\chi_n}(k_\theta)(\rho_\partial(\mathbf{m})\psi)(k_\theta^{-1}h)\,\mathrm{d}m_K(k_\theta)$$

for all $\mathbf{m}\in\mathfrak{sl}_2(\mathbb{R})$. Hence Proposition 7.7 shows that

$$\bigl(\pi|_K\bigr)_*\bigl(\overline{\chi_n}\bigr)\pi_*(\psi)v = \pi_*(\psi_n)v$$

is a smooth vector. The first claim in the lemma now follows from

$$\pi_*(\psi)v = \sum_{n\in\mathbb{Z}}\bigl(\pi|_K\bigr)_*\bigl(\overline{\chi_n}\bigr)\pi_*(\psi)v = \lim_{N\to\infty}\sum_{n=-N}^{N}\pi_*(\psi_n)v.$$

Fix some $n\in\mathbb{Z}$ and suppose now that

$$\mathcal{V}_n = \{v\in\mathcal{H}_\pi \mid v \text{ has } K\text{-weight } n\}$$

is finite-dimensional. Using Proposition 1.49 for a smooth approximate identity and each vector in a basis of $\mathcal{V}_n$, we can find some $\psi\in C_c^\infty(G)$ such that $\pi_*(\psi)v$ is close to $v$ for each of the basis vectors $v$ of $\mathcal{V}_n$. Since $(\pi|_K)_*(\overline{\chi_n})$ is the orthogonal projection onto $\mathcal{V}_n$, it follows that

$$\pi_*(\psi_n)|_{\mathcal{V}_n}\colon \mathcal{V}_n \longrightarrow \mathcal{V}_n$$

is as close to the identity on $\mathcal{V}_n$ as we desire. In particular, we may ensure that $\pi_*(\psi_n)(\mathcal{V}_n)=\mathcal{V}_n$, and the final claim of the lemma follows from the first part of the proof. □

### 9.2.4 A Differential Equation for Matrix Coefficients

We continue our journey towards the description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ by examining matrix coefficients of $K$-eigenvectors in irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$. In fact we now show how the Casimir eigenvalue $\alpha_\pi$ for some unitary representation $\pi\in\widehat{\mathrm{SL}_2(\mathbb{R})}$, and the mere existence of a smooth unit vector $v\in\mathcal{H}_\pi$ of $K$-weight $n$, determines its matrix coefficient $\varphi_v^\pi$.

**Lemma 9.18 (The differential equation for the matrix coefficient).** *Let $\pi$ be a unitary representation of the group $\mathrm{SL}_2(\mathbb{R})$. Suppose that $v\in\mathcal{H}_\pi$ is a smooth $K$-eigenvector with weight $n\in\mathbb{Z}$ and Casimir eigenvalue $\alpha\in\mathbb{R}$. Then*

$$\varphi_v^\pi\bigl(k_\theta a_t k_\psi\bigr) = \mathrm{e}^{\mathrm{i}n(\theta+\psi)}\varphi_v^\pi(a_t)$$

*for all* $k_\theta a_t k_\psi \in \mathrm{SL}_2(\mathbb{R})$ *and the smooth function* $\phi\colon \mathbb{R} \to \mathbb{C}$ *defined by*

$$\phi(t) = \varphi_v^\pi(a_t)$$

*for* $t \in \mathbb{R}$ *satisfies the second order linear differential equation*

$$\phi''(t) + 2\tfrac{\cosh(2t)}{\sinh(2t)}\phi'(t) + \left(1 - \alpha + \tfrac{n^2}{\cosh^2 t}\right)\phi(t) = 0$$

*of degree two for all* $t \in \mathbb{R}\smallsetminus\{0\}$*. Moreover,* $\phi$ *is real-valued and satisfies*

$$\phi(t) = \phi(-t)$$

*for all* $t \in \mathbb{R}$*.*

The proofs of Lemma 9.18 and the following Proposition 9.19 are rather elementary, but parts of them may require good motivation by the reader. To gain this motivation, the reader may at first skip the two proofs in order to see how these results are used to help us describe $\widehat{\mathrm{SL}_2(\mathbb{R})}$ in Section 9.2.5.

Proof of Lemma 9.18. The first part of the lemma follows simply because $v$ is a $K$-eigenvector of weight $n$. Indeed, we have

$$\varphi_v^\pi\big(k_\theta a_t k_\psi\big) = \left\langle \pi_{a_t}\pi_{k_\psi} v, \pi_{k_{-\theta}} v\right\rangle = \left\langle \pi_{a_t}(\mathrm{e}^{\mathrm{i}n\psi}v), \mathrm{e}^{-\mathrm{i}n\theta}v\right\rangle = \mathrm{e}^{\mathrm{i}n(\theta+\psi)}\varphi_v^\pi(a_t)$$

for all $k_\theta a_t k_\psi \in \mathrm{SL}_2(\mathbb{R})$. We note that

$$k_{\pi/2} a_t k_{-\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^{-t} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = a_{-t},$$

for all $t \in \mathbb{R}$. For $\phi = \varphi_v^\pi$ this implies

$$\phi(-t) = \varphi_v^\pi(a_{-t}) = \varphi_v^\pi\big(k_{\pi/2} a_t k_{-\pi/2}\big) = \varphi_v^\pi(a_t) = \phi(t)$$

for all $t \in \mathbb{R}$ by Lemma 9.18. In other words, $\phi$ is an even function on $\mathbb{R}$. Moreover,

$$\overline{\phi(t)} = \overline{\langle \pi_{a_t} v, v\rangle} = \langle v, \pi_{a_t} v\rangle = \langle \pi_{a_{-t}} v, v\rangle = \phi(-t) = \phi(t)$$

for all $t \in \mathbb{R}$, which also shows that $\phi$ only takes values in $\mathbb{R}$.

To obtain the differential equation for $\phi(t) = \varphi_v^\pi(a_t)$ for $t \in \mathbb{R}\smallsetminus\{0\}$, we will combine the information that $v$ has $K$-weight $n \in \mathbb{Z}$, the assumption $\pi_\partial(\Omega)v = \alpha v$, and the formula

$$\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + 2\mathbf{e}\circ\mathbf{f} + 2\mathbf{f}\circ\mathbf{e}$$

in Lemma 9.2. For this, recall that $v$ having $K$-weight $n$ implies $\pi_\partial(\mathbf{k})v = \mathrm{i}nv$ and $\pi_\partial(\mathbf{k}^{\circ 2})v = -n^2 v$.

We start by calculating how to express $\phi'$ and $\phi''$ as matrix coefficients. Indeed,

$$\begin{aligned}\phi'(t) &= \lim_{h\to 0}\frac{1}{h}\big(\phi(t+h)-\phi(t)\big)\\ &= \lim_{h\to 0}\frac{1}{h}\left\langle \pi_{a_t}(\pi_{a_h}v-v),v\right\rangle = \left\langle \pi_{a_t}\pi_\partial(\mathbf{a})v,v\right\rangle\end{aligned}$$

and, similarly,

$$\phi''(t) = \lim_{h\to 0}\frac{1}{h}\left\langle \pi_{a_t}\big(\pi_{a_h}\pi_\partial(\mathbf{a})v-\pi_\partial(\mathbf{a})v\big),v\right\rangle = \left\langle \pi_{a_t}\pi_\partial(\mathbf{a}^{\circ 2})v,v\right\rangle$$

for all $t\in\mathbb{R}$.

We note that $\mathbf{a}^{\circ 2}$ is one of the terms in $\Omega$, but that we do not yet have an interpretation of the term $2\mathbf{e}\circ\mathbf{f}+2\mathbf{f}\circ\mathbf{e}$ in terms of $\phi$. To obtain such an interpretation, we use the consequence of $v$ having $K$-weight $n$ mentioned above to express $-n^2\phi$ in three different ways. Indeed, we have

$$\begin{aligned}-n^2\phi(t) &= -n^2\left\langle \pi_{a_t}v,v\right\rangle = \left\langle \pi_{a_t}(-n^2v),v\right\rangle\\ &= \left\langle \pi_{a_t}\pi_\partial(\mathbf{k}^{\circ 2})v,v\right\rangle;\\ -n^2\phi(t) &= -\left\langle \pi_{a_t}\mathrm{i}nv,\mathrm{i}nv\right\rangle = -\left\langle \pi_{a_t}\pi_\partial(\mathbf{k})v,\pi_\partial(\mathbf{k})v\right\rangle = \left\langle \pi_\partial(\mathbf{k})\pi_{a_t}\pi_\partial(\mathbf{k})v,v\right\rangle\\ &= \left\langle \pi_{a_t}\pi_\partial\big(\mathrm{Ad}_{a_{-t}}(\mathbf{k})\circ\mathbf{k}\big)v,v\right\rangle;\\ -n^2\phi(t) &= \left\langle \pi_{a_t}v,-n^2v\right\rangle = \left\langle \pi_{a_t}v,\pi_\partial(\mathbf{k}^{\circ 2})v\right\rangle = \left\langle \pi_\partial(\mathbf{k}^{\circ 2})\pi_{a_t}v,v\right\rangle\\ &= \left\langle \pi_{a_t}\pi_\partial\Big(\big(\mathrm{Ad}_{a_{-t}}(\mathbf{k})\big)^{\circ 2}\Big)v,v\right\rangle\end{aligned}$$

for all $t\in\mathbb{R}$. We recall that $\mathbf{k}=-\mathbf{e}+\mathbf{f}$ and note that

$$\mathrm{Ad}_{a_{-t}}(\mathbf{k}) = -\mathrm{e}^{-2t}\mathbf{e}+\mathrm{e}^{2t}\mathbf{f}$$

for all $t\in\mathbb{R}$. We put these two formulas into the three expressions above, expand the resulting parentheses, and obtain from this that

$$\begin{aligned}-n^2\phi(t) &= \left\langle \pi_{a_t}\pi_\partial\big(\mathbf{e}^{\circ 2}-\mathbf{e}\circ\mathbf{f}-\mathbf{f}\circ\mathbf{e}+\mathbf{f}^{\circ 2}\big)v,v\right\rangle\\ &= \left\langle \pi_{a_t}\pi_\partial\big(\mathrm{e}^{-2t}\mathbf{e}^{\circ 2}-\mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f}-\mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e}+\mathrm{e}^{2t}\mathbf{f}^{\circ 2}\big)v,v\right\rangle\\ &= \left\langle \pi_{a_t}\pi_\partial\big(\mathrm{e}^{-4t}\mathbf{e}^{\circ 2}-\mathbf{e}\circ\mathbf{f}-\mathbf{f}\circ\mathbf{e}+\mathrm{e}^{4t}\mathbf{f}^{\circ 2}\big)v,v\right\rangle\end{aligned}$$

for all $t\in\mathbb{R}$.

As our aim is to find a formula involving $2\mathbf{e}\circ\mathbf{f}+2\mathbf{f}\circ\mathbf{e}$, we wish to rid ourselves of all expressions involving the two terms $\mathbf{e}^{\circ 2}$ and $\mathbf{f}^{\circ 2}$. As we have three formulas for $-n^2\phi$, this is an exercise in linear algebra. In fact, multiplying the first line by 1, the second by $-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)$, the third by 1, and taking the sum gives

$$-n^2\phi(t)\big(1-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)+1\big)=n^2\phi(t)\big(\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}\big)=4n^2\sinh^2 t\,\phi(t)$$

on the left-hand side.

On the right-hand side, we use the similarities between the three formulas and obtain the expression

$$\big\langle \pi_{a_t}\pi_\partial(\mathbf{m}_t)v,v\big\rangle\,,$$

where $\mathbf{m}_t$ is the element of $\mathfrak{E}_{\leqslant 2}$ defined by

$$\begin{aligned}\mathbf{m}_t=\big(\mathbf{e}^{\circ 2}-\mathbf{e}\circ\mathbf{f}-\mathbf{f}\circ\mathbf{e}+\mathbf{f}^{\circ 2}\big)&\\ -\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)\big(\mathrm{e}^{-2t}\mathbf{e}^{\circ 2}-\mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f}-\mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e}+\mathrm{e}^{2t}\mathbf{f}^{\circ 2}\big)&\\ +\big(\mathrm{e}^{-4t}\mathbf{e}^{\circ 2}-\mathbf{e}\circ\mathbf{f}-\mathbf{f}\circ\mathbf{e}+\mathrm{e}^{4t}\mathbf{f}^{\circ 2}\big).&\end{aligned}$$

Our choice of the three coefficients 1, $-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)$, and 1 was made so that the coefficient in front of $\mathbf{e}^{\circ 2}$ is

$$1-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)\mathrm{e}^{-2t}+\mathrm{e}^{-4t}=1-1-\mathrm{e}^{-4t}+\mathrm{e}^{-4t}=0$$

and the coefficient in front of $\mathbf{f}^{\circ 2}$ is

$$1-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)\mathrm{e}^{2t}+\mathrm{e}^{4t}=1-\mathrm{e}^{4t}-1+\mathrm{e}^{4t}=0$$

also. Therefore,

$$\begin{aligned}\mathbf{m}_t&=\big(-\mathbf{e}\circ\mathbf{f}-\mathbf{f}\circ\mathbf{e}\big)-\big(\mathrm{e}^{2t}+\mathrm{e}^{-2t}\big)\big(-\mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f}-\mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e}\big)+\big(-\mathbf{e}\circ\mathbf{f}-\mathbf{f}\circ\mathbf{e}\big)\\&=\big(-1+1+\mathrm{e}^{-4t}-1\big)\mathbf{e}\circ\mathbf{f}+\big(-1+\mathrm{e}^{4t}+1-1\big)\mathbf{f}\circ\mathbf{e}\\&=\big(\mathrm{e}^{-4t}-1\big)\mathbf{e}\circ\mathbf{f}+\big(\mathrm{e}^{4t}-1\big)\mathbf{f}\circ\mathbf{e}.\end{aligned}$$

Using

$$\begin{aligned}\mathrm{e}^{4t}-1&=\mathrm{e}^{2t}\big(\mathrm{e}^{2t}-\mathrm{e}^{-2t}\big)=2\mathrm{e}^{2t}\sinh(2t),\\ \mathrm{e}^{-4t}-1&=\mathrm{e}^{-2t}\big(\mathrm{e}^{2t}-\mathrm{e}^{-2t}\big)=-2\mathrm{e}^{-2t}\sinh(2t),\end{aligned}$$

we also have

$$\mathbf{m}_t=-2\mathrm{e}^{-2t}\sinh(2t)\mathbf{e}\circ\mathbf{f}+2\mathrm{e}^{2t}\sinh(2t)\mathbf{f}\circ\mathbf{e}.$$

To summarize, we have shown that

$$\begin{aligned}4n^2\sinh^2 t\,\phi(t)&=\big\langle\pi_{a_t}\pi_\partial(\mathbf{m}_t)v,v\big\rangle\\&=2\sinh(2t)\,\big\langle\pi_{a_t}\pi_\partial\big(-\mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f}+\mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e}\big)v,v\big\rangle\,.\end{aligned}$$

Dividing by $2\sinh(2t)=4\sinh t\cosh t$, we obtain

$$n^2\frac{\sinh t}{\cosh t}\phi(t)=\big\langle\pi_{a_t}\pi_\partial\big(-\mathrm{e}^{-2t}\mathbf{e}\circ\mathbf{f}+\mathrm{e}^{2t}\mathbf{f}\circ\mathbf{e}\big)v,v\big\rangle\,.$$

Next we note that for $s_1, s_2 \in \mathbb{R}$, we have

$$s_1 \mathbf{e} \circ \mathbf{f} + s_2 \mathbf{f} \circ \mathbf{e} = \frac{s_1 + s_2}{2}(\mathbf{e} \circ \mathbf{f} + \mathbf{f} \circ \mathbf{e}) + \frac{s_1 - s_2}{2} \underbrace{[\mathbf{e}, \mathbf{f}]}_{=\mathbf{a}},$$

which, with the choice $s_1 = -\mathrm{e}^{-2t}$ and $s_2 = \mathrm{e}^{2t}$, gives

$$\begin{aligned} n^2 \frac{\sinh t}{\cosh t} \phi(t) &= \left\langle \pi_{a_t} \pi_\partial \left( \frac{\mathrm{e}^{2t} - \mathrm{e}^{-2t}}{2} (\mathbf{e} \circ \mathbf{f} + \mathbf{f} \circ \mathbf{e}) - \frac{\mathrm{e}^{2t} + \mathrm{e}^{-2t}}{2} \mathbf{a} \right) v, v \right\rangle \\ &= \frac{\sinh(2t)}{2} \left\langle \pi_{a_t} \pi_\partial (2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}) v, v \right\rangle - \cosh(2t) \phi'(t). \end{aligned}$$

Dividing by $\frac{\sinh(2t)}{2} = \sinh t \cosh t$, we also obtain

$$\left\langle \pi_{a_t} \pi_\partial (2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}) v, v \right\rangle = \frac{n^2}{\cosh^2 t} \phi(t) + 2 \frac{\cosh(2t)}{\sinh(2t)} \phi'(t).$$

Using the relations $\pi_\partial(\Omega) v = \alpha v$ and $\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + 2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}$, we finally arrive at the differential equation

$$\begin{aligned} \alpha \phi(t) &= \left\langle \pi_{a_t}(\alpha v), v \right\rangle \\ &= \left\langle \pi_{a_t} \pi_\partial(\Omega) v, v \right\rangle \\ &= \phi(t) + \phi''(t) + \left\langle \pi_{a_t} \pi_\partial (2\mathbf{e} \circ \mathbf{f} + 2\mathbf{f} \circ \mathbf{e}) v, v \right\rangle \\ &= \phi''(t) + \left( 1 + \frac{n^2}{\cosh^2 t} \right) \phi(t) + 2 \frac{\cosh(2t)}{\sinh(2t)} \phi'(t), \end{aligned}$$

as in the lemma. □

**Proposition 9.19 (Determining the matrix coefficient).** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. Suppose that $v \in \mathcal{H}_\pi$ is a smooth $K$-eigenvector of weight $n \in \mathbb{Z}$ and Casimir eigenvalue $\alpha \in \mathbb{R}$. Then the matrix coefficient $\varphi_v^\pi$ is uniquely determined by $\alpha$, $n$, and $\|v\|$.*

We would like to apply the uniqueness part of the theorem of Picard–Lindelöf[(19)] to the function $\phi(t) = \varphi_v^\pi(a_t)$ for $t \in \mathbb{R}$ introduced in Lemma 9.18. In fact $\phi(0) = \|v\|^2$ and $\phi'(0) = 0$ (see Exercise 9.20) give two initial conditions for the second-order differential equation satisfied by $\phi$. Unfortunately, applying the Picard–Lindelöf theorem is not straightforward, as the differential equation is really only defined on the domain $\mathbb{R} \smallsetminus \{0\}$.

**Exercise 9.20.** Show that the function $\phi$ defined in Lemma 9.18 satisfies $\phi'(0) = 0$.

PROOF OF PROPOSITION 9.19. Let $\pi$, $v$, $\alpha$, $n$, and $\phi$ be as in Lemma 9.18. We briefly discuss the structure of all real-valued solutions to the second-order linear differential equation

$$y'' + 2\frac{\cosh(2t)}{\sinh(2t)}y' + \left(1 - \alpha + \frac{n^2}{\cosh^2 t}\right) y = 0 \tag{9.15}$$

for $t \in \mathbb{R}\smallsetminus\{0\}$. Restricting to the connected component $(-\infty, 0)$ of $\mathbb{R}\smallsetminus\{0\}$, a corollary of the theorem of Picard–Lindelöf shows that

$$\mathscr{F} = \{y\colon (-\infty, 0) \to \mathbb{R} \ \mid y \text{ solves (9.15) for all } t \in (-\infty, 0)\}$$

is a two-dimensional real vector space. In fact this corresponds to the independence and sufficiency (to determine the unique solution) of the two initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$ for some $t_0 < 0$. By Lemma 9.18 we have

$$y_v = \phi|_{(-\infty,0)} \in \mathscr{F}.$$

We claim that $\mathscr{F}$ contains an element $y_\infty$ with

$$\lim_{t\nearrow 0} y_\infty'(t) = +\infty. \tag{9.16}$$

Assuming the claim for now, we see that $y_v, y_\infty \in \mathscr{F}$ are linearly independent, and so they form a basis for $\mathscr{F}$. Suppose now that $\tau$ and $w$ are another unitary representation and vector as in Lemma 9.18 with the same $K$-weight $n$, the same Casimir eigenvalue $\alpha$, and the same norm $\|v\| = \|w\|$. Then

$$y_w(t) = \varphi_w^\tau(a_{-t})$$

for $t \in (-\infty, 0)$ defines another element of $\mathscr{F}$. It follows that

$$y_w = s_v y_v + s_\infty y_\infty$$

for some $s_v, s_\infty \in \mathbb{R}$. Recall that $\varphi_v^\pi(a_t)$ and $\varphi_w^\tau(a_t)$ defining $y_v$ and $y_w$ are both even by Lemma 9.18. If $s_\infty \neq 0$ we may use Exercise 9.20 and (9.16) to obtain the contradiction

$$0 = \lim_{t\nearrow 0} |y_w'(t)| = \lim_{t\nearrow 0} |s_v y_v' + s_\infty y_\infty'| = \infty.$$

Hence $s_\infty = 0$, and so

$$\|w\|^2 = \lim_{t\nearrow 0} y_w(t) = \lim_{t\nearrow 0} s_v y_v(t) = s_v \|v\|^2$$

shows that $s_v = 1$, and so $y_w = y_v$.

Using Lemma 9.18, we see that the claim implies the proposition.

It remains to construct $y_\infty \in \mathscr{F}$ as in the claim. We note that since the coefficient $2\frac{\cosh 2t}{\sinh 2t}$ of $y'$ in the differential equation (9.15) goes to infinity in absolute value as $t \to \infty$, it is natural to expect the claim to hold. The following elementary proof of the claim precisely relies on this property of (9.15) and is simply an exercise in real analysis.

To bound the effect of the term involving $y$ in (9.15), we define

$$M = \max_{t\in[-1,0]} \left|1 - \alpha + \frac{n^2}{\cosh^2 t}\right|$$

and choose $t_0 \in [-1, 0)$ so that

$$\left|\frac{\cosh(2t)}{\sinh(2t)}\right| \geqslant M + 1$$

for all $t \in [t_0, 0)$. Using the existence part of the theorem of Picard–Lindelöf, we define $y_\infty$ as the solution of (9.15) on $(-\infty, 0)$ with the initial value conditions

$$\begin{cases} y_\infty(t_0) = 0, \\ y'_\infty(t_0) = 1. \end{cases}$$

We note that this gives

$$y''_\infty(t_0) = -2\frac{\cosh(2t_0)}{\sinh(2t_0)} > 0$$

by (9.15). We will show that

$$\lim_{t\nearrow 0} y_\infty(t) = \infty.$$

For this, we first define

$$B = \bigl\{t \in [t_0, 0) \mid y'_\infty \geqslant y_\infty \geqslant 0\bigr\},$$

so that $t_0 \in B$. Moreover, since $y'_\infty(t_0) > y_\infty(t_0) = 0$, there exists some $\delta_0$ with $[t_0, t_0 + \delta_0) \subseteq B$ and $y_\infty(t_0 + \delta_0) > 0$. We also define

$$s = \sup\bigl\{t \in [-t_0, 0) \mid [0, t) \subseteq B\bigr\}$$

and note that $[t_0, s) \subseteq B$. Now consider the derivative of $y'_\infty - y_\infty$, which is given by

$$\begin{aligned} y''_\infty(t) - y'_\infty(t) &= \left|2\frac{\cosh(2t)}{\sinh(2t)}\right| y'_\infty(t) - \left(1 - \alpha + \frac{n^2}{\cosh^2 t}\right) y_\infty(t) - y'_\infty(t) \\ &= \underbrace{\left(\left|2\frac{\cosh(2t)}{\sinh(2t)}\right| - 1\right)}_{\geqslant M} y'_\infty(t) - \underbrace{\left(1 - \alpha + \frac{n^2}{\cosh^2 t}\right)}_{\leqslant M} y_\infty(t), \end{aligned}$$

where the indicated estimates hold for all $t \in [t_0, 0)$. For $t \in [t_0, s)$ we obtain from our definitions of $t_0$ and of $B$ that

$$y''_\infty(t) - y'_\infty(t) \geqslant M y'_\infty(t) - M y_\infty(t) \geqslant 0.$$

However, this shows that $t \mapsto y'_\infty(t) - y_\infty(t)$ is monotone non-decreasing on $[t_0, s)$. Suppose for a moment that $s < 0$. Then monotonicity of $y'_\infty - y_\infty$ and of $y_\infty$ on $[t_0, s]$ imply

$$\begin{cases} y'_\infty(s) - y_\infty(s) \geqslant y'_\infty(t_0) - y_\infty(t_0) = 1 > 0, \\ y_\infty(s) \geqslant y_\infty(t_0 + \delta_0) > 0. \end{cases}$$

However, this also implies the existence of some $\delta > 0$ with $[s, s+\delta) \subseteq B$ and contradicts the definition of $s$. Therefore we have $s = 0$.

Equivalently, we have shown that

$$y'_\infty(t) \geqslant y_\infty(t) \geqslant 0$$

for all $t \in [t_0, 0)$. With this we now estimate the growth of $y'_\infty$ on $[t_0, 0)$. Indeed,

$$\begin{aligned} y''_\infty(t) &= \left| 2\frac{\cosh(2t)}{\sinh(2t)} \right| y'_\infty(t) - \left(1 - \alpha + \frac{n^2}{\cosh^2 t}\right) y_\infty(t) \\ &\geqslant \left| \frac{\cosh(2t)}{\sinh(2t)} \right| y'_\infty(t) + \underbrace{M y'_\infty(t) - M y_\infty(t)}_{\geqslant 0} \geqslant 0 \end{aligned}$$

for all $t \in [t_0, 0)$ implies that $y'_\infty$ is monotone non-decreasing.

With $y'_\infty(t_0) = 1$, this gives $y'_\infty(t) \geqslant 1$ for all $t \in [t_0, 0)$, which leads to

$$y''_\infty(t) \geqslant \left| \frac{\cosh(2t)}{\sinh(2t)} \right| \geqslant \frac{c}{|t|}$$

for all $t \in [t_0, 0)$ and some absolute constant $c > 0$. Therefore

$$\lim_{t \nearrow 0} y''_\infty(t) = \lim_{t \nearrow 0} y'_\infty(t) = \infty.$$

This proves the claim, and hence the proposition. □

### 9.2.5 The Unitary Dual of $\mathbf{SL_2}(\mathbb{R})$

The following corollary to Proposition 9.19 will be our main tool for the classification of the elements of $\widehat{\mathrm{SL}_2(\mathbb{R})}$.

**Corollary 9.21 (Isomorphisms).** *Let $\pi$ and $\tau$ be irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$. Suppose that the Casimir eigenvalues $\alpha_\pi = \alpha_\tau$ agree, and that there exists some $n \in \mathbb{Z}$ such that both $\mathcal{H}_\pi$ and $\mathcal{H}_\tau$ contain a $K$-eigenvector of weight $n$. Then $\pi \cong \tau$.*

Proof. Let $v \in \mathcal{H}_\pi$ and $w \in \mathcal{H}_\tau$ be $K$-eigenvectors of weight $n$. Without loss of generality, we may assume that $\|v\| = \|w\| = 1$. By Proposition 9.19 this implies $\phi_v^\pi = \phi_w^\tau$. However, Proposition 1.63 now shows that

$$\mathcal{H}_\pi = \langle v \rangle_\pi \cong \langle w \rangle_\tau = \mathcal{H}_\tau$$

are isomorphic as unitary representations of $\mathrm{SL}_2(\mathbb{R})$. □

We are now in a position to completely describe the unitary dual of $\mathrm{SL}_2(\mathbb{R})$. Since $\mathrm{SL}_2(\mathbb{R})$ has the non-trivial centre $\{\pm I\}$, the first distinction we can make between irreducible unitary representations of $\mathrm{SL}_2(\mathbb{R})$ is by use of the central character as in Corollary 1.32. Since the centre of $\mathrm{SL}_2(\mathbb{R})$ is given by $\{\pm I\}$, we say that $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$ is *even* if $\pi_{-I} = I$ and *odd* if $\pi_{-I} = -I$. The second distinction is in terms of the 'infinitesimal character' obtained by applying Proposition 9.8 to all central elements of the universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{sl}_2(\mathbb{R})$. Actually this centre is generated† by $\mathbb{1}_{\mathfrak{E}}$ and the Casimir element $\Omega$ of Section 9.2.1. Hence the infinitesimal character is in our case simply the Casimir eigenvalue $\alpha_\pi \in \mathbb{R}$. The third and final distinction is in terms of which $K$-weights are present in the representation $\pi$.

The following result of Bargmann [2] contains all the possibilities of these three aspects, and introduces the final two types of irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$; these will be studied in detail in Sections 9.3 and 9.5.

We depict $\widehat{\mathrm{SL}_2(\mathbb{R})}$ geometrically in Figure 9.1, where we draw even representations on the top half and odd ones on the bottom half. We also present $\widehat{\mathrm{SL}_2(\mathbb{R})}$ as a list in Table 9.1.

**Table 9.1:** The different types of irreducible unitary representations and their main properties.

| Notation | Name | Tempered? | Casimir eigenvalue | $K$-weights |
|---|---|---|---|---|
| $\delta^{n,\pm}$ for $n \geqslant 2$ | discrete series representation | ✓ | $(n-1)^2$ | $\pm(n+2\mathbb{N}_0)$ |
| $\delta^{1,\pm}$ | mock discrete series representation | ✓ | 0 | $\pm(1+2\mathbb{N}_0)$ |
| $\pi^{\xi,\mathrm{e}}$ for $\xi \geqslant 0$ | even principal series representation | ✓ | $-\xi^2$ | $2\mathbb{Z}$ |
| $\pi^{\xi,\mathrm{o}}$ for $\xi > 0$ | odd principal series representation | ✓ | $-\xi^2$ | $1+2\mathbb{Z}$ |
| $\gamma^s$ for $s \in (0,1)$ | complementary series representation | ✗ | $s^2$ | $2\mathbb{Z}$ |
| $\mathbb{1}$ | trivial representation | ✗ | 1 | $\{0\}$ |

† We do not have to know this (see Exercise 9.3) if we simply define the infinitesimal character as the Casimir eigenvalue.

**Theorem 9.22 (Unitary dual of $\mathrm{SL}_2(\mathbb{R})$).** *Suppose that $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$ is even. Then one of the following four possibilities holds:*

- $\alpha_\pi = 1$ *and* $\pi = \mathbb{1}$ *is the trivial representation.*
- $\alpha_\pi = (n-1)^2$ *for some* $n \in 2\mathbb{N}$, *and* $\pi = \delta^{n,\pm}$ *is either the holomorphic or the anti-holomorphic discrete series representation with terminal weight* $\pm n$ *(see Section 8.4).*
- $\alpha_\pi = -\xi^2 \leqslant 0$ *for some* $\xi \in [0,\infty)$ *and* $\pi = \pi^{\xi,\mathrm{e}}$ *is the even principal series representation for the parameter* $\xi$ *(see Section 9.3).*
- $\alpha_\pi = s^2$ *for some* $s \in (0,1)$, *and* $\pi = \gamma^s$ *is the complementary series representation for the parameter* $s$ *(see Section 9.5).*

*Suppose that $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$ is odd. Then one of the following three possibilities holds:*

- $\alpha_\pi = 0$ *and* $\pi = \delta^{1,\pm}$ *is either the holomorphic or the anti-holomorphic mock discrete series representations (see Section 8.4).*
- $\alpha_\pi = (n-1)^2$ *for some* $n \in (2\mathbb{N}+1)$, *and* $\pi = \delta^{n,\pm}$ *is either the holomorphic or the anti-holomorphic discrete series representation with terminal weight* $\pm n$ *(see Section 8.4).*
- $\alpha_\pi = -\xi^2$ *for some* $\xi \in (0,\infty)$ *and* $\pi = \pi^{\xi,\mathrm{o}}$ *is the odd principal series representation for the parameter* $\xi$ *(see Section 9.3).*

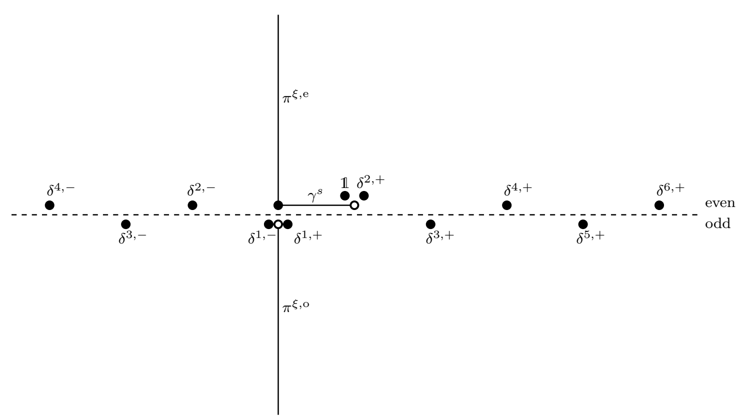


**Fig. 9.1:** The graphical representation of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ as a subset of $\mathbb{C}$ (with the representations $\pi^{0,\mathrm{e}}, \delta^{1,-}, \delta^{1,+}$ at the origin and both $\mathbb{1}$ and $\delta^{2,+}$ at $1 \in \mathbb{C}$) also has the property that $\alpha_\pi$ is the square of the position of $\pi$ when drawn in $\mathbb{C}$ (except for the artificial small gap between the even and odd representations, and problems arising from $\{\pi^{0,\mathrm{e}}, \delta^{1,+}, \delta^{1,-}\}$ and $\{\delta^{2,+}, \mathbb{1}\}$ which should be drawn at the same point).

Proof of Theorem 9.22. In the following we let $\pi$ be an irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$. By Corollary 9.12, the closure of $\pi_\partial(\Omega)$ is multiplication by $\alpha_\pi$ for some $\alpha_\pi \in \mathbb{R}$. By Lemma 9.17, there also exists a smooth $K$-eigenvector $v \in \mathcal{H}_\pi$ with weight $n \in \mathbb{Z}$ and unit length $\|v\| = 1$. By Corollary 9.21, we have that $(n, \alpha_\pi)$ uniquely determines $\pi$ up to isomorphism. Hence the question is really which $(n, \alpha_\pi) \in \mathbb{Z} \times \mathbb{R}$ are possible (in general, and within each irreducible representation). As already explained, Corollary 9.14 gives the constraint

$$\alpha_\pi \leqslant (n \pm 1)^2; \tag{9.17}$$

(see the discussion leading to (9.14), and Figure 9.2).

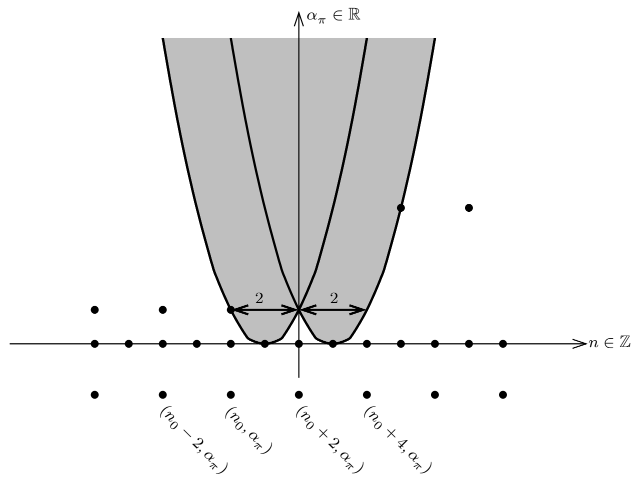


**Fig. 9.2:** The shaded region is the union of the region defined by the two inequalities $\alpha_\pi > (n+1)^2$ and $\alpha_\pi > (n-1)^2$. By (9.17), we know that this region cannot contain a pair $(n, \alpha_\pi)$ corresponding to a non-zero $K$-eigenvector for some representation $\pi \in \widehat{\mathrm{SL}_2(\mathbb{R})}$. In addition, we see how Proposition 9.13 creates additional pairs from one pair. We also note that the two parabolas have width 2 precisely at height $\alpha_\pi = 1$.

We first go through the list of representations that we have already encountered.

- If $\pi = \mathbb{1}_G$ is the trivial representation on $\mathbb{C}$, then $v = 1 \in \mathbb{C}$ has $K$-weight $n = 0$ and Casimir eigenvalue 1, since $\pi_\partial(\Omega) = \pi_\partial(\mathbb{1}_{\mathfrak{E}}) = 1$. By the above, it follows that $\mathbb{1}_G$ is characterized by the pair $(n = 0, \alpha = 1)$.
- For the holomorphic discrete series representation $\delta^{n,+}$ or the anti-holomorphic discrete series representation $\delta^{n,-}$ with $n \geqslant 2$ we have, by Theorem 8.23, that $\delta^{n,\pm}$ contains vectors with $K$-weights $\pm(n + 2\mathbb{N}_0)$ and only these. Moreover, Corollary 9.15 gives $\alpha_{\delta^{n,\pm}} = (n-1)^2$.

- This also holds similarly for the mock discrete series representations $\delta^{1,\pm}$ with $K$-weights $\pm(1+2\mathbb{N}_0)$ and $\alpha_{\delta^{1,\pm}} = 0$ (see Theorem 8.30 and Corollary 9.15).

We next describe the remaining irreducible unitary representations listed in the theorem and in Table 9.1.

- Suppose $\pi$ is an irreducible unitary representation with $\alpha_\pi < 0$. Now let $v_n \in \mathcal{H}_\pi$ be a smooth $K$-eigenvector of weight $n \in \mathbb{Z}$. Using the raising and lowering operators in Proposition 9.13, we can define two other $K$-eigenvectors $\pi_\partial(\mathbf{r}^\pm)v_n$ of $K$-weights $n \pm 2$. By Corollary 9.14, we have
$$\|\pi_\partial(\mathbf{r}^\pm)v_n\|^2 = \tfrac{1}{4}\big((n\pm 1)^2 - \alpha_\pi\big)\|v_n\|^2. \tag{9.18}$$
Since we assume $\alpha_\pi < 0$, it follows that $(n\pm1)^2 - \alpha_\pi > 0$ and so that $\pi$ contains smooth non-zero $K$-eigenvalues of $K$-weights $n \pm 2$. Iterating this shows that $\pi$ contains $K$-eigenvectors with $K$-weights $m$ for all $m \in n + 2\mathbb{Z}$. Suppose next that $\tau$ is another irreducible unitary representation with $\alpha_\tau = \alpha_\pi$. If $\pi$ and $\tau$ are both even (or both odd) irreducible representations, this, together with the first argument of the proof, implies that $\pi \cong \tau$. In other words, if there is an even (or, similarly, an odd) irreducible unitary representation $\pi$ with a given $\alpha_\pi < 0$, then $\alpha_\pi$ uniquely determines $\pi$ up to isomorphism. We will show in Section 9.3 that for any $\xi > 0$ there exists an irreducible unitary representation $\pi^{\xi,\mathrm{e}}$ with $\alpha_{\pi^{\xi,\mathrm{e}}} = -\xi^2$ and $K$-weights in $2\mathbb{Z}$, and an irreducible unitary representation $\pi^{\xi,\mathrm{o}}$ with $\alpha_{\pi^{\xi,\mathrm{o}}} = -\xi^2$ and $K$-weights in $1+2\mathbb{Z}$. These are the even and odd principal series representations.
- The discussion above almost applies to the case where the Casimir eigenvalue $\alpha_\pi = 0$. Indeed, if $\pi$ is an even irreducible unitary representation with $\alpha_\pi = 0$, then the $K$-weight $n$ is even, $n \pm 1$ is odd, and hence
$$(n\pm1)^2 - \alpha_\pi = (n\pm1)^2 > 0.$$
Applying the argument above, it follows that $\pi$ must contain $K$-eigenvectors for all even $K$-weights, and that $\pi$ is uniquely determined up to isomorphism. We will show in Section 9.3 that the even principal series representation $\pi^{0,\mathrm{e}}$ is this irreducible unitary representation with $\alpha_{\pi^{0,\mathrm{e}}} = 0$. We note that the argument above fails in the odd case precisely when $n = \pm 1$. Moreover, we already found the two odd irreducible unitary representations $\delta^{1,\pm}$ with vanishing Casimir eigenvalue and with $K$-weights in $\pm(1+2\mathbb{N}_0)$ respectively.
- Suppose now that $\pi$ is an even irreducible unitary representation with
$$\alpha_\pi \in (0,1).$$
We note that
$$(n\pm1)^2 - \alpha_\pi \geqslant 1 - \alpha_\pi > 0$$

for all even $n \in \mathbb{Z}$. Hence the argument above applies once more, $\pi$ contains $K$-eigenvectors for all even $K$-weights, and is uniquely determined up to isomorphism by $\alpha_\pi$. We will show in Section 9.5 that this so-called complementary series representation $\gamma^s$ for $s \in (0,1)$ with $\alpha_{\gamma^s} = s^2 \in (0,1)$ exists.

It remains to show that the cases above give all possible irreducible unitary representations. So let $\pi$ be an irreducible unitary representation with Casimir eigenvalue $\alpha_\pi$ and let $v_n \in \mathcal{H}_\pi$ be a smooth $K$-eigenvector of $K$-weight $n \in \mathbb{Z}$. We will now use Figure 9.2 to repeat and extend the argument that we used above for the principal series representation. By Proposition 9.13 and Corollary 9.14, the vectors $\pi_\partial(\mathbf{r}^\pm)v_n$ have $K$-weight $n \pm 2$ and satisfy (9.18). In particular, this implies that $\alpha_\pi \leqslant (n \pm 1)^2$, or equivalently that $(n, \alpha_\pi)$ does not belong to the 'forbidden' shaded region in Figure 9.2. Moreover, if $(n, \alpha_\pi)$ does not belong to either of the two parabolas defined by $\alpha = (n \pm 1)^2$, then we may replace $n$ by $n \pm 2$ and iterate this argument to obtain further eigenvectors with different $K$-weights. If, however, $\alpha_\pi = (n+1)^2$ (or $\alpha_\pi = (n-1)^2$), then (9.18) shows that $\pi_\partial(\mathbf{r}^+)v_n = 0$ (or $\pi_\partial(\mathbf{r}^-)v_n = 0$, respectively).

This argument creates a chain of points $(n, \alpha_\pi)$ avoiding the forbidden region in Figure 9.2, possibly with end points belonging to either of the parabolas defined by $\alpha_\pi = (n \pm 1)^2$. There are a few possibilities for this chain of points, as follows.

- $\alpha_\pi < 0$, and the chain is bi-infinite.
- $\alpha_\pi = 0$, $n$ is even, and the chain is bi-infinite.
- $\alpha_\pi = 0$, $n$ is odd, and the chain is one of $\pm(1 + 2\mathbb{N}_0) \times \{0\}$.
- $\alpha_\pi \in (0,1)$, $n$ is even, and the chain is bi-infinite and jumps over the two shaded regions that have width less than two at the height $\alpha_\pi$.
- However, $\alpha_\pi \in (0,1]$ and $n$ is odd is impossible, since the chain starting on either side would lead to the creation of one of the points $(\pm 1, \alpha_\pi)$ inside the forbidden region.
- $\alpha_\pi = 1$ and $n$ even has three such chains, one starting at $(2,1)$ going to the right, one starting at $(-2,1)$ going to the left, and one consisting of $(0,1)$ only.
- $\alpha_\pi > 1$ and $n > 0$ creates a half-infinite chain going to the right. It cannot go infinitely far to the left, as the forbidden region has width larger than 4 and the gaps in the chain have size 2. Hence the chain has to stop with a point on the right parabola. Replacing $n$ by this minimal $K$-weight, we see that $\alpha_\pi = (n-1)^2 > 1$.
- $\alpha_\pi > 1$ and $n < 0$ gives rise to a half-infinite chain going to the left.

We leave it to the reader to match the cases above to the irreducible unitary representations discussed earlier and appearing in Table 9.1, which concludes the proof. □

**Corollary 9.23.** *Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ and suppose that $v \in \mathcal{H}_\pi$ is a smooth $K$-eigenvector of weight $n \in \mathbb{Z}$ and Casimir eigenvalue $\alpha \in \mathbb{R}$. Then the restriction of $\pi$ to the cyclic subspace $\langle v \rangle_\pi$ is irreducible, and its type is determined by $(n, \alpha_\pi)$.*

Proof. The last part of the proof of Theorem 9.22 shows that every pair $(n, \alpha_\pi)$ that does not lead via raising and lowering to a pair inside the forbidden region in Figure 9.2 is achieved by a unit vector $w \in \mathcal{H}_\rho$ for one of the irreducible representations $\rho$ of $\mathrm{SL}_2(\mathbb{R})$. By Proposition 9.19 this shows that $\varphi_v^\pi = \varphi_w^\rho$, which implies $\langle v \rangle_\pi \cong \langle w \rangle_\rho = \mathcal{H}_\rho$ by Proposition 1.63. □

Using the description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ it is possible to derive similar descriptions of related groups.

**Exercise 9.24.** (a) Describe $\widehat{\mathrm{PSL}_2(\mathbb{R})}$.
(b) Describe $\widehat{\mathrm{GL}_2(\mathbb{R})^o}$.

For the following exercise, the reader may have to also use the concrete properties of the principal and complementary series representations in Sections 9.3 and 9.5.

**Exercise 9.25.** Let $G = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid |\det g| = 1\}$, and note that $G = \mathrm{SL}_2(\mathbb{R}) \sqcup r\mathrm{SL}_2(\mathbb{R})$ where $r$ is the diagonal matrix $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with $r^2 = I$.
(a) Show that $G$ has two one-dimensional representations.
(b) Describe how $r$ interacts with $K$-weights.
(c) Use the (mock) discrete series representation to define for every $n \in \mathbb{N}$ an irreducible unitary representation $\delta^{n,G}$ whose restriction to $\mathrm{SL}_2(\mathbb{R})$ is equal to $\delta^{n,+} \oplus \delta^{n,-}$.
(d) Extend the principal series representation from $\mathrm{SL}_2(\mathbb{R})$ to $G$. Twist these by the non-trivial character to obtain a second non-isomorphic series of representations of $G$.
(e) Repeat (d) for the complementary series representation.
(f) Show that $\widehat{G}$ comprises precisely the representations found in (a), (c), (d), and (e).
(g) Describe $\widehat{\mathrm{GL}_2(\mathbb{R})}$.

**Exercise 9.26.** Let $G$ denote a locally compact $\sigma$-compact metric group. Show that

$$\widehat{\mathrm{SL}_2(\mathbb{R}) \times G} \cong \widehat{\mathrm{SL}_2(\mathbb{R})} \times \widehat{G}$$

(in a natural sense, as in Proposition 5.21).

### 9.2.6 (Non-)Spherical Representations

**Definition 9.27.** We say that a unitary representation $\pi$ of $\mathrm{SL}_2(\mathbb{R})$ is *spherical* if $\mathcal{H}_\pi$ contains a non-trivial vector invariant under $K$ (equivalently, of $K$-weight 0). Otherwise we say that $\pi$ is *non-spherical*.

**Essential Exercise 9.28.** Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. Show that $\pi$ is non-spherical if

$$\int_K \varphi_v^\pi(k)\,\mathrm{d}m_K(k) = 0 \tag{9.19}$$

for all $v \in \mathcal{H}_\pi$.

**Corollary 9.29 (Non-spherical representations are tempered).** *Every non-spherical unitary representation of* $\mathrm{SL}_2(\mathbb{R})$ *is tempered, and so has decay exponent* $1-\varepsilon$ *for all* $\varepsilon > 0$.

PROOF. By Theorems 8.23 and 8.30 the discrete series representations and mock discrete series representations are tempered. By the discussions in the next section, the odd principal series representations $\pi^{\xi,\mathrm{o}}$ are also tempered for $\xi \in \mathbb{R}\smallsetminus\{0\}$. Hence Theorem 9.22 (and Table 9.1), imply that every non-spherical irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$ is tempered.

Suppose now that $\rho$ is a non-spherical unitary representation of $\mathrm{SL}_2(\mathbb{R})$. By Exercise 9.28, being non-spherical is equivalent to the vanishing of the integral in (9.19). Using the definition of weak containment, it follows that every irreducible unitary representation of $\mathrm{SL}_2(\mathbb{R})$ that is weakly contained in $\rho$ must be non-spherical also. By our discussion above, this shows that every irreducible unitary representation weakly contained in $\rho$ is tempered. We now combine the definition of temperedness, the definition of weak containment, and Proposition 4.36. By the latter, we know that for any unit vector $v \in \mathcal{H}_\rho$ the matrix coefficient $\varphi_v^\rho$ can be approximated in the compact-open topology by sums of the form $\sum_{j=1}^n \varphi_{v_j}^{\pi_j}$ for some irreducible unitary representations $\pi_j \prec \rho$ and vectors $v_j \in \mathcal{H}_{\pi_j}$ for $j = 1,\dots,n$; since $\pi_j \prec \lambda_{\mathrm{SL}_2(\mathbb{R})}$ we can approximate $\varphi_{v_j}^{\pi_j}$ for $j = 1,\dots,n$ by some sum of diagonal matrix coefficients for the regular representation. Putting these together, we obtain the same property for $\varphi_v^\rho$, which gives $\rho \prec \lambda_{\mathrm{SL}_2(\mathbb{R})}$. The last claim now follows from Theorem 8.41. □

## 9.3 The Principal Series Representations

We will now modify the representation $\pi^0$ from Section 8.5.2, which will give rise to the even and odd principal series representations appearing in Theorem 9.22. Along the way we will also explain the connection to Example 1.6.

**Definition 9.30 (Principal series representation).** For a given $\xi \in \mathbb{R}$, we define the character $\chi_\xi$ on

$$B = \{a_t u_x \mid t, x \in \mathbb{R}\}$$

by

$$\chi_\xi(a_t u_x) = \mathrm{e}^{\mathrm{i}\xi t}$$

for all $a_t u_x \in B$. The representation $\pi^\xi$ of $G = \mathrm{SL}_2(\mathbb{R})$ is defined by

$$(\mathcal{H}_\xi, \pi^\xi) = \mathrm{Ind}_B^G(\mathbb{C}, \chi_\xi),$$

or, more concretely, by the left-regular representation on the space $\mathcal{H}_\xi$ of those functions $f\colon G \to \mathbb{C}$ with the following properties:

(1) $f$ is measurable,
(2) $f(gb) = \overline{\chi_\xi(b)}\Delta_B(b)^{\frac{1}{2}} f(g)$ for all $g \in G$ and $b \in B$, and
(3) $\|f|_K\|_{L^2(K)} < \infty$.

The *even principal series representation* $\pi^{\xi,\mathrm{e}} = \pi^{\xi,\mathrm{even}}$ (for frequency parameter $\xi$) is defined as the restriction of $\pi^\xi$ to the subspace

$$\mathcal{H}_\xi^{\mathrm{even}} = \{f \in \mathcal{H}_\xi \mid f(-g) = f(g) \text{ for all } g \in G\}.$$

Similarly, the *odd principal series representation* $\pi^{\xi,\mathrm{o}} = \pi^{\xi,\mathrm{odd}}$ (for frequency parameter $\xi$) is defined as the restriction of $\pi^\xi$ to the subspace

$$\mathcal{H}_\xi^{\mathrm{odd}} = \{f \in \mathcal{H}_\xi \mid f(-g) = -f(g) \text{ for all } g \in G\}.$$

Let us summarize the properties of the even and odd principal series representations that we will prove in this section.

**Theorem 9.31 (Even and odd principal series representations).** *The representations $\pi^\xi$, $\pi^{\xi,\mathrm{e}}$, and $\pi^{\xi,\mathrm{o}}$ are unitary representations with Casimir eigenvalue $-\xi^2$ for any $\xi \in \mathbb{R}$. The representation $\pi^{\xi,\mathrm{e}}$ is irreducible for any $\xi \in \mathbb{R}$ and $\pi^{\xi,\mathrm{o}}$ is irreducible for all $\xi \in \mathbb{R}\smallsetminus\{0\}$. Moreover, $\pi^{-\xi,\mathrm{e}}$ is isomorphic to $\pi^{\xi,\mathrm{e}}$, and $\pi^{-\xi,\mathrm{o}}$ is isomorphic to $\pi^{\xi,\mathrm{o}}$ for all $\xi \in \mathbb{R}$, and $\pi^{0,\mathrm{o}}$ is isomorphic to the sum $\delta^{1,+} \oplus \delta^{1,-}$ of the holomorphic and anti-holomorphic mock discrete series representations. Finally, all of these representations are tempered with almost decay exponent 1 and are not discrete series representations.*

Proof of unitarity in Theorem 9.31. Recall from Example 1.6 that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{S}^1 \subseteq \mathbb{R}^2$ via

$$\mathbb{S}^1 \ni v \longmapsto g\bullet v = \frac{1}{\|gv\|} gv$$

for $g \in \mathrm{SL}_2(\mathbb{R})$ and $v \in \mathbb{S}^1$, and let $m$ again denote the normalized length measure so that the Radon–Nikodym derivative is given by

$$\frac{\mathrm{d}g_* m}{\mathrm{d}m}(v) = \|g^{-1}v\|^{-2}.$$

Let $\xi \in \mathbb{R}$ and note that the map

$$\mathrm{SL}_2(\mathbb{R}) \times \mathbb{S}^1 \ni (g, v) \longmapsto c(g, v) = \|g^{-1}v\|^{-1+\mathrm{i}\xi}$$

satisfies the equation

$$c(g_1,v)c(g_2,g_1^{-1}\bullet v) = \|g_1^{-1}v\|^{-1-\mathrm{i}\xi}\left\|g_2^{-1}\frac{g_1^{-1}v}{\|g_1^{-1}v\|}\right\|^{-1-\mathrm{i}\xi}$$
$$= \|(g_1g_2)^{-1}v\|^{-1-\mathrm{i}\xi} = c(g_1g_2,v)$$

for $g_1,g_2\in \mathrm{SL}_2(\mathbb{R})$ and $v\in\mathbb{S}^1$. By Proposition 1.5 the formula

$$\pi_g^{\mathbb{S}^1,\xi}(f)(v) = \|g^{-1}v\|^{-1-\mathrm{i}\xi}f(g^{-1}\bullet v)$$

for $g\in\mathrm{SL}_2(\mathbb{R})$, $f\in L^2_m(\mathbb{S}^1)$, and $v\in\mathbb{S}^1$ defines a unitary representation $\pi^{\mathbb{S}^1,\xi}$ of $\mathrm{SL}_2(\mathbb{R})$ on $L^2_m(\mathbb{S}^1)$.

We now show that $\pi^\xi$ is $\pi^{\mathbb{S}^1,\xi}$ in disguise. In fact, we define for $f\in L^2_m(\mathbb{S}^1)$ the function

$$U(f)\colon \mathrm{SL}_2(\mathbb{R})\ni g\longmapsto U(f)(g) = \|ge_1\|^{-1-\mathrm{i}\xi}f(g\bullet e_1).$$

Then $\|U(f)\|_{L^2(K)} = \|f\|_{L^2_m(\mathbb{S}^1)}$ since the normalized Haar measure $m_K$ is mapped under the action to the normalized length measure $m$ on $\mathbb{S}^1$. Moreover, $g\in\mathrm{SL}_2(\mathbb{R})$ and $b=a_tu_x\in B$ imply $be_1=\mathrm{e}^te_1$ and so

$$\begin{aligned}U(f)(gb) &= \|gbe_1\|^{-1-\mathrm{i}\xi}f(gb\bullet e_1)\\ &= \mathrm{e}^{-t-\mathrm{i}\xi t}\|ge_1\|^{-1-\mathrm{i}\xi}f(g\bullet v)\\ &= \overline{\chi_\xi(b)}\Delta_B(b)^{\frac12}U(f)(g)\end{aligned}$$

by (8.22), which shows that $U(f)\in\mathcal{H}_\xi$. We note that since $f\in L^2_m(\mathbb{S}^1)$ was arbitrary, this shows in particular that every $F\in L^2(K)$ has an extension to an element of $\mathcal{H}_\xi$. Moreover, by the Iwasawa decomposition and Definition 9.30(2) this extension is also uniquely determined.

Finally, we let $g_0\in\mathrm{SL}_2(\mathbb{R})$ and calculate

$$U(f)(g_0^{-1}g) = \|g_0^{-1}ge_1\|^{-1-\mathrm{i}\xi}f(g_0^{-1}g\bullet e_1)$$

and

$$\begin{aligned}U\bigl(\pi_{g_0}^{\mathbb{S}^1,\xi}f\bigr)(g) &= \|ge_1\|^{-1-\mathrm{i}\xi}\bigl(\pi_{g_0}^{\mathbb{S}^1,\xi}f\bigr)(g\bullet e_1)\\ &= \underbrace{\|ge_1\|^{-1-\mathrm{i}\xi}\|g_0^{-1}(g\bullet e_1)\|^{-1-\mathrm{i}\xi}}_{\|g_0^{-1}ge_1\|^{-1-\mathrm{i}\xi}}f(g_0^{-1}g\bullet e_1).\end{aligned}$$

Together these show that $U\colon L^2_m(\mathbb{S}^1)\to\mathcal{H}_\xi$ is an equivariant isomorphism, and hence that $\pi^\xi$ is a unitary representation.

Since $-I$ belongs to the centre of $\mathrm{SL}_2(\mathbb{R})$, the subspaces $\mathcal{H}_\xi^{\mathrm{even}}$, $\mathcal{H}_\xi^{\mathrm{odd}}$ of $\mathcal{H}_\xi$ are closed invariant subspaces. It follows that $\pi^\xi$, $\pi^{\xi,\mathrm{e}}$, $\pi^{\xi,\mathrm{o}}$ are well-defined unitary representations of $\mathrm{SL}_2(\mathbb{R})$. □

**Exercise 9.32.** As an alternative, use Corollary 8.36 to show that $\pi^\xi$ defines a unitary representation of $\mathrm{SL}_2(\mathbb{R})$.

For the proof of irreducibility, we will use the following lemma.

**Lemma 9.33 (Casimir eigenvalue for $\pi^\xi$).** *Let $\xi \in \mathbb{R}$. Then the closure of $\pi^\xi_\partial(\Omega)$ is equal to multiplication by $\alpha_\xi = -\xi^2$. Moreover, for every $n \in \mathbb{Z}$ the extension of $\chi_{-n} \in L^2(K)$ to an element $F_{\xi,n} \in \mathcal{H}_\xi$ has $K$-weight $n$, and is given by*

$$F_{\xi,n}(k_\psi a_t u_x) = \mathrm{e}^{-\mathrm{i}n\psi - \mathrm{i}\xi t - t} \tag{9.20}$$

*for all $k_\psi a_t u_x \in KAU = G$. These functions satisfy*

$$\pi^\xi_\partial(\mathbf{r}^+)F_{\xi,n} = \frac{n+1+\mathrm{i}\xi}{2}F_{\xi,n+2},$$
$$\pi^\xi_\partial(\mathbf{r}^-)F_{\xi,n} = \frac{-n+1+\mathrm{i}\xi}{2}F_{\xi,n-2},$$

*and*

$$\pi^\xi_\partial(\mathbf{a})F_{\xi,n} = \frac{n+1+\mathrm{i}\xi}{2}F_{\xi,n+2} + \frac{-n+1+\mathrm{i}\xi}{2}F_{\xi,n-2} \tag{9.21}$$

*for all $n \in \mathbb{Z}$.*

Proof. For any $n \in \mathbb{Z}$, we define $F_{\xi,n} \in \mathcal{H}_\xi$ by setting

$$F_{\xi,n}|_K = \chi_{-n} \in L^2(K)$$

to be the character defined by $-n$ and extending it by the defining properties of its elements to an element of $\mathcal{H}_\xi$. Using the formula $\Delta_B(a_t u_x) = \mathrm{e}^{-2t}$ for all $a_t u_x \in AN = B$ and the definition of $\mathcal{H}_\xi$, this gives (9.20).

To see that $F_{\xi,n}$ has $K$-weight $n$, we calculate

$$\pi^\xi_{k_\psi}(F_{\xi,n})(k_\theta) = F_{\xi,n}(k_\psi^{-1}k_\theta) = \mathrm{e}^{\mathrm{i}n\psi - \mathrm{i}n\theta} = \mathrm{e}^{\mathrm{i}n\psi}F_{\xi,n}(k_\theta)$$

for all $k_\psi, k_\theta \in K$. Since the characters $\chi_{-n}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $L^2(K)$, it follows that the functions $F_{\xi,n}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $\mathcal{H}_\xi$. We note that each $F_{\xi,n}$ is a smooth function on $G$, which implies, by dominated convergence, that it is also a smooth vector for $\pi^\xi$. Alternatively, the latter also follows from Lemma 9.17.

Next we wish to calculate $\pi^\xi_\partial(\mathbf{a})F_{\xi,n}$. For $t \in \mathbb{R}$ we have

$$\exp(t\mathbf{a}) = \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix},$$

and

$$\big(\pi^\xi_{\exp(t\mathbf{a})}F_{\xi,n}\big)(k_\theta) = F_{\xi,n}\big(\exp(-t\mathbf{a})k_\theta\big) = F_{\xi,n}\left(\begin{pmatrix} \mathrm{e}^{-t} & \\ & \mathrm{e}^{t} \end{pmatrix} k_\theta\right).$$

In order to apply the definition of $F_{\xi,n}$ in (9.20), we need to write the argument in the form

$$\begin{pmatrix} \mathrm{e}^{-t} & \\ & \mathrm{e}^{t} \end{pmatrix} k_\theta = k_\psi a_{t_0} u_{x_0},$$

where in fact we are only interested in the angle parameter $\psi = \psi(t, \theta)$ and the diagonal parameter $t_0 = t_0(t, \theta)$ considered as functions in $t$ and $\theta$. As in the proof of the estimate for the Harish-Chandra spherical function in Proposition 8.39, we obtain $\psi$ and $t_0$ by using polar co-ordinates in $\mathbb{R}^2$. Indeed,

$$k_\psi a_{t_0} u_{x_0} e_1 = \mathrm{e}^{t_0} \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} \tag{9.22}$$

must equal

$$\begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{t} \end{pmatrix} k_\theta e_1 = \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{t} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-t}\cos\theta \\ \mathrm{e}^{t}\sin\theta \end{pmatrix}. \tag{9.23}$$

Therefore

$$\mathrm{e}^{2t_0} = \mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta.$$

Since we will later take the partial derivative with respect to $t$ at $t = 0$, we calculate from this that

$$\begin{aligned} 2\mathrm{e}^{2t_0}\frac{\partial}{\partial t}t_0 = \frac{\partial}{\partial t}(\mathrm{e}^{2t_0}) &= \frac{\partial}{\partial t}\big(\mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta\big) \\ &= -2\mathrm{e}^{-2t}\cos^2\theta + 2\mathrm{e}^{2t}\sin^2\theta. \end{aligned}$$

For $t = 0$, this gives, with $t_0(0, \theta) = 0$ for all $\theta \in \mathbb{R}$,

$$\frac{\partial}{\partial t}\bigg|_{t=0}(t_0) = -\cos^2\theta + \sin^2\theta = -\cos(2\theta) = -\tfrac{1}{2}\big(\mathrm{e}^{2\theta\mathrm{i}} + \mathrm{e}^{-2\theta\mathrm{i}}\big). \tag{9.24}$$

For the angle $\psi = \psi(\theta, t)$, we obtain from (9.22) and (9.23) that

$$\tan\psi = \frac{\mathrm{e}^{t}\sin\theta}{\mathrm{e}^{-t}\cos\theta} = \mathrm{e}^{2t}\tan\theta,$$

$$(1 + \tan^2\psi)\frac{\partial}{\partial t}\psi = \frac{\partial}{\partial t}(\tan\psi) = \frac{\partial}{\partial t}\big(\mathrm{e}^{2t}\tan\theta\big) = 2\mathrm{e}^{2t}\tan\theta.$$

Setting $t = 0$ and using in addition $\psi(0, \theta) = \theta$ for all $\theta \in \mathbb{R}$, we obtain

$$\frac{\partial}{\partial t}\bigg|_{t=0}(\psi) = \frac{2\tan\theta}{1 + \tan^2\theta} = 2\sin\theta\cos\theta = \sin 2\theta = \tfrac{1}{2\mathrm{i}}\big(\mathrm{e}^{2\theta\mathrm{i}} - \mathrm{e}^{-2\theta\mathrm{i}}\big). \tag{9.25}$$

Combining (9.24), (9.25), and using again $t_0(0, \theta) = 0$ and $\psi(0, \theta) = \theta$ for all $\theta \in \mathbb{R}$, this gives

$$\begin{aligned}
\pi_\partial^\xi(\mathbf{a})F_{\xi,n}(k_\theta) &= \frac{\partial}{\partial t}\bigg|_{t=0}\big(F_{\xi,n}(k_\psi a_{t_0} u_{x_0})\big) = \frac{\partial}{\partial t}\bigg|_{t=0}\big(\mathrm{e}^{-\mathrm{i}n\psi-\mathrm{i}\xi t_0-t_0}\big)\\
&= \mathrm{e}^{-\mathrm{i}n\theta}\left(-\mathrm{i}n\Big(\frac{\partial}{\partial t}\bigg|_{t=0}\psi\Big)-(\mathrm{i}\xi+1)\Big(\frac{\partial}{\partial t}\bigg|_{t=0}t_0\Big)\right)\\
&= \mathrm{e}^{-\mathrm{i}n\theta}\left(-\mathrm{i}n\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{2\theta\mathrm{i}}-\mathrm{e}^{-2\theta\mathrm{i}}\right)+(\mathrm{i}\xi+1)\frac{1}{2}\left(\mathrm{e}^{2\theta\mathrm{i}}+\mathrm{e}^{-2\theta\mathrm{i}}\right)\right)\\
&= \left(\frac{n+1+\mathrm{i}\xi}{2}\right)\mathrm{e}^{-\mathrm{i}(n+2)\theta}+\left(\frac{-n+1+\mathrm{i}\xi}{2}\right)\mathrm{e}^{-\mathrm{i}(n-2)\theta}\\
&= \left(\frac{n+1+\mathrm{i}\xi}{2}\right)F_{\xi,n+2}(k_\theta)+\left(\frac{-n+1+\mathrm{i}\xi}{2}\right)F_{\xi,n-2}(k_\theta)
\end{aligned}$$

for all $k_\theta \in K$. To summarize, we have shown (9.21).

Recalling that $\mathbf{a} = \mathbf{r}^+ + \mathbf{r}^-$ and that, by Proposition 9.13, $\pi_\partial(\mathbf{r}^\pm)F_{\xi,n}$ has weight $n \pm 2$, we obtain

$$\pi_\partial^\xi(\mathbf{r}^+)F_{\xi,n} = \frac{n+1+\mathrm{i}\xi}{2}F_{\xi,n+2}$$

and

$$\pi_\partial^\xi(\mathbf{r}^-)F_{\xi,n} = \frac{-n+1+\mathrm{i}\xi}{2}F_{\xi,n-2},$$

as claimed in the lemma.

Using the formula for $\Omega$ in (9.13) in terms of $\mathbf{r}^+$, $\mathbf{r}^-$, and $\mathbf{k}$, we obtain with $\pi_\partial^\xi(\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})F_{\xi,n} = (1-n)F_{\xi,n}$ that

$$\begin{aligned}
\pi_\partial^\xi(\Omega)F_{\xi,n} &= 4\pi_\partial^\xi(\mathbf{r}^+\circ\mathbf{r}^-)F_{\xi,n} + \pi_\partial^\xi\left((\mathbb{1}_{\mathfrak{E}}+\mathrm{i}\mathbf{k})^{\circ 2}\right)F_{\xi,n}\\
&= 2\pi_\partial^\xi(\mathbf{r}^+)(-n+1+\mathrm{i}\xi)F_{\xi,n-2} + (1-n)^2F_{\xi,n}\\
&= (n-1+\mathrm{i}\xi)(-n+1+\mathrm{i}\xi)F_{\xi,n} + (1-n)^2F_{\xi,n}\\
&= \big(-(n-1)^2-\xi^2+(1-n)^2\big)F_{\xi,n}\\
&= -\xi^2F_{\xi,n}
\end{aligned}$$

for all $n \in \mathbb{Z}$. Since the functions $F_{\xi,n}$ for $n \in \mathbb{Z}$ form an orthonormal basis of $\mathcal{H}^\xi$, the lemma follows. □

Proof of irreducibility claims in Theorem 9.31. For $\xi \in \mathbb{R}$ let $\rho = \pi^{\xi,\mathrm{e}}$ be the restriction of $\pi^\xi$ to

$$\mathcal{H}_\rho = \mathcal{H}_\xi^{\mathrm{even}} = \left\langle F_{\xi,2n} \mid n \in \mathbb{Z}\right\rangle = \{f \in \mathcal{H}_\xi \mid \pi_{-I}^\xi f = f\},$$

or let $\rho = \pi^{\xi,\mathrm{o}}$ be the restriction of $\pi^\xi$ to

$$\mathcal{H}_\rho = \mathcal{H}_\xi^{\mathrm{odd}} = \left\langle F_{\xi,2n+1} \mid n \in \mathbb{Z}\right\rangle = \{f \in \mathcal{H}_\xi \mid \pi_{-I}^\xi f = -f\}.$$

Suppose $\mathcal{V} < \mathcal{H}_\rho$ is a non-trivial closed $\rho$-invariant subspace. Since $K = \mathrm{SO}_2(\mathbb{R}) < \mathrm{SL}_2(\mathbb{R})$ is compact and abelian, $\mathcal{V}$ contains a $K$-eigenfunction.

As $\mathcal{H}_\rho$ is given as the linear hull of orthonormal $K$-eigenfunctions of different weights, it follows that $F_{\xi,n} \in \mathcal{V}$ for some $n \in \mathbb{Z}$. By Lemma 9.33, this implies

$$\pi_\partial^\xi(\mathbf{r}^+)F_{\xi,n} = \left(\frac{n+1+\mathrm{i}\xi}{2}\right)F_{\xi,n+2} \in \mathcal{V}$$

and

$$\pi_\partial^\xi(\mathbf{r}^-)F_{\xi,n} = \left(\frac{-n+1+\mathrm{i}\xi}{2}\right)F_{\xi,n-2} \in \mathcal{V}.$$

If $\xi \neq 0$, then certainly $\frac{\pm n+1+\mathrm{i}\xi}{2} \neq 0$, and we obtain $F_{\xi,n+2}, F_{\xi,n-2} \in \mathcal{V}$. Moreover, in this case we can iterate this argument and obtain

$$\mathcal{H}_\rho = \left\langle F_{\xi,n+2k} \mid k \in \mathbb{Z}\right\rangle \subseteq \mathcal{V} \leqslant \mathcal{H}_\rho,$$

which implies that $\rho$ is irreducible.

If $\xi = 0$ and $\rho = \pi^{0,\mathrm{e}}$ is the even principal series representation, then the $K$-weight $n$ of $F_{\xi,n}$ is even, $\frac{\pm n+1}{2}$ is non-zero, and we again obtain $F_{\xi,n-2}, F_{\xi,n+2} \in \mathcal{V}$. Once more we can iterate this and thus obtain $\mathcal{V} = \mathcal{H}_0^{\mathrm{even}}$, and so deduce that $\pi^{0,\mathrm{e}}$ is irreducible. □

While the above was independent of Section 9.2, for the following step we are going to use the Bargmann classification (Theorem 9.22).

Proof of isomorphism claims in Theorem 9.31. Let $\xi \in \mathbb{R}$. By Lemma 9.33 the representations $\pi^{\xi,\mathrm{e}}$ and $\pi^{\xi,\mathrm{o}}$ have Casimir eigenvalue $-\xi^2$. By the previous part of the proof, we know that these are irreducible with the exception of $\pi^{0,\mathrm{o}}$. By Theorem 9.22 there is however only one even irreducible representation with Casimir eigenvalue $-\xi^2$, which gives $\pi^{\xi,\mathrm{e}} \cong \pi^{-\xi,\mathrm{e}}$. Similarly, for $\xi \in \mathbb{R}\smallsetminus\{0\}$ we have $\pi^{\xi,\mathrm{o}} \cong \pi^{-\xi,\mathrm{o}}$.

Let us now discuss $\pi^{0,\mathrm{o}}$ with Casimir eigenvalue 0. By Corollary 9.15 we also have $\alpha_{\delta^{1,+}} = \alpha_{\delta^{1,-}} = 0$. By the construction of $\pi^{0,\mathrm{o}}$, it contains all odd $K$-weights. Let $v_1 \in \mathcal{H}_{\pi^{0,\mathrm{o}}}$ be a unit vector with $K$-weight 1, and let $e_0 \in \mathcal{H}_{\delta^{1,+}}$ be the unit vector with $K$-weight 1 as in Lemma 8.25 (see also the paragraph after Theorem 8.30). By Proposition 9.19, we conclude that $\varphi_{v_1}^{\pi^{0,\mathrm{o}}} = \varphi_{e_0}^{\delta^{1,+}}$. However, Proposition 1.63 now implies that the cyclic representations $\langle v_1\rangle_{\pi^{0,\mathrm{o}}}$ and $\langle e_0\rangle_{\delta^{1,+}} = \mathcal{H}_{\delta^{1,+}}$ are isomorphic. In other words, we have shown that $\delta^{1,+} < \pi^{0,\mathrm{o}}$ (up to isomorphisms). Using a unit vector $v_{-1} \in \mathcal{H}_{\pi^{0,\mathrm{o}}}$ of $K$-weight $-1$, we obtain, from the same argument, that $\delta^{1,-} < \pi^{0,\mathrm{o}}$. Together we have $\delta^{1,+} \oplus \delta^{1,-} < \pi^{0,\mathrm{o}}$. However, since in $\delta^{1,+} \oplus \delta^{1,-}$ and $\pi^{0,\mathrm{o}}$ each odd $K$-weight appears with multiplicity one, we must have equality. □

Proof of integrability and decay properties in Theorem 9.31. Let $\xi \in \mathbb{R}$ and $n \in \mathbb{Z}$. By (9.20) we have $|F_{\xi,n}| = F_{0,0}$. For $m, n \in \mathbb{Z}$ this implies

$$\begin{aligned}\left|\left\langle \pi_g^\xi F_{\xi,m}, F_{\xi,n}\right\rangle\right| &= \left|\int_K F_{\xi,m}(g^{-1}k)\overline{F_{\xi,n}(k)}\,\mathrm{d}m_K(k)\right| \\ &\leqslant \int_K F_{0,0}(g^{-1}k)F_{0,0}(k)\,\mathrm{d}m_K(k) = \Xi(g)\end{aligned}$$

for $g \in \mathrm{SL}_2(\mathbb{R})$. By Proposition 8.39 and Theorem 8.41 we deduce that $\pi^\xi$ is tempered with almost decay exponent at least 1 for any $\xi \in \mathbb{R}$.

Also by Proposition 8.39 we have $\Xi \notin L^2(\mathrm{SL}_2(\mathbb{R}))$, which shows by Theorem 8.2 that $\pi^{0,\mathrm{e}}$ is not a discrete series representation. For the odd representation recall the isomorphism $\pi^{0,\mathrm{o}} \cong \delta^{1,+} \oplus \delta^{1,-}$ and that by Theorem 8.30 both $\delta^{1,+}$ and $\delta^{1,-}$ are also not discrete series representations.

So suppose now that $\xi \in \mathbb{R}\smallsetminus\{0\}$ and $n \in \mathbb{Z}$. Let $\alpha = -\xi^2 < 0$. We recall that by Lemma 9.18 the function $\phi(t) = \varphi^{\pi^\xi}_{F_{\xi,n}}(a_t)$ satisfies for $t > 0$ the second order linear differential equation

$$\phi'' + f_1\phi' + f_0\phi = 0, \tag{9.26}$$

where $f_1(t) = 2 + \mathrm{O}(\mathrm{e}^{-2t})$ and $f_0(t) = 1 - \alpha + \mathrm{O}(\mathrm{e}^{-2t})$ for $t \geqslant 1$. Also recall that $\phi'(t) = \left\langle \pi^\xi_{a_t}\pi_\partial(\mathbf{a})F_{\xi,n}, F_{\xi,n}\right\rangle$ (essentially by definition; also see the proof of Lemma 9.18). To see that $\pi^{\xi,\mathrm{e}}$ and $\pi^{\xi,\mathrm{o}}$ are not discrete series representations, we will show that $\varphi^{\pi^\xi}_{F_{\xi,n}}$ and $\varphi^{\pi^\xi}_{\pi_\partial(\mathbf{a})F_{\xi,n}}$ are not both square integrable for any $n \in \mathbb{Z}$ and apply Theorem 8.2. Equivalently, we will show that $\phi$ and $\phi'$ are not both square integrable on $[1, \infty)$ with respect to $\mathrm{e}^{2t}\,\mathrm{d}t$ (see (8.11)). We define $y(t) = \mathrm{e}^t\phi(t)$, so that

$$\begin{aligned}y'(t) &= \mathrm{e}^t(\phi'(t) + \phi(t)),\\ y''(t) &= \mathrm{e}^t(\phi''(t) + 2\phi'(t) + \phi(t)),\end{aligned}$$

and we wish to show that $y$ and $y'$ are not both square integrable on $[1, \infty)$ with respect to $\mathrm{d}t$. The differential equation (9.26) for $y$ therefore takes the form

$$\underbrace{y'' - 2y' + y}_{=\mathrm{e}^t\phi''} + f_1\underbrace{(y' - y)}_{\mathrm{e}^t\phi'} + f_0\underbrace{y}_{\mathrm{e}^t\phi} = 0,$$

or

$$y'' + F_1y' + F_0y = 0$$

for $F_1 = f_1 - 2 = \mathrm{O}(\mathrm{e}^{-2t})$ and $F_0 = 1 - f_1 + f_0 = -\alpha + \mathrm{O}(\mathrm{e}^{-2t})$. Finally, we define $z = (y')^2 - \alpha y^2$ so that

$$\begin{aligned} z' &= 2y'(y'' - \alpha y) \\ &= 2y'(-F_1 y' - F_0 y - \alpha y) \\ &= -2F_1(y')^2 - 2(F_0 + \alpha)yy' \\ &= \mathrm{O}(\mathrm{e}^{-2t})z, \end{aligned} \tag{9.27}$$

where in the last step we applied the asymptotics for $F_1$ and $F_0$ and bounded both $(y')^2$ and $2yy'$ by $z = y^2 + (y')^2$. As $\pi^\xi$ is tempered, we may use a multiple of $\Xi$ to bound $\phi$ and $\phi'$. By Proposition 8.39 we have $\Xi(a_t) \ll t\mathrm{e}^{-t}$ for $t \geqslant 1$, which gives $|y(t)| \ll t$, $|y'(t)| \ll t$, and $z(t) \ll t^2$ for $t \geqslant 1$. Therefore (9.27) implies that

$$z(t) - z(t_0) = \int_{t_0}^{t} z'(s)\,\mathrm{d}s = \mathrm{O}\left(\int_{t_0}^{t} \mathrm{e}^{-2s} s^2\,\mathrm{d}s\right)$$

for $t \geqslant t_0 \geqslant 1$. However, this implies that $\lim_{t\to\infty} z(t)$ exists. In particular,

$$S(t_0) = \sup\{z(t) \mid t \geqslant t_0\} < \infty$$

for all $t_0 \geqslant 1$. Note that we have $S(t_0) \geqslant z(t_0) > 0$ as $z(t_0) = 0$ would give $\phi(t_0) = \phi'(t_0) = 0$ and contradict the uniqueness of the solution to (9.26) due to the Picard–Lindelöf theorem. Using (9.27) again, there exists a constant $C$ so that

$$|z(t) - z(t_1)| \leqslant C \int_{t_1}^{t} \mathrm{e}^{-2s}\,\mathrm{d}s S(t_0) \leqslant C\frac{1}{2}\mathrm{e}^{-2t_1} S(t_0)$$

for all $t \geqslant t_1 \geqslant t_0$. Now choose $t_0 \geqslant 1$ so that $C\mathrm{e}^{-2t_0} \leqslant \frac{1}{2}$, choose $t_1 \geqslant t_0$ so that $z(t_1) > \frac{1}{2}S(t_0)$. Then

$$|z(t) - z(t_1)| \leqslant \frac{1}{4}S(t_0) \leqslant \frac{1}{2}z(t_1)$$

for $t > t_1$, which implies that $\lim_{t\to\infty} z(t) \geqslant \frac{1}{2}z(t_1)$. However, this also implies that

$$\int_1^\infty \left(|\alpha| y^2 + (y')^2\right)\,\mathrm{d}t = \int_1^\infty z\,\mathrm{d}t = \infty.$$

Hence $y$ and $y'$ are not both square-integrable on $[1, \infty)$. As discussed above, this shows that both $\pi^{\xi,\mathrm{e}}$ and $\pi^{\xi,\mathrm{o}}$ are not discrete series representations.

Finally, recall from Lemma 8.45 that the integrability exponent $p_\pi$ of a unitary representation $\pi$ of $\mathrm{SL}_2(\mathbb{R})$ and its almost decay exponent $\kappa_\pi$ satisfy $p_\pi \leqslant \frac{2}{\kappa_\pi}$. The above showed for any principal series representation $\pi$ that $p_\pi \geqslant 2$ and $\kappa_\pi \geqslant 1$, which implies $\kappa_\pi = 1$. □

**Exercise 9.34.** (a) Analyze the above argument to show that for any $\xi \in \mathbb{R}\smallsetminus\{0\}$ and $n \in \mathbb{Z}$ there exists a constant $C_{\xi,n}$ so that

$$\left|\varphi^{\pi^\xi}_{F_{\xi,n}}(g)\right| \ll C_{\xi,n}\|g\|^{-1}.$$

(b) Show that $C_{\xi,n} \to 0$ as $\xi \to 0$, which makes the conclusion in part (a) less interesting.

## 9.4 Two Koopman Representations of $\mathrm{SL}_2(\mathbb{R})$*

We wish to show here how the principal series representations can naturally occur as components of other unitary representations. Since $\mathrm{SL}_2(\mathbb{R})$ acts both on the Euclidean plane $\mathbb{R}^2$ and on the hyperbolic plane $\mathbb{H}$, preserving area measure on the space in each case, this already gives rise to two natural unitary representations of $\mathrm{SL}_2(\mathbb{R})$. As we will see, the case of $\mathbb{R}^2$ will be relatively straightforward to analyze. On the other hand, understanding the case of $\mathbb{H}$ will require more work, but we will motivate the formulas arising.

### 9.4.1 The Koopman Representation on $\mathbb{R}^2$

Almost by definition, the group $\mathrm{SL}_2(\mathbb{R})$ acts continuously on $\mathbb{R}^2$, preserving the two-dimensional Lebesgue measure $m = m_{\mathbb{R}^2}$. Using Proposition 1.3, this gives rise to a Koopman representation $\pi^{\mathbb{R}^2}$ of $\mathrm{SL}_2(\mathbb{R})$ on $L^2_m(\mathbb{R}^2)$, where

$$\pi^{\mathbb{R}^2}_g(f)(x) = f(g^{-1}x)$$

for $g \in \mathrm{SL}_2(\mathbb{R})$, $f \in L^2_m(\mathbb{R}^2)$, and $x \in \mathbb{R}^2$.

Using polar coordinates

$$(r,\theta) \in (0,\infty) \times [0,2\pi)$$

for $\mathbb{R}^2\smallsetminus\{0\}$ with $\mathrm{d}m = r\,\mathrm{d}r\,\mathrm{d}\theta$, we make the following definition.

**Definition 9.35 (Radial Mellin transform).** For a function $f \in L^2_m(\mathbb{R}^2)$, an element $h \in \mathrm{SL}_2(\mathbb{R})$, and a frequency parameter $\xi \in \mathbb{R}$, we define the *radial Mellin transform* of $f$ at $(h,\xi)$ by

$$\widehat{f}^{\,\mathrm{rad}}(h,\xi) = \int_0^\infty f(rhe_1)r^{\mathrm{i}\xi}\,\mathrm{d}r, \tag{9.28}$$

where

$$e_1 = \begin{pmatrix}1\\0\end{pmatrix}$$

denotes the first basis vector of $\mathbb{R}^2$.

Just as in the case of the usual Fourier transform, the integral in (9.28) may not make sense as a Lebesgue integral. Thus we also need to discuss the meaning of this expression more carefully (which we will do in the proof of Proposition 9.39). The following lemma, together with the definition of the principal series representation in Definition 9.30 reveal why (9.28) is really the right definition.

**Lemma 9.36 (Equivariance properties).** *For $f \in C_c(\mathbb{R}^2)$ the radial Mellin transform $\widehat{f}^{\mathrm{rad}}(h,\xi)$ is well-defined and satisfies*

$$\begin{cases} \widehat{\pi_g^{\mathbb{R}^2}(f)}^{\mathrm{rad}}(h,\xi) = \widehat{f}^{\mathrm{rad}}(g^{-1}h,\xi) \\ \widehat{f}^{\mathrm{rad}}(hb,\xi) = \overline{\chi_\xi(b)}\Delta_B(b)^{\frac{1}{2}}\widehat{f}^{\mathrm{rad}}(h,\xi) \end{cases}$$

*for all $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$, $g \in \mathrm{SL}_2(\mathbb{R})$, and $b \in B = AU$.*

Proof. It is clear that for $f \in C_c(\mathbb{R}^2)$, the domain of integration in 9.28 can be chosen to be a compact interval, which gives the first claim in the lemma.

Now fix some $g, h \in \mathrm{SL}_2(\mathbb{R})$ and $\xi \in \mathbb{R}$. Then

$$\begin{aligned} \widehat{\pi_g^{\mathbb{R}^2}(f)}^{\mathrm{rad}}(h,\xi) &= \int_0^\infty \bigl(\pi_g^{\mathbb{R}^2}(f)\bigr)(rhe_1)r^{\mathrm{i}\xi}\,\mathrm{d}r \\ &= \int_0^\infty f(g^{-1}rhe_1)r^{\mathrm{i}\xi}\,\mathrm{d}r \\ &= \widehat{f}^{\mathrm{rad}}(g^{-1}h,\xi), \end{aligned}$$

as claimed.

For the second claim, we calculate for $b = a_t u_x \in B = AU$ that

$$\begin{aligned} \widehat{f}^{\mathrm{rad}}(ha_tu_x,\xi) &= \int_0^\infty f\bigl(rh\underbrace{a_tu_xe_1}_{=\mathrm{e}^te_1}\bigr)r^{\mathrm{i}\xi}\,\mathrm{d}r \\ &= \int_0^\infty f\bigl(\widetilde{r}he_1\bigr)\bigl(\widetilde{r}\mathrm{e}^{-t}\bigr)^{\mathrm{i}\xi}\mathrm{e}^{-t}\,\mathrm{d}\widetilde{r} \\ &= \mathrm{e}^{-\mathrm{i}\xi t}\mathrm{e}^{-t}\widehat{f}^{\mathrm{rad}}(h,\xi), \end{aligned}$$

where we used the substitution $\widetilde{r} = \mathrm{e}^t r$ with $\mathrm{d}\widetilde{r} = \mathrm{e}^t\,\mathrm{d}r$. The lemma follows by recalling that $\Delta_B(a_tu_x) = \mathrm{e}^{-2t}$ and $\chi_\xi(a_tu_x) = \mathrm{e}^{\mathrm{i}\xi t}$ for all $a_tu_x \in B$. □

In addition to the correct equivariance properties as shown above, the radial Mellin transform is also isometric in the following sense.

**Lemma 9.37 (Isometry).** *For $f \in C_c(\mathbb{R}^2)$ we have*

$$\|f\|_{L^2(\mathbb{R}^2)} = \Bigl\|\widehat{f}^{\mathrm{rad}}|_{K\times\mathbb{R}}\Bigr\|_{L^2(K\times\mathbb{R})},$$

*where we equip $K \times \mathbb{R}$ with the Haar measure $\mathrm{d}m_K\,\mathrm{d}\xi = \frac{1}{2\pi}\,\mathrm{d}\theta\,\mathrm{d}\xi$.*

Proof. We first note that

$$\begin{aligned}\|f\|_{L^2(\mathbb{R}^2)}^2 &= \int_0^{2\pi}\int_0^\infty \bigl|f\bigl(rk_\theta e_1\bigr)\bigr|^2 r\,\mathrm{d}r\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\int_0^\infty \bigl|rf\bigl(rk_\theta e_1\bigr)\bigr|^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\theta.\end{aligned}$$

We now define, for $\theta \in [0,2\pi)$, the function $F_\theta\colon \mathbb{R}\to\mathbb{C}$ by

$$F_\theta(s) = \mathrm{e}^s f\bigl(\mathrm{e}^s k_\theta e_1\bigr)$$

for all $s\in\mathbb{R}$, and note that $F_\theta$ corresponds, roughly speaking, to the restriction of $f$ to the ray from 0 at angle $\theta$ to the positive $x$-axis. Using the fact that $f$ has compact support in $\mathbb{R}^2$ and is bounded near 0, we see that $F_\theta \in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ for all $\theta\in\mathbb{R}$. Using the substitution $r=\mathrm{e}^s$ with $\frac{\mathrm{d}r}{r} = \mathrm{d}s$ we see that

$$\|f\|_{L^2(\mathbb{R}^2)}^2 = \int_0^{2\pi} \underbrace{\int_{-\infty}^\infty |F_\theta(s)|^2\,\mathrm{d}s}_{=\|F_\theta\|_{L^2(\mathbb{R})}^2}\,\mathrm{d}\theta.$$

Next we use the fact that

$$\|F_\theta\|_{L^2(\mathbb{R})} = \|\widecheck{F_\theta}\|_{L^2(\mathbb{R})},$$

where $\widecheck{F_\theta}$ again denotes the Fourier back transform. Using the definitions and the substitution $r=\mathrm{e}^s$ with $\mathrm{d}r = \mathrm{e}^s\,\mathrm{d}s$ again, we obtain

$$\begin{aligned}\widecheck{F_\theta}(\zeta) &= \int_{-\infty}^\infty F_\theta(s)\mathrm{e}^{2\pi\mathrm{i}\zeta s}\,\mathrm{d}s\\ &= \int_{-\infty}^\infty \mathrm{e}^s f(\mathrm{e}^s k_\theta e_1)\mathrm{e}^{2\pi\mathrm{i}\zeta s}\,\mathrm{d}s\\ &= \int_0^\infty f(rk_\theta e_1)r^{2\pi\mathrm{i}\zeta}\,\mathrm{d}r = \widehat{f}^{\,\mathrm{rad}}(k_\theta,2\pi\zeta) \qquad (9.29)\end{aligned}$$

for all $\zeta\in\mathbb{R}$. Together, we obtain

$$\begin{aligned}\|f\|_{L^2(\mathbb{R}^2)}^2 &= \int_0^{2\pi}\|\widecheck{F_\theta}\|_{L^2(\mathbb{R})}^2\,\mathrm{d}\theta\\ &= \int_0^{2\pi}\int_{-\infty}^\infty |\widehat{f}^{\,\mathrm{rad}}(k_\theta,2\pi\zeta)|\,\mathrm{d}\zeta\,\mathrm{d}\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^\infty |\widehat{f}^{\,\mathrm{rad}}(k_\theta,\xi)|\,\mathrm{d}\xi\,\mathrm{d}\theta = \|\widehat{f}^{\,\mathrm{rad}}|_{K\times\mathbb{R}}\|_{L^2(K\times\mathbb{R})}^2\end{aligned}$$

by using the substitution $\xi = 2\pi\zeta$. □

Definition 9.30 and Lemmas 9.36 and 9.37 suggest the following definition.

**Definition 9.38 (Integrals of principal series representations).** For any $\sigma$-finite measure $\mu$ on $\mathbb{R}$, we define the space $\mathcal{H}_\mu$ of all functions

$$F\colon \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R} \to \mathbb{C}$$

satisfying the following properties:

(1) $F$ is measurable;
(2) $F(hb, \xi) = \overline{\chi_\xi(b)} \Delta_B(b)^{\frac{1}{2}} F(h, \xi)$ for all $h \in \mathrm{SL}_2(\mathbb{R})$, $b \in B$, $\xi \in \mathbb{R}$; and
(3) $\|f|_{K\times\mathbb{R}}\|_{L^2(K\times\mathbb{R}, m_K\times\mu)} < \infty$.

The unitary representation[†]

$$\pi^\mu = \int_{\mathbb{R}} \pi^\xi \,\mathrm{d}\mu(\xi)$$

is defined by the left regular representation on the first component; that is,

$$\pi_g^\mu(F)(h, \xi) = F(g^{-1}h, \xi)$$

for all $F \in \mathcal{H}_\mu$, $g, h \in \mathrm{SL}_2(\mathbb{R})$, and $\xi \in \mathbb{R}$. Moreover, we also define

$$\pi^{\mu,\mathrm{e}} = \int_{\mathbb{R}} \pi^{\xi,\mathrm{e}} \,\mathrm{d}\mu(\xi)$$

and

$$\pi^{\mu,\mathrm{o}} = \int_{\mathbb{R}} \pi^{\xi,\mathrm{o}} \,\mathrm{d}\mu(\xi)$$

to be the restrictions of $\pi^\mu$ to the subspaces

$$\mathcal{H}_\mu^{\mathrm{even}} = \{F \in \mathcal{H}_\mu \mid F(-g, x) = F(g, x) \text{ for all } g, x\}$$

and

$$\mathcal{H}_\mu^{\mathrm{odd}} = \{F \in \mathcal{H}_\mu \mid F(-g, x) = -F(g, x) \text{ for all } g, x\}$$

respectively.

**Proposition 9.39 (Spectral decomposition of $\pi^{\mathbb{R}^2}$).** *The Koopman representation of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{R}^2$ *is isomorphic to*

$$\pi^m = \pi^{m,\mathrm{e}} \oplus \pi^{m,\mathrm{o}} = \int_{\mathbb{R}} \pi^{\xi,\mathrm{e}} \,\mathrm{d}\xi \oplus \int_{\mathbb{R}} \pi^{\xi,\mathrm{o}} \,\mathrm{d}\xi,$$

*where we use the Lebesgue measure* $\mu = m$ *on* $\mathbb{R}$.

[†] We did not discuss the integral of unitary representations, but believe that the notation is justified in this case.

Proof. By Lemmas 9.36 and 9.37, the radial Mellin transform

$$C_c(\mathbb{R}^2) \ni f \longmapsto \widehat{f}^{\,\mathrm{rad}} \in \mathcal{H}_m$$

is well-defined, equivariant, and an isometry. Hence it extends by the density of $C_c(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$ to a well-defined, equivariant isometry

$$L^2(\mathbb{R}^2) \ni f \longmapsto \widehat{f}^{\,\mathrm{rad}} \in \mathcal{H}_m.$$

We keep referring to $\widehat{f}^{\,\mathrm{rad}}$ as the radial Mellin transform of $f \in L^2(\mathbb{R}^2)$.

It remains to show that this map is onto. For this, assume that $f_K \in C(K)$ and $f_\mathbb{R} \in C_c(\mathbb{R})$. We note that

$$f_K \otimes \widecheck{f_\mathbb{R}} \colon K \times \mathbb{R} \longrightarrow \mathbb{C}$$

can be extended using property (2) in Definition 9.38 to an element of $\mathcal{H}_m$. We claim that $f_K \otimes \widecheck{f_\mathbb{R}} = \widehat{f}^{\,\mathrm{rad}}$ for some $f \in C_c(\mathbb{R}^2)$. Also recall that the subspaces $C(K) \subseteq L^2(K)$ and $\widecheck{C_c(\mathbb{R})} \subseteq L^2(\mathbb{R}, m)$ are dense (for the latter, apply Theorem 2.15). Varying $f_K$ and $f_\mathbb{R}$, we can then, for example, approximate any function of the form $\mathbb{1}_{B_K} \otimes \mathbb{1}_{B_\mathbb{R}}$ extended to an element of $\mathcal{H}_\mu$, where $B_K \subseteq K$ and $B_\mathbb{R} \subseteq \mathbb{R}$ are measurable with finite measures. For this reason, the claim implies that the image of the radial Mellin transform (extended to $L^2(\mathbb{R}^2)$) is indeed all of $\mathcal{H}_\mu$.

To prove the claim, we reuse the argument from the proof of Lemma 9.37. Let $f_K \in C(K)$ and $f_\mathbb{R} \in C_c(\mathbb{R})$ be as above. We define $f \in C_c(\mathbb{R}^2)$ using polar coordinates by

$$f(rk_\theta e_1) = \tfrac{1}{2\pi} f_K(\theta) r^{-1} f_\mathbb{R}\left(\tfrac{1}{2\pi}\log r\right).$$

For this $f$, the function $F_\theta$ for $\theta \in [0, 2\pi)$ appearing in the proof of Lemma 9.37 becomes

$$F_\theta(s) = \mathrm{e}^s f(\mathrm{e}^s k_\theta e_1) = \tfrac{1}{2\pi} f_K(\theta) f_\mathbb{R}\left(\tfrac{1}{2\pi}s\right)$$

for $s \in \mathbb{R}$. Hence by (9.29) and the substitution $\widetilde{s} = \frac{1}{2\pi}s$ we have

$$\begin{aligned}\widehat{f}^{\,\mathrm{rad}}(k_\theta, 2\pi\zeta) = \widecheck{F_\theta}(\zeta) &= \tfrac{1}{2\pi} f_K(\theta) \int_R f_\mathbb{R}\left(\tfrac{1}{2\pi}s\right) \mathrm{e}^{2\pi \mathrm{i} s\zeta}\,\mathrm{d}s \\ &= f_K(\theta)\int_\mathbb{R} f_\mathbb{R}(\widetilde{s})\mathrm{e}^{2\pi\mathrm{i}\widetilde{s}2\pi\zeta}\,\mathrm{d}\widetilde{s} = f_K(\theta)\widecheck{f_\mathbb{R}}(2\pi\zeta)\end{aligned}$$

for all $\theta \in [0, 2\pi)$ and $\zeta \in \mathbb{R}$. Equivalently, we have $\widehat{f}^{\,\mathrm{rad}} = f_K \otimes \widecheck{f_\mathbb{R}}$ as claimed, which gives the proposition. □

**Exercise 9.40.** (a) Show that $\pi^\mu$ as in Definition 9.38 is indeed a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ for any $\sigma$-finite measure $\mu$ on $\mathbb{R}$.

(b) Show that $\pi^\mu$ is tempered.

**Exercise 9.41.** Use Fourier inversion on $\mathbb{R}$ to prove a Fourier inversion formula that expresses $f \in C_c^\infty(\mathbb{R}^2)$ as an integral over values of $\widehat{f}^{\,\mathrm{rad}}$.

**Exercise 9.42.** Is the centralizer of the Koopman representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{R})$ abelian? Prove your claim. Can you identify the centralizer?

### 9.4.2 Moving a Loudspeaker to Infinity

To better understand the formulas required for the hyperbolic Fourier transform we wish to discuss a physical interpretation of the Fourier transform on the two planes $\mathbb{R}^2$ and $\mathbb{H}$. As this is just meant as a motivation for the formal definitions coming later, we leave the details of these calculations as exercises.

To begin with, we imagine a loudspeaker $L$, which we assume will produce the desired sound for any given frequency and amplitude. We imagine the sound wave being represented by a $\mathbb{C}$-valued function $f_L$ on the plane, where $|f_L|^2$ represents the energy of the wave, and the argument of $f_L$ represents the phase shift of the wave. We also imagine that there is no energy loss in the passage of the wave through the medium. This physical interpretation suggests that $|f_L(P_0)|^2$ is inversely proportional to the length of the circle with centre $L$ containing a point $P$, as illustrated in Figure 9.3.

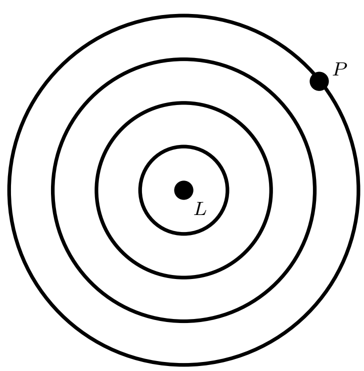


**Fig. 9.3:** The sound waves emanate from the loudspeaker $L$ and decrease in loudness. The concentric circles indicate the phase of $f_L$.

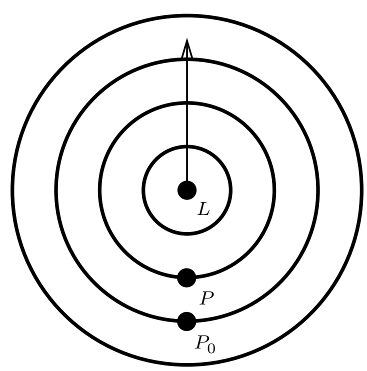


**Fig. 9.4:** We move the loudspeaker $L$ upwards, and turn up the volume.

We now fix some origin $P_0$ in the plane, and move the loudspeaker $L$ further away in some pre-determined direction, say upwards as in Figure 9.4. This of course means that we do not hear the sound much at $P_0$ if $L$ is already far away. To get round this problem, we simultaneously turn up the volume at $L$ so that $|f_L(P_0)|^2 = 1$. We now wish to move $L$ to infinity and describe what will happen to $f_L$ if we do so. However, to do this we have to distinguish between the cases of the Euclidean and hyperbolic planes.

**Euclidean plane:** In the Euclidean plane, the circle of radius $r$ has circumference $2\pi r$. If $P_0$ belongs to a circle of radius $r_0$ (that is, if $r_0 = \|P_0 - L\|$) and $P$ has distance $\|P - P_0\|$ to $P_0$, then $P$ belongs to a circle of radius $r$ with $\Delta r = r - r_0$ satisfying $|\Delta r| \leqslant \|P - P_0\|$ (by the triangle inequality). Hence

$$|f_L(P_0)|^2 \cdot 2\pi r_0 = |f_L(P)|^2 \cdot 2\pi r.$$

Letting $L$ go to infinity, we have $r_0 \to \infty$ and $\frac{2\pi r}{2\pi r_0} = \frac{r_0+\Delta r}{r_0} \to 1$. Therefore the limiting sound distribution $f$ will have the property that $|f|^2$ is constant and equal to 1. Moreover, the concentric circles degenerate to equidistant parallel lines, so that in the limit we may obtain in this way the function

$$f(x, y) = \mathrm{e}^{\mathrm{i}\xi y}$$

for $P = (x, y) \in \mathbb{R}^2$, where $\xi$ represents the frequency of the wave. Allowing different frequencies and different directions along which $L$ is moved, one obtains in this way any character $\chi_{(\xi_1,\xi_2)}$ for $(\xi_1, \xi_2) \in \mathbb{R}^2$, which we may think of as the elementary waves on $\mathbb{R}^2$.

This suggests the following interpretation for the Fourier transform of a function on $\mathbb{R}^2$. Given $f$, we first test the correlation of $f$ against all elementary waves. Next we imagine infinitely many loudspeakers at infinity in all directions using various frequencies with well-chosen amplitudes. Fourier inversion now tells us that these then create the prescribed sound distribution $f$ by superposition of the so-created (and correctly amplified) elementary waves.

**Exercise 9.43.** (a) For a given frequency $\xi \in \mathbb{R}$ and two points $P_0, L \in \mathbb{R}^2$, calculate the function $f_L$ representing a wave of frequency $\xi$ emanating from $L$ with $f_L(P_0) = 1$.
(b) Calculate the limit $f$ of $f_L$ as $L = (0, y) \to \infty$.

**Hyperbolic plane:** To get some intuition for the hyperbolic Fourier transform, we repeat the above discussion on $\mathbb{H}$, which will lead to the functions that will take over the role of characters in the more formal discussions of the following sections.

We again imagine the loudspeaker $L$ moving to infinity along the upward oriented geodesic and consider equidistant concentric circles with centre $L$, as in Figure 9.5. We note that in the limit we obtain circles (in the Euclidean sense, within $\mathbb{C} \supseteq \mathbb{D}$) touching the boundary. These are not hyperbolic geodesics; instead these curves are called *horocycles*.

To understand the limit function $f$ of the sound distribution $f_L$ for $L$ going to the boundary, we need to calculate the circumference of a circle of radius $R$. To simplify matters, we let the centre be $0 \in \mathbb{D}$ as in Lemma 8.15.

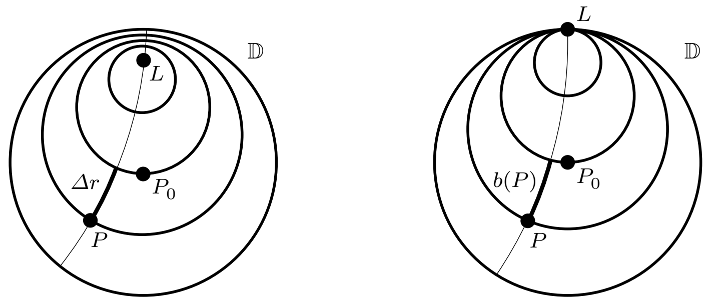


**Fig. 9.5:** A calculation reveals that any Möbius transformation $z \mapsto \frac{\alpha z+\beta}{\gamma z+\delta}$ on $\overline{\mathbb{C}}$ maps lines and circles to lines and circles. With this, it is straightforward to verify that concentric hyperbolic circles with centre $L$ appear in the disk model of the hyperbolic plane as circles, with $L$ appearing closer to the circle near the boundary. If $L$ is moved to the boundary, these circles degenerate to circles touching the boundary.

By (8.12) the Euclidean radius of this circle is given by $\rho = \tanh\bigl(\frac{R}{2}\bigr)$. Hence the circumference can be calculated using the path

$$[0, 2\pi] \ni \theta \longmapsto \rho e^{i\theta},$$

which, by definition of the Riemannian metric in (8.6), gives

$$\int_0^{2\pi} \frac{2}{(1-\rho^2)}\rho \, d\theta = \frac{4\pi \tanh\bigl(\frac{R}{2}\bigr)}{\bigl(1-\tanh^2\bigl(\frac{R}{2}\bigr)\bigr)} = 4\pi \frac{\frac{\sinh(\frac{R}{2})}{\cosh(\frac{R}{2})} \cdot \cosh^2\bigl(\frac{R}{2}\bigr)}{\bigl(\cosh^2\bigl(\frac{R}{2}\bigr) - \sinh^2\bigl(\frac{R}{2}\bigr)\bigr)}$$
$$= 4\pi \sinh\bigl(\tfrac{R}{2}\bigr)\cosh\bigl(\tfrac{R}{2}\bigr) = 2\pi \sinh R.$$

We let $L$ go to infinity along the Northward geodesic, so that

$$r_0 = \mathsf{d}(P_0, L) \to \infty.$$

For a third point $P \in$ d, we let $r = \mathsf{d}(P, L)$. We also define the 'relative distance' from $L$ compared to $P_0$ by setting it equal to

$$\varDelta r = \mathsf{d}(P, L) - \mathsf{d}(P_0, L) = r - r_0,$$

see also Figure 9.5. Note that $\varDelta r = r - r_0$ satisfies $|\varDelta r| \leqslant \mathsf{d}(P, P_0)$. With $\sinh R \sim e^R$ as $R \to \infty$, and

$$|f_L(P_0)|^2 2\pi \sinh r_0 = |f_L(P)|^2 2\pi \sinh r,$$

we obtain

$$\frac{|f_L(P)|^2}{|f_L(P_0)|^2} = \frac{\sinh r_0}{\sinh(r_0 + \varDelta r)} \sim e^{-\varDelta r}$$

as $r_0 \to \infty$. Since we normalize the loudness of $L$ along the way to have $|f_L(P_0)|^2 = 1$, we expect that the limit sound wave satisfies

$$|f(P)| = \mathrm{e}^{-\frac{1}{2}b(P)}$$

for the limiting function $b$ of $\varDelta r$. Putting the phase with frequency $\xi$ into the discussions, we expect that functions of the form[†]

$$f(P) = \mathrm{e}^{(-\frac{1}{2}+\frac{\mathrm{i}}{2}\xi)b(P)}$$

are the relevant elementary waves on the hyperbolic plane. This is indeed the case, so we will have to define the so-called Busemann function $b(P)$ more carefully (we also refer to Busemann's monograph [7]).

By varying both the frequency and the position of the loudspeakers on the boundary of the hyperbolic plane, we again expect that any sound distribution on the plane can be produced as a superposition of elementary waves.

**Exercise 9.44.** Repeat (a) and (b) from Exercise 9.43 for $\mathbb{H}$.

### 9.4.3 The Busemann Function

In the upper half-plane model $\mathbb{H}$, the desired function takes a particularly easy form. Indeed, if we move the loudspeaker $L$ simply up to the point $\infty$ in $\partial\mathbb{H}$, then the concentric circles degenerate to horizontal lines near i, as in Figure 9.6.

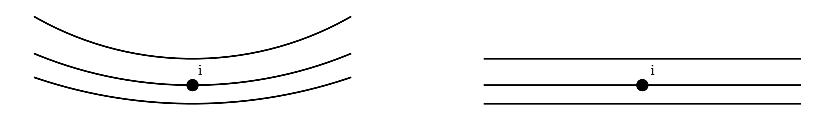


**Fig. 9.6:** On the left we see that the concentric circles with centre $L = y\mathrm{i}$ for large $y$ are almost horizontal Euclidean lines. On moving $y$ to $\infty$, these become horizontal Euclidean lines or *horizontal horocycles* in the hyperbolic plane $\mathbb{H}$.

**Definition 9.45 (Busemann function for $\infty \in \partial\mathbb{H}$).** The *Busemann function* on $\mathbb{H}$ with respect to $\infty \in \partial\mathbb{H}$ (and origin $\mathrm{i} \in \mathbb{H}$) is defined for $z \in \mathbb{H}$ by

$$b_\infty^{\mathbb{H}}(z) = -\log \Im(z).$$

[†] As earlier, we normalize the meaning of frequency in the following discussions to simplify some of the formulas arising.

We note that the point $\infty \in \partial\mathbb{H}$ should be thought of as being the point 'at infinity' that is higher up than any $z \in \mathbb{H}$. Roughly speaking, $b_\infty^{\mathbb{H}}(z)$ is comparing the distance of $z$ and of $\mathrm{i}$ to $\infty$ (both of which are of course infinite). More precisely, $b_\infty^{\mathbb{H}}(z)$ measures the hyperbolic distance between the horizontal horocycle at $z = x + \mathrm{i}y \in \mathbb{H}$ and the horizontal horocycle at our designated origin $\mathrm{i}$, given by

$$\mathsf{d}(\mathrm{i}y, \mathrm{i}) = \left| \int_1^y \frac{\mathrm{d}y}{y} \right| = |\log y|.$$

We should think of $b_\infty^{\mathbb{H}}(z)$ as an oriented relative distance, since $b_\infty^{\mathbb{H}}(z) > 0$ means that $z$ is further from $\infty$ than $\mathrm{i}$ is, while $b_\infty^{\mathbb{H}}(z) < 0$ means that $\mathrm{i}$ is further away from $\infty$.

Using the discussion of Section 9.4.2 as in Figure 9.7, we are now led to the following definition.

**Definition 9.46 (Hyperbolic wave).** We define the *hyperbolic wave function* coming from $\infty$ with frequency $\xi \in \mathbb{R}$ (normalized for the origin $\mathrm{i} \in \mathbb{H}$) to be

$$\chi_{\infty,\xi}(z) = \mathrm{e}^{(-\frac{1}{2}+\frac{\mathrm{i}}{2}\xi) b_\infty^{\mathbb{H}}(z)} = \Im(z)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi}$$

for all $z \in \mathbb{H}$.

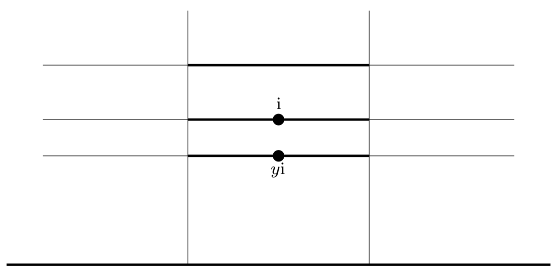


**Fig. 9.7:** We think of a hyperbolic wave coming from $\infty$ with horizontal horocycles being the wave fronts. Since an interval of Euclidean length 1 of the horizontal horocycle with vertical coordinate $y \in (0, \infty)$ has hyperbolic length $\frac{1}{y}$, the energy of the wave spreads over a larger region as it spreads down and so its intensity decreases, as in Definition 9.46.

We note that by definition of Möbius transformations in (8.3) and the description of $\varDelta_B$ in (8.22) we have

$$\begin{aligned} \chi_{\infty,\xi}(b^{-1} \bullet z) &= \Im\big(\mathrm{e}^{-2t}(z - x)\big)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi} \\ &= \mathrm{e}^{-t(1-\mathrm{i}\xi)} \Im(z)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi} = \varDelta_B(b)^{\frac{1}{2}} \chi_\xi(b) \chi_{\infty,\xi}(z) \end{aligned} \tag{9.30}$$

for all $b = u_x a_t \in B$ and $z \in \mathbb{H}$. These formulas suggest a possible link between the hyperbolic wave of frequency $\xi$ and the principal series representation $\pi^\xi$ corresponding to the frequency parameter $\xi$ (see Section 9.3), which we will explain after defining the hyperbolic Fourier transform.

We also note that the identifications

$$B \ni b = u_x a_t \longmapsto bK \in \mathrm{SL}_2(\mathbb{R})/K \longmapsto z = b \centerdot \mathrm{i} = x + \mathrm{e}^{2t}\mathrm{i} \in \mathbb{H}$$

are measure-preserving by our choice of the Haar measure $m_B$ on $B$ in Section 8.3.5 and the normalization $m_K(K) = 1$. We will also simply write $m$ for the Haar measure $m = m_B \times m_K$ on $\mathrm{SL}_2(\mathbb{R}) = BK$, and will write $\int_G \cdot \,\mathrm{d}m$ for integration over $G = \mathrm{SL}_2(\mathbb{R})$.

### 9.4.4 The Hyperbolic Fourier Transform

We recall from Section 8.3.1 that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ by Möbius transformations preserves the hyperbolic area measure defined by

$$\mathrm{dvol} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

By Proposition 1.3 this gives rise to a Koopman unitary representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{H})$ defined by

$$\big(\pi_g^{\mathbb{H}}(f)\big)(z) = f(g^{-1} \centerdot z)$$

for $g \in \mathrm{SL}_2(\mathbb{R})$, $f \in L^2(\mathbb{H})$, and $z \in \mathbb{H}$. We note that $\pi_{-I}^{\mathbb{H}} = I$ since

$$(-I)\centerdot z = \frac{-1z + 0}{0z - 1} = z$$

for all $z \in \mathbb{H}$, so that $\pi^{\mathbb{H}}$ is an even representation. We also recall that we may use $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/K$ to identify $L^2(\mathbb{H})$ with the subspace of $L^2(\mathrm{SL}_2(\mathbb{R}))$ consisting of all right $K$-invariant functions. Under this identification, $\pi^{\mathbb{H}}$ becomes the restriction of the left regular representation $\lambda^{\mathrm{SL}_2(\mathbb{R})}$ to this subspace. In particular, $\pi^{\mathbb{H}}$ is tempered, and has uniform decay exponent 1 by Theorem 8.31.

In analogy to the definition of the radial Mellin transform in Definition 9.35, and motivated by the discussions in Sections 9.4.2 and 9.4.3, we are led to the following definition.

**Definition 9.47 (Hyperbolic Fourier transform).** We define the hyperbolic Fourier transform of $f$ at $(h, \xi)$ for $f \in L^2_{\mathrm{vol}}(\mathbb{H})$, $h \in \mathrm{SL}_2(\mathbb{R})$, and a frequency parameter $\xi \in \mathbb{R}$, by

$$\widehat{f}^{\text{hyp}}(h,\xi) = \int_{\mathbb{H}} f(h\bullet z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z) = \int_0^\infty \int_{-\infty}^\infty f(h\bullet(x+\mathrm{i}y))y^{\frac{1}{2}+\frac{\mathrm{i}}{2}\xi}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

As with the (radial) Fourier transform, this may not be a well-defined Lebesgue integral but, as we will see, can be defined for almost every $(h,\xi)$ by an isometric extension of the transform on $C_c(\mathbb{H})$. We note that the measure-preserving substitution $w = h\bullet z$ in the definition implies that

$$\widehat{f}^{\text{hyp}}(h,\xi) = \int_{\mathbb{H}} f(w)\overline{\chi_{\infty,\xi}(h^{-1}\bullet w)}\,\mathrm{dvol}(w). \tag{9.31}$$

Hence we will think of $\widehat{f}^{\text{hyp}}(h,\xi)$ as the *correlation* of $f$ with the hyperbolic wave function $\mathbb{H} \ni w \mapsto \chi_{\infty,\xi}(h^{-1}\bullet w)$ which we think of as 'coming from $h\bullet\infty$ normalized for the point $h\bullet\mathrm{i}$'.

**Lemma 9.48 (Equivariance properties).** *For $f \in C_c(\mathbb{H})$ the hyperbolic Fourier transform $\widehat{f}^{\text{hyp}}$ is well-defined and satisfies*

$$\widehat{\pi_g^{\mathbb{H}} f}^{\text{hyp}}(h,\xi) = \widehat{f}^{\text{hyp}}(g^{-1}h,\xi)$$

*and*

$$\widehat{f}^{\text{hyp}}(hb,\xi) = \overline{\chi_\xi(b)}\Delta_B(b)^{\frac{1}{2}}\widehat{f}^{\text{hyp}}(h,\xi)$$

*for all $g \in \mathrm{SL}_2(\mathbb{R})$, $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$ and $b \in B = AU$.*

In other words, the lemma says that for any $\xi \in \mathbb{R}$ the map

$$C_c(\mathbb{H}) \ni f \longmapsto \widehat{f}^{\text{hyp}}(\cdot,\xi) \in \mathcal{H}_\xi^{\text{even}}$$

is equivariant between the Koopman representation and the principal series representation $\pi^{\xi,\mathrm{e}}$.

Proof of Lemma 9.48. For $g,h \in \mathrm{SL}_2(\mathbb{R})$ and $\xi \in \mathbb{R}$ we have

$$\begin{aligned}\widehat{\pi_g^{\mathbb{H}}(f)}^{\text{hyp}}(h,\xi) &= \int_{\mathbb{H}} \big(\pi_g^{\mathbb{H}} f\big)(h\bullet z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z)\\ &= \int_{\mathbb{H}} f(g^{-1}h\bullet z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z)\\ &= \widehat{f}^{\text{hyp}}(g^{-1}h,\xi).\end{aligned}$$

Moreover, for $b = u_x a_t \in B$ we also have

$$\begin{aligned}\widehat{f}^{\,\mathrm{hyp}}(hb,\xi) &= \int_{\mathbb{H}} f\bigl(h\raisebox{0.1ex}{\scriptsize$\bullet$}(b\raisebox{0.1ex}{\scriptsize$\bullet$}z)\bigr)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z)\\ &= \int_{\mathbb{H}} f(h\raisebox{0.1ex}{\scriptsize$\bullet$}w)\overline{\chi_{\infty,\xi}(b^{-1}\raisebox{0.1ex}{\scriptsize$\bullet$}w)}\,\mathrm{dvol}(w)\\ &= \Delta_B(b)^{\frac{1}{2}}\overline{\chi_\xi(b)}\int_{\mathbb{H}} f(h\raisebox{0.1ex}{\scriptsize$\bullet$}w)\overline{\chi_{\infty,\xi}(w)}\,\mathrm{dvol}(w)\end{aligned}$$

by using the measure-preserving substitution $w = b\raisebox{0.1ex}{\scriptsize$\bullet$}z$ and (9.30), which proves the lemma. □

We wish to explain Lemma 9.48 in another, more convenient, way using convolutions. For this, we let $g = kb \in KB = \mathrm{SL}_2(\mathbb{R})$ with $b = u_x a_t \in B$ and obtain

$$\chi_{\infty,\xi}(g^{-1}\raisebox{0.1ex}{\scriptsize$\bullet$}\mathrm{i}) = \chi_{\infty,\xi}(b^{-1}\raisebox{0.1ex}{\scriptsize$\bullet$}\mathrm{i}) = \mathrm{e}^{-t+\mathrm{i}t\xi} = \overline{F_\xi(b)} = \overline{F_\xi(g)}$$

where $F_\xi = F_{\xi,0} \in \mathcal{H}_\xi^{\mathrm{even}}$ is defined in (9.20). We also identify $\chi_{\infty,\xi}$ with the right $K$-invariant function

$$\chi_{\infty,\xi}\colon \mathrm{SL}_2(\mathbb{R}) \ni g \longmapsto \chi_{\infty,\xi}(g\raisebox{0.1ex}{\scriptsize$\bullet$}\mathrm{i}).$$

Recalling that $\mathrm{SL}_2(\mathbb{R})$ is unimodular, we can use the involution of Section 1.4.1 to put the above into the form

$$\chi_{\infty,\xi}^{*} = F_\xi. \tag{9.32}$$

The identification between functions on $\mathbb{H}$ with right $\mathrm{SO}_2(\mathbb{R})$-invariant functions on $\mathrm{SL}_2(\mathbb{R})$ allows us to use convolutions in $L^1(\mathrm{SL}_2(\mathbb{R}))$ as discussed in Section 1.4.1 for functions on $\mathbb{H}$. We will however also use convolutions of functions in $C_c(\mathrm{SL}_2(\mathbb{R}))$ and $C(\mathrm{SL}_2(\mathbb{R}))$, giving rise to functions in $C(\mathrm{SL}_2(\mathbb{R}))$ (see Exercise 1.46).

**Lemma 9.49 (Convolution formula).** *For a function $f \in C_c(\mathbb{H})$ we have*

$$\widehat{f}^{\,\mathrm{hyp}}(h,\xi) = f * F_\xi(h) = \int_G f(g\raisebox{0.1ex}{\scriptsize$\bullet$}\mathrm{i})F_\xi(g^{-1}h)\,\mathrm{d}m_G(g)$$

*for all $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$.*

We note that Lemma 9.49 implies both claims of Lemma 9.48. Indeed, $f * F_\xi \in \mathcal{H}_\xi$ since $F_\xi \in \mathcal{H}_\xi$ and $\mathcal{H}_\xi$ is defined by a formula using the right regular representation restricted to $B$, which commutes with the left convolution. Similarly, the equivariance under the Koopman representation (or, equivalently, under the left regular representation) also follows from the properties of convolutions.

Proof of Lemma 9.49. For a function $f \in C_c(\mathbb{H})$ and $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$ we have

$$\begin{aligned}\widehat{f}^{\text{hyp}}(h,\xi) &= \int_{\mathbb{H}} f(w)\overline{\chi_{\infty,\xi}}(h^{-1}\bullet w)\,\mathrm{dvol}(w) && \text{(by (9.31))}\\ &= \int_G f(g\bullet \mathrm{i})\chi^*_{\infty,\xi}(g^{-1}h)\,\mathrm{d}m(g)\\ &= \int_G f(g\bullet \mathrm{i})F_\xi(g^{-1}h)\,\mathrm{d}m(g) = f * F_\xi(h), && \text{(by (9.32))}\end{aligned}$$

where we also extended integration from $w = g\bullet\mathrm{i} \in \mathbb{H}$ to $g \in \mathrm{SL}_2(\mathbb{R})$ using $m_K(K) = 1$. □

**Lemma 9.50 (Rapid decay of transform).** *For any function $f \in C_c^\infty(\mathbb{H})$, we have*

$$\big\|\mathbb{R} \ni \xi \longmapsto \xi^\ell \widehat{f}^{\text{hyp}}(I,\xi)\big\|_\infty < \infty$$

*for any $\ell \in \mathbb{N}_0$.*

PROOF. We first recall that for $F \in C_c^\infty(\mathbb{R})$ we have

$$\|\mathbb{R} \ni \xi \longmapsto \xi^\ell \check{F}(\xi)\|_\infty \ll_\ell \|F^{(\ell)}\|_1 \tag{9.33}$$

by partial integration and induction on $\ell$ (see, for example, [24, Prop. 9.43]). To apply this, we rewrite the definition of $\widehat{f}^{\text{hyp}}$ using the substitution $y = \mathrm{e}^{2t}$ with $\frac{\mathrm{d}y}{y} = 2\,\mathrm{d}t$, which gives

$$\begin{aligned}\widehat{f}^{\text{hyp}}(I,\xi) &= \int_{\mathbb{H}} f(z)\overline{\chi_{\infty,\xi}(z)}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}\\ &= \int_0^\infty \int_{-\infty}^\infty f(x+\mathrm{i}y)\,\mathrm{d}x\, y^{-\frac{1}{2}} y^{\frac{\mathrm{i}}{2}\xi}\frac{\mathrm{d}y}{y}\\ &= \int_{-\infty}^\infty \underbrace{\int_{-\infty}^\infty f\big(x+\mathrm{i}\mathrm{e}^{2t}\big)\,\mathrm{d}x\, 2\mathrm{e}^{-t}}_{=F(t)}\,\mathrm{e}^{\mathrm{i}t\xi}\,\mathrm{d}t = \widehat{F}\big(\tfrac{1}{2\pi}\xi\big).\end{aligned}$$

Differentiation under the integral sign shows that the function $F \in C_c(\mathbb{R})$ is indeed smooth, so that (9.33) proves the lemma. □

### 9.4.5 The Hyperbolic Fourier Inversion Formula

As explained at the end of Section 9.4.2, we expect to be able to write a given function on $\mathbb{H}$ as a superposition of elementary waves $z \mapsto \chi_{\infty,\xi}(k^{-1}\bullet z)$ of various frequencies $\xi$ emanating from the boundary points $k\bullet\infty \in \partial\mathbb{H}$. For this the hyperbolic Fourier transform $\widehat{f}^{\text{hyp}}(k,\xi)$ for a pair $(k,\xi) \in K \times \mathbb{R}$ should be related to the desired volume at $k\bullet\infty \in \partial\mathbb{H}$ for frequency $\xi$. Assuming smoothness and compact support of the original function ensures that the desired integral representation converges.

**Theorem 9.51 (Hyperbolic Fourier inversion).** *Let* $f \in C_c^\infty(\mathbb{H})$*. Then*

$$f(z) = \frac{1}{16\pi} \int_{\mathbb{R}} \int_K \widehat{f}^{\,\mathrm{hyp}}(k,\xi)\chi_{\infty,\xi}(k^{-1}\bullet z)\,\mathrm{d}m_K(k)\xi \tanh\frac{\pi\xi}{2}\,\mathrm{d}\xi$$

*for all* $z \in \mathbb{H}$*.*

The proof will rely on some elementary integral manipulations, Fourier inversion on $\mathbb{R}$ (applied in a surprising way), and a contour integration to determine the correct volume amplification factor $\xi \tanh\bigl(\frac{\pi\xi}{2}\bigr)$ for $\xi \in \mathbb{R}$. To reduce the complexity of the problem, we first consider a special class of functions.

**Definition 9.52 (Spherical functions).** A function $f\colon \mathbb{H} \to \mathbb{C}$ is called *spherical* if $f(k\bullet z) = f(z)$ for all $k \in K$ and $z \in \mathbb{H}$.

We note that due to the equivariance property in Lemma 9.48 the hyperbolic Fourier transform of a spherical function is again invariant under $K$. Because of this, for a spherical function $f$ we will also use the simplified notation

$$\widehat{f}^{\,\mathrm{hyp}}(\xi) = \int_{\mathbb{H}} f(z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z)$$

satisfying

$$\widehat{f}^{\,\mathrm{hyp}}(k,\xi) = \int_{\mathbb{H}} f(k\bullet z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z) = \widehat{f}^{\,\mathrm{hyp}}(\xi) \tag{9.34}$$

for all $(k,\xi) \in K \times \mathbb{R}$ by Definitions 9.47 and 9.52 above. Recall that Lemma 9.48 also shows that the function $\widehat{f}^{\,\mathrm{hyp}}(\cdot,\xi)$ belongs to $\mathcal{H}_\xi^{\mathrm{even}}$. With this, (9.34) becomes

$$\widehat{f}^{\,\mathrm{hyp}}(\cdot,\xi) = \widehat{f}^{\,\mathrm{hyp}}(\xi)F_\xi(\cdot), \tag{9.35}$$

where $F_\xi$ is the extension of $\mathbb{1}_K$ to an element of $\mathcal{H}_\xi^{\mathrm{even}}$ (the case $n = 0$ in (9.20)).

The following lemma gives another connection between the hyperbolic Fourier transform and $\pi^\xi$, or more precisely its matrix coefficient $\phi_\xi = \varphi_{F_\xi}^{\pi^\xi}$.

**Lemma 9.53 (Matrix coefficient giving symmetry).** *Let* $f \in C_c(\mathbb{H})$ *be a spherical function. Then*

$$\widehat{f}^{\,\mathrm{hyp}}(\xi) = \int_{\mathbb{H}} f\phi_\xi\,\mathrm{dvol}(z) \tag{9.36}$$

*for all* $\xi \in \mathbb{R}$*. Moreover, we have* $\phi_\xi = \phi_{-\xi}$ *and*

$$\widehat{f}^{\,\mathrm{hyp}}(\xi) = \widehat{f}^{\,\mathrm{hyp}}(-\xi)$$

*for all* $\xi \in \mathbb{R}$.

PROOF. Using (9.34), the normalized Haar measure $m_K$ on $K$, the convolution formula in Lemma 9.49, and Fubini's theorem we have

$$\begin{aligned}\widehat{f}^{\text{hyp}}(\xi) &= \int_K \widehat{f}^{\text{hyp}}(k,\xi)\,\mathrm{d}m_K(k)\\ &= \int_K \int_G f(g\cdot\mathrm{i})F_\xi(g^{-1}k)\,\mathrm{d}m_G(g)\,\mathrm{d}m_K(k)\\ &= \int_G f(g\cdot\mathrm{i}) \underbrace{\int_K F_\xi(g^{-1}k)F_\xi(k)\,\mathrm{d}m_K(k)}_{=\langle\pi_g^\xi F_\xi, F_\xi\rangle=\phi_\xi(g)}\,\mathrm{d}m_G(g),\end{aligned}$$

which proves (9.36).

Finally, by Theorem 9.31, $\pi^{\xi,\mathrm{e}}$ and $\pi^{-\xi,\mathrm{e}}$ are unitarily isomorphic. Since the vector $F_{\pm\xi} \in \mathcal{H}_{\pm\xi}^{\text{even}}$ is, up to scalar multiples, the unique $K$-fixed vector and both have unit length, it follows that

$$\phi_{-\xi} = \varphi_{F_{-\xi}}^{\pi^{-\xi,\mathrm{e}}} = \varphi_{F_\xi}^{\pi^{\xi,\mathrm{e}}} = \phi_\xi.$$

Together with (9.36), this gives the lemma. Alternatively, one may also apply Proposition 9.19 and Lemma 9.33. □

We note that we are going to use the assumption that $f \in C_c^\infty(\mathbb{H})$ is spherical to reduce the number of free variables in the proof of the Fourier inversion formula. In fact, $f$ is spherical if and only if it can be written as

$$f(z) = F_d\bigl(\mathsf{d}_{\text{hyp}}(z,\mathrm{i})\bigr)$$

for $z \in \mathbb{H}$, where $F_d\colon [0,\infty) \to \mathbb{C}$ is a function, and $\mathsf{d}_{\text{hyp}}$ denotes the hyperbolic metric on $\mathbb{H}$ (see Lemma 8.16). This follows from the transitivity of the action of $K$ on every circle with centre i.

In terms of the upcoming integral substitutions, it is better to consider instead of $\mathsf{d}_{\text{hyp}}(z,\mathrm{i})$ the closely related expression

$$r(z) = \cosh^2\bigl(\tfrac{1}{2}\mathsf{d}_{\text{hyp}}(z,\mathrm{i})\bigr) = \tfrac{1}{2}\cosh\bigl(\mathsf{d}_{\text{hyp}}(z,\mathrm{i})\bigr) + \tfrac{1}{2} \tag{9.37}$$

for $z \in \mathbb{H}$, where we have used the identity

$$\cosh^2 t = \left(\frac{\mathrm{e}^t + \mathrm{e}^{-t}}{2}\right)^2 = \frac{1}{2}\cosh(2t) + \frac{1}{2}$$

for $t = \frac{1}{2}\mathsf{d}_{\text{hyp}}(z,\mathrm{i})$. Once more every spherical function $f\colon \mathbb{H} \to \mathbb{C}$ can be written in the form

$$f(z) = F(r(z))$$

for all $z \in \mathbb{H}$ and for some function $F\colon [1,\infty) \to \mathbb{C}$.

**Lemma 9.54 (Inversion at i for spherical functions).** *For a spherical function $f \in C_c^\infty(\mathbb{H})$ we have*

$$f(\mathrm{i}) = \frac{1}{16\pi^2} \int_{\mathbb{R}} \widehat{f}^{\,\mathrm{hyp}}(\xi)\xi \int_{\mathbb{R}} \frac{\sin \xi t}{\sinh t} \,\mathrm{d}t \,\mathrm{d}\xi.$$

Proof. As explained above, for the given spherical function $f \in C_c^\infty(\mathbb{H})$ there exists a continuous function $F \in C_c([1,\infty))$ with $f(z) = F(r(z))$ for all $z \in \mathbb{H}$. In fact we can define $F$ by restricting $f$ to $\{y\mathrm{i} \mid y > 0\}$ and an appropriate coordinate change. Indeed, if $y = \mathrm{e}^{2t}$, then $\mathsf{d}_{\mathrm{hyp}}(\mathrm{e}^{2t}\mathrm{i}, \mathrm{i}) = 2|t|$ and $r(z) = \cosh^2 t$, which leads to the definition

$$F(r) = f\Big(\mathrm{i}\mathrm{e}^{2 \operatorname{arcosh} \sqrt{r}}\Big)$$

for $r \in [1,\infty)$. Equivalently, we have $F(\cosh^2 t) = f(\mathrm{i}\mathrm{e}^{2t})$ for $t \in \mathbb{R}$, and, since $f$ is spherical, more generally, $F(r(z)) = f(z)$ for all $z \in \mathbb{H}$.

Using the formula for $\mathsf{d}_{\mathrm{hyp}}(\cdot,\cdot)$ in Lemma 8.16 we also have, for $r(z)$ as in (9.37),

$$\begin{aligned} r(z) &= \frac{1}{2}\left(1 + \frac{\|z-\mathrm{i}\|^2}{2y}\right) + \frac{1}{2} \\ &= \frac{1}{2}\left(1 + \frac{x^2}{2y} + \frac{(y-1)^2}{2y}\right) + \frac{1}{2} \\ &= \frac{1}{2}\left(1 + \frac{x^2}{2y} + \frac{y+y^{-1}}{2} - 1\right) + \frac{1}{2} = \frac{1}{2} + \frac{y+y^{-1}}{4} + \frac{x^2}{4y}. \end{aligned}$$

Below we will also use the coordinates $(x,t) \in \mathbb{R}^2$ for $z = x + \mathrm{i}\mathrm{e}^{2t} \in \mathbb{H}$, which gives

$$r(z) = \frac{1}{2} + \frac{\mathrm{e}^{2t} + \mathrm{e}^{-2t}}{4} + \frac{1}{4}x^2\mathrm{e}^{-2t} = \cosh^2 t + \frac{1}{4}x^2\mathrm{e}^{-2t}.$$

For our functions $f$ and $F$, this gives

$$F\left(\cosh^2 t + \tfrac{1}{4}x^2\mathrm{e}^{-2t}\right) = f(x + \mathrm{i}\mathrm{e}^{2t})$$

for $x,t \in \mathbb{R}$.

We now use this identity in the formula for the hyperbolic Fourier transform, and obtain

$$\begin{aligned} \widehat{f}^{\,\mathrm{hyp}}(\xi) &= \int_{\mathbb{H}} f(x+\mathrm{i}y) y^{\frac{1}{2}+\frac{1}{2}\mathrm{i}\xi} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} F\left(\cosh^2 t + \tfrac{1}{4}x^2\mathrm{e}^{-2t}\right) \mathrm{e}^{t+\mathrm{i}\xi t}\mathrm{e}^{-4t}\,\mathrm{d}x\, 2\mathrm{e}^{2t}\,\mathrm{d}t \\ &= 2\int_{\mathbb{R}}\int_{\mathbb{R}} F\left(\cosh^2 t + \tfrac{1}{4}x^2\mathrm{e}^{-2t}\right)\,\mathrm{d}x\,\mathrm{e}^{-t}\mathrm{e}^{\mathrm{i}\xi t}\,\mathrm{d}t \end{aligned}$$

by using $y = \mathrm{e}^{2t}$ and $\mathrm{d}y = 2\mathrm{e}^{2t}\,\mathrm{d}t$. We now set $u = \frac{1}{2}\mathrm{e}^{-t}x$ with $2\,\mathrm{d}u = \mathrm{e}^{-t}\,\mathrm{d}x$ to arrive at

$$\widehat{f}^{\text{hyp}}(\xi) = 4\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi t}\int_{\mathbb{R}} F\left(\cosh^2 t + u^2\right)\,\mathrm{d}u\,\mathrm{d}t. \tag{9.38}$$

We use the inner integral to define the function

$$\Phi(s) = \int_{\mathbb{R}} F\left(s + u^2\right)\,\mathrm{d}u$$

for $s \in [1,\infty)$ and the composition

$$\Psi(t) = \Phi(\cosh^2 t) = \int_{\mathbb{R}} F\left(\cosh^2 t + u^2\right)\,\mathrm{d}u$$

for $t \in \mathbb{R}$. We note that $\Phi \in C_c([1,\infty))$ and $\Psi \in C_c(\mathbb{R})$.

With the function $\Psi$ to hand, we realize that $\widehat{f}^{\text{hyp}}$ is essentially the Fourier transform of $\Psi$. More precisely, we may reformulate (9.38) as

$$\widehat{f}^{\text{hyp}}(\xi) = 4\check{\Psi}\left(\frac{1}{2\pi}\xi\right) \tag{9.39}$$

for all $\xi \in \mathbb{R}$.

By Lemma 9.50, we know that $\widehat{f}^{\text{hyp}}$ decays rapidly. Hence we may apply Fourier inversion on $\mathbb{R}$, and the substitution $\xi = 2\pi\zeta$ to obtain from (9.39) that

$$\Psi(t) = \int_{\mathbb{R}} \check{\Psi}(\zeta)\mathrm{e}^{-2\pi\mathrm{i}\zeta t}\,\mathrm{d}\zeta = \frac{1}{2\pi}\int_{\mathbb{R}} \check{\Psi}\left(\frac{1}{2\pi}\xi\right)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}\xi = \frac{1}{8\pi}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}\xi$$

for all $t \in \mathbb{R}$. By Lemma 9.53 we have $\widehat{f}^{\text{hyp}}(\xi) = \widehat{f}^{\text{hyp}}(-\xi)$, which turns the above into

$$\Phi(\cosh^2 t) = \Psi(t) = \frac{1}{8\pi}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\cos(\xi t)\,\mathrm{d}\xi$$

Using the rapid decay of $\widehat{f}^{\text{hyp}}$ in Lemma 9.50 again, we may differentiate under the integral to obtain

$$\Psi'(t) = -\frac{1}{8\pi}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\xi\sin(\xi t)\,\mathrm{d}\xi \tag{9.40}$$

for $t \in \mathbb{R}$. Using the chain rule for $\Psi(t) = \Phi(\cosh^2 t)$, we also have

$$\Psi'(t) = \Phi'(\cosh^2 t)2\cosh t\sinh t$$

for $t \in \mathbb{R}\smallsetminus\{0\}$. We divide this by $2\sinh t$ to obtain, with (9.40),

$$\Phi'(\cosh^2 t)\cosh t = \frac{\Psi'(t)}{2\sinh t} = -\frac{1}{16\pi}\int_{\mathbb{R}} \widehat{f}^{\text{hyp}}(\xi)\xi\frac{\sin\xi t}{\sinh t}\,\mathrm{d}\xi \tag{9.41}$$

for $t \in \mathbb{R} \smallsetminus \{0\}$.

Moreover, note that

$$\lim_{t \to 0} \frac{\sin \xi t}{\sinh t} = \xi$$

and

$$\left| \frac{\sin \xi t}{\sinh t} \right| = \left| \frac{\sin \xi t}{t} \right| \frac{t}{\sinh t} \leqslant |\xi| \frac{t}{\sinh t} \leqslant |\xi| \tag{9.42}$$

by the mean-value theorem applied to the function $\mathbb{R} \ni t \mapsto \sin \xi t$. Together with Lemma 9.50, this shows that $\Phi'$ extends continuously from $(1, \infty)$ to $[1, \infty)$. It follows from the mean value theorem for $\Phi$ that $\Phi$ also has a one-sided derivative as $s = 1$.

The Fourier inversion formulas above allow us to obtain the value

$$\Phi(1) = \Psi(0) = \int_{\mathbb{R}} F(1 + u^2) \, \mathrm{d}u$$

from the hyperbolic Fourier transform, but we wish instead to obtain the value $F(1) = f(\mathrm{i})$ of the integrand $F$ in the definition of $\Phi$. To obtain this, we rely on some stunning but elementary integration trickery. In fact, we claim that one can recover $F$ from $\Phi$ by the formula

$$F(s) = -\frac{1}{\pi} \int_{\mathbb{R}} \Phi'(s + v^2) \, \mathrm{d}v \tag{9.43}$$

for all $s \geqslant 1$.

To see this, note first that $f \in C_c^\infty(\mathbb{H})$ implies that $F|_{(1,\infty)}$ is smooth, that

$$\Phi'(s) = \int_{\mathbb{R}} F'(s + u^2) \, \mathrm{d}u$$

and

$$\Phi'(s + v^2) = \int_{\mathbb{R}} F'(s + u^2 + v^2) \, \mathrm{d}u$$

for all $s > 1$ and $v \in \mathbb{R}$. Integrating the latter over $v \in \mathbb{R}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \Phi'(s + v^2) \, \mathrm{d}v &= \int_{\mathbb{R}^2} F'(s + u^2 + v^2) \underbrace{\mathrm{d}u \, \mathrm{d}v}_{R \, \mathrm{d}R \, \mathrm{d}\theta} \\ &= 2\pi \int_0^\infty F'(s + R^2) R \, \mathrm{d}R \\ &= \pi \int_0^\infty F'(s + \rho) \, \mathrm{d}\rho, \end{aligned}$$

where we used polar coordinates $(R, \theta)$ for $(u, v) \in \mathbb{R}^2$ and the substitution $\rho = R^2$. For the latter integral we may now apply the fundamental theorem of calculus. Since $F \in C_c([1,\infty))$, this takes the form

$$\int_0^\infty F'(s+\rho)\,\mathrm{d}\rho = -F(s)$$

for all $s > 1$, which proves the claim in (9.43) for $s > 1$. Since $F$ and $\Phi'$ both lie in $C_c([1,\infty))$, this extends by continuity to $s = 1$.

We now set $s = 1$ in (9.43), substitute $v = \sinh t$ with $\mathrm{d}v = \cosh t\,\mathrm{d}t$, and combine the resulting integral with (9.41), which leads to

$$\begin{aligned} F(1) &= -\frac{1}{\pi}\int_{\mathbb{R}} \Phi'(\underbrace{1+v^2}_{=\cosh^2 t})\,\mathrm{d}v \\ &= -\frac{1}{\pi}\int_{\mathbb{R}} \Phi'(\cosh^2 t)\cosh t\,\mathrm{d}t \\ &= \frac{1}{16\pi^2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \widehat{f}^{\,\mathrm{hyp}}(\xi)\xi\frac{\sin\xi t}{\sinh t}\,\mathrm{d}\xi\right)\,\mathrm{d}t. \end{aligned}$$

Recall that by Lemma 9.48 we have

$$\left|\widehat{f}^{\,\mathrm{hyp}}(\xi)\right| \ll_f \frac{1}{(1+\xi^2)^2}.$$

Together with (9.42), we obtain

$$\left|\widehat{f}^{\,\mathrm{hyp}}(\xi)\xi\frac{\sin\xi t}{\sinh t}\right| \leqslant \frac{\xi^2}{(1+\xi^2)^2}\frac{t}{\sinh t},$$

which is easily seen to be integrable over $\mathbb{R}^2$. This implies that the integrand above lies in $L^1(\mathbb{R}^2)$. Hence we may apply Fubini's theorem and obtain, with

$$f(\mathrm{i}) = F(1) = \frac{1}{16\pi^2}\int_{\mathbb{R}} \widehat{f}^{\,\mathrm{hyp}}(\xi)\xi\int_{\mathbb{R}}\frac{\sin\xi t}{\sinh t}\,\mathrm{d}t\,\mathrm{d}\xi,$$

the lemma. □

Lemma 9.54 explains why the following result is of interest to us.

**Lemma 9.55 (Volume factor).** *For $\xi \in \mathbb{R}$ we have*

$$\int_{\mathbb{R}} \frac{\sin\xi t}{\sinh t}\,\mathrm{d}t = \pi\tanh\left(\frac{\pi\xi}{2}\right). \tag{9.44}$$

PROOF. To prove (9.44), we will apply the Cauchy integral formula to the meromorphic function $f$ defined by

$$f(z) = \frac{\mathrm{e}^{\mathrm{i}\xi z}}{\sinh z}. \tag{9.45}$$

We start with some elementary observations about the function $f$. For the point $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ we may apply the reverse triangle inequality to see that

$$|\sinh z| = \left|\frac{\mathrm{e}^x\mathrm{e}^{\mathrm{i}y} - \mathrm{e}^{-x}\mathrm{e}^{-\mathrm{i}y}}{2}\right| \geqslant \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2} = \sinh x.$$

By symmetry, this gives

$$|\sinh z| \geqslant |\sinh x|, \tag{9.46}$$

which implies that $f(x + \mathrm{i}y)$ defined in (9.45) decays rapidly for $|x| \to \infty$ as long as $|y|$ is bounded. The estimate (9.46) also shows that $\sinh(x + \mathrm{i}y)$ can only vanish when $x = 0$. Since $\sinh(\mathrm{i}y) = \mathrm{i}\sin y$ for $y \in \mathbb{R}$, we see that $f$ has poles precisely at the points in $\mathbb{Z}\pi\mathrm{i}$. At 0 the residue of $f$ is given by 1. Finally, we have

$$\sinh(z + \pi\mathrm{i}) = \frac{\mathrm{e}^{z+\pi\mathrm{i}} + \mathrm{e}^{-z-\pi\mathrm{i}}}{2} = -\sinh z$$

for $z \in \mathbb{C}$, which implies that

$$f(z + \pi\mathrm{i}) = \frac{\mathrm{e}^{\mathrm{i}\xi(z+\pi\mathrm{i})}}{-\sinh z} = -\mathrm{e}^{-\xi\pi} f(z) \tag{9.47}$$

for $z \in \mathbb{C}\smallsetminus(\mathbb{Z}\pi\mathrm{i})$.

We now integrate $f$ over the closed path

$$\gamma = \gamma_{\mathrm{bottom}} \sqcup \gamma_{\mathrm{right}} \sqcup \gamma_{\mathrm{top}} \sqcup \gamma_{\mathrm{top}} \sqcup \gamma_{\mathrm{left}}$$

indicated in Figure 9.8.

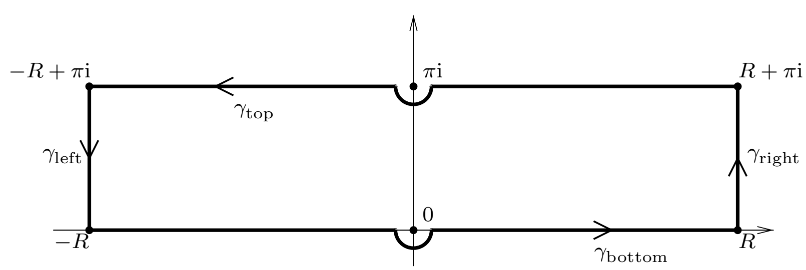


**Fig. 9.8:** The closed path $\gamma = \gamma_{R,\varepsilon}$ consists of four pieces. The first path $\gamma_{\mathrm{bottom}}$ goes from $-R$ to $R$ but avoids the pole at 0 (but including it inside the contour) by following a semi-circle of radius $\varepsilon$ around 0. The paths $\gamma_{\mathrm{right}}$, $\gamma_{\mathrm{top}}$, and $\gamma_{\mathrm{left}}$ go as indicated via $R + \pi\mathrm{i}$ and $-R + \pi\mathrm{i}$ back to $-R$, again avoiding the pole at $\pi\mathrm{i}$ (but leaving it outside the contour).

The description of the poles of $f$ given above now implies that

$$\oint_{\gamma} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}$$

independent of $R > 1$ and $\varepsilon \in (0,1)$. The decay properties of $f$ discussed above also imply that

$$\lim_{R\to\infty} \oint_{\gamma_{\text{right}}} f(z)\,\mathrm{d}z = \lim_{R\to\infty} \oint_{\gamma_{\text{left}}} f(z)\,\mathrm{d}z = 0.$$

Moreover, we defined the paths $\gamma_{\text{bottom}}$ and $\gamma_{\text{top}}$ so that $\gamma_{\text{bottom}} + \pi\mathrm{i}$ is equal to $\gamma_{\text{top}}$ except for the orientation, which is reversed. Together with (9.47) this gives

$$\oint_{\gamma_{\text{top}}} f(z)\,\mathrm{d}z = -\oint_{\gamma_{\text{bottom}}} f(z+\pi\mathrm{i})\,\mathrm{d}z = \mathrm{e}^{-\xi\pi} \oint_{\gamma_{\text{bottom}}} f(z)\,\mathrm{d}z$$

again independently of $R$ and $\varepsilon$. Putting this together, we obtain

$$\begin{aligned} 2\pi\mathrm{i} &= \lim_{R\to\infty} \oint_{\gamma} f(z)\,\mathrm{d}z \\ &= \lim_{R\to\infty} \bigl(1+\mathrm{e}^{-\xi\pi}\bigr) \oint_{\gamma_{\text{bottom}}} f(z)\,\mathrm{d}z \\ &= \bigl(1+\mathrm{e}^{-\xi\pi}\bigr)\left(\int_{-\infty}^{-\varepsilon} \frac{\mathrm{e}^{\mathrm{i}\xi t}}{\sinh t}\,\mathrm{d}t + \int_{\varepsilon}^{\infty} \frac{\mathrm{e}^{\mathrm{i}\xi t}}{\sinh t}\,\mathrm{d}t + \oint_{\gamma_\varepsilon} f(z)\,\mathrm{d}z\right), \end{aligned} \tag{9.48}$$

where $\gamma_\varepsilon \colon [\pi, 2\pi] \ni t \mapsto \varepsilon\mathrm{e}^{\mathrm{i}t}$ is the semi-circular path appearing in $\gamma_{\text{bottom}}$. To understand the asymptotics of

$$\oint_{\gamma_\varepsilon} f(z)\,\mathrm{d}z$$

as $\varepsilon$ decreases to 0, we note that $f(z) = \frac{1}{z} + h(z)$ for a function $h$ that is holomorphic at 0. By continuity of $h$, we have

$$\lim_{\varepsilon\searrow 0} \oint_{\gamma_\varepsilon} h(z)\,\mathrm{d}z = 0,$$

so we only have to calculate

$$\oint_{\gamma_\varepsilon} \frac{1}{z}\,\mathrm{d}z = \int_{\pi}^{2\pi} \frac{1}{\varepsilon\mathrm{e}^{\mathrm{i}t}} \varepsilon\mathrm{i}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t = \pi\mathrm{i}.$$

We now take the imaginary part in (9.48) and let $\varepsilon$ decrease to 0, which gives

$$2\pi = \left(1 + \mathrm{e}^{-\xi\pi}\right)\left(\int_{-\infty}^{\infty} \frac{\sin \xi t}{\sinh t}\,\mathrm{d}t + \pi\right) = \left(1 + \mathrm{e}^{-\xi\pi}\right)\int_{-\infty}^{\infty} \frac{\sin \xi t}{\sinh t}\,\mathrm{d}t + \pi + \mathrm{e}^{-\xi\pi}\pi.$$

Solving this equation for the integral gives

$$\int_{-\infty}^{\infty} \frac{\sin \xi t}{\sinh t}\,\mathrm{d}t = \frac{1 - \mathrm{e}^{-\xi\pi}}{1 + \mathrm{e}^{-\xi\pi}}\pi = \frac{\mathrm{e}^{\xi\pi/2} - \mathrm{e}^{-\xi\pi/2}}{\mathrm{e}^{\xi\pi/2} + \mathrm{e}^{-\xi\pi/2}}\pi,$$

and hence the lemma. □

Having obtained the hyperbolic Fourier inversion formula at i and for spherical functions in the last two lemmas, we are now in a position to prove the general case.

Proof of Theorem 9.51. Let $f \in C_c^\infty(\mathbb{H})$. Combining Lemmas 9.54 and 9.55, we see that if $f \in C_c^\infty(\mathbb{H})$ is spherical, then

$$f(\mathrm{i}) = \frac{1}{16\pi}\int_{\mathbb{R}} \widehat{f}^{\,\mathrm{hyp}}(\xi)\xi \tanh\left(\frac{\pi\xi}{2}\right)\mathrm{d}\xi.$$

To use this for a general $f \in C_c^\infty(\mathbb{H})$, we define

$$f_{\mathrm{sph}}(z) = \int_K f(k\bullet z)\,\mathrm{d}m_K(k).$$

Since $k\bullet\mathrm{i} = \mathrm{i}$ for all $k \in K$, and we may differentiate under the integral sign, it follows that $f_{\mathrm{sph}} \in C_c^\infty(\mathbb{H})$ has $f_{\mathrm{sph}}(\mathrm{i}) = f(\mathrm{i})$. Applying Lemma 9.55 to this spherical function, we obtain

$$f(\mathrm{i}) = f_{\mathrm{sph}}(\mathrm{i}) = \frac{1}{16\pi}\int_{\mathbb{R}} \widehat{f}_{\mathrm{sph}}^{\,\mathrm{hyp}}(\xi)\xi \tanh\left(\frac{\pi\xi}{2}\right)\mathrm{d}\xi. \tag{9.49}$$

Using Fubini's theorem and the definition of the hyperbolic Fourier transform, we also have

$$\begin{aligned}\widehat{f}_{\mathrm{sph}}^{\,\mathrm{hyp}}(\xi) &= \int_{\mathbb{H}} f_{\mathrm{sph}}(z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z)\\ &= \int_K \int_{\mathbb{H}} f(k\bullet z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z)\,\mathrm{d}m_K(k)\\ &= \int_K \widehat{f}^{\,\mathrm{hyp}}(k,\xi)\,\mathrm{d}m_K(k).\end{aligned}$$

Putting this into (9.49), we obtain

$$f(\mathrm{i})=\frac{1}{16\pi}\int_{\mathbb{R}}\int_K \widehat{f}^{\,\mathrm{hyp}}(k,\xi)\xi\tanh\left(\frac{\pi\xi}{2}\right)\,\mathrm{d}m_K(k)\,\mathrm{d}\xi.$$

Now let $h\in\mathrm{SL}_2(\mathbb{R})$ be arbitrary and define $\widetilde{f}=\pi_h^{\mathbb{H}}f$. Applying the previous formula to $\widetilde{f}$ and the equivariance claim in Lemma 9.48, we obtain

$$\begin{aligned}f(h^{-1}\bullet\mathrm{i})=\bigl(\pi_h^{\mathbb{H}}f\bigr)(\mathrm{i})&=\frac{1}{16\pi}\int_{\mathbb{R}}\int_K \widehat{\pi_h^{\mathbb{H}}f}^{\,\mathrm{hyp}}(k,\xi)\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}m_K(k)\,\mathrm{d}\xi\\&=\frac{1}{16\pi}\int_{\mathbb{R}}\int_K \widehat{f}^{\,\mathrm{hyp}}(h^{-1}k,\xi)\,\mathrm{d}m_K(k)\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi.\end{aligned}$$

Next we note that

$$\begin{aligned}\int_K \widehat{f}^{\,\mathrm{hyp}}(h^{-1}k,\xi)\,\mathrm{d}m_K(k)&=\left\langle \pi_h^{\xi,\mathrm{e}}\widehat{f}^{\,\mathrm{hyp}}(\cdot,\xi),F_\xi\right\rangle_{\mathcal{H}_\xi}\\&=\left\langle \widehat{f}^{\,\mathrm{hyp}}(\cdot,\xi),\pi_{h^{-1}}^{\xi,\mathrm{e}}F_\xi\right\rangle_{\mathcal{H}_\xi}\\&=\int_K \widehat{f}^{\,\mathrm{hyp}}(k,\xi)\overline{F_\xi(hk)}\,\mathrm{d}m_K(k)\\&=\int_K \widehat{f}^{\,\mathrm{hyp}}(k,\xi)\chi_{\infty,\xi}(k^{-1}h^{-1}\bullet\mathrm{i})\,\mathrm{d}m_K(k)\end{aligned}$$

by (9.32). Combining this with the above, and setting $h^{-1}\bullet\mathrm{i}=z$, we obtain

$$f(z)=\frac{1}{16\pi}\int_{\mathbb{R}}\int_K \widehat{f}^{\,\mathrm{hyp}}(k,\xi)\chi_{\infty,\xi}(k^{-1}\bullet z)\,\mathrm{d}m_K(k)\xi\tanh\bigl(\tfrac{\pi\xi}{2}\bigr)\,\mathrm{d}\xi,$$

which gives the theorem. □

### 9.4.6 The Hyperbolic Fourier Transform in the Disc Model

We recall from Section 8.3.3 that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is conjugated to the action of $\mathrm{SU}_{1,1}(\mathbb{R})$ on $\mathbb{D}$ by the map

$$\Phi\colon\overline{\mathbb{C}}\ni w\longmapsto\begin{pmatrix}1&\mathrm{i}\\\mathrm{i}&1\end{pmatrix}\bullet w=\frac{w+\mathrm{i}}{\mathrm{i}w+1}\in\overline{\mathbb{C}}$$

with $\Phi(\mathbb{D})=\mathbb{H}$, $\Phi(0)=\mathrm{i}$ being our choice of origin in $\mathbb{H}$, and $\Phi(\mathrm{i})=\infty$. Using this, we can move the Busemann function to $\mathbb{D}$.

**Definition 9.56 (Busemann function for $\mathrm{i}\in\partial\mathbb{D}$).** The Busemann function on $\mathbb{D}$ with respect to $\mathrm{i}\in\partial\mathbb{D}$ (and origin $0\in\mathbb{D}$) is defined by

$$b_{\mathrm{i}}^{\mathbb{D}}(w)=b_\infty^{\mathbb{H}}(\Phi(w))=-\log\left(\frac{1-|w|^2}{|w-\mathrm{i}|^2}\right).$$

We now verify that the two formulas in Definition 9.56 above are in fact equivalent. Indeed, for $w = x + \mathrm{i}y$ we have

$$\begin{aligned}\Im \Phi(w) &= \Im \frac{w+\mathrm{i}}{\mathrm{i}w+1} = \Im \frac{(x+\mathrm{i}y+\mathrm{i})(-\mathrm{i}x-y+1)}{(\mathrm{i}x-y+1)(-\mathrm{i}x-y+1)} \\ &= \Im \frac{-\mathrm{i}x^2-\mathrm{i}y^2+\mathrm{i}y-\mathrm{i}y+\mathrm{i}}{x^2+(y-1)^2} = \frac{1-x^2-y^2}{x^2+(y-1)^2} = \frac{1-|w|^2}{|w-\mathrm{i}|^2}\end{aligned}$$

as claimed.

Next we recall that

$$k_\theta = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \in K < \mathrm{SU}_{1,1}(\mathbb{R})$$

rotates $\mathbb{D}$ so that our origin $0 \in \mathbb{D}$ is fixed, and $k_\theta \bullet \mathrm{i} = \mathrm{e}^{-2\mathrm{i}\theta}\mathrm{i}$. This suggests the following more general definitions.

**Definition 9.57 (Busemann functions and hyperbolic waves on $\mathbb{D}$).** The *Busemann function on* $\mathbb{D}$ with respect to $p \in \partial\mathbb{D}$ (and origin $0 \in \mathbb{D}$) is defined by

$$b_p^{\mathbb{D}}(w) = -\log\left(\frac{1-|w|^2}{|w-p|^2}\right).$$

Moreover, the hyperbolic wave coming from $p$ and with frequency $\xi \in \mathbb{R}$ (normalized for the origin $0 \in \mathbb{D}$) is defined by the function

$$\chi_{p,\xi}(w) = \mathrm{e}^{(-\frac{1}{2}+\frac{\mathrm{i}}{2}\xi) b_p^{\mathbb{D}}(w)} = \left(\frac{1-|w|^2}{|w-p|^2}\right)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi}.$$

We imagine that the hyperbolic wave $\chi_{p,\xi}(w)$ is the sound produced by a loudspeaker at $p \in \partial\mathbb{D}$ using frequency $\xi \in \mathbb{R}$. The following reformulation of Theorem 9.51 establishes our goal to obtain any function $f \in C_c^\infty(\mathbb{D})$ as a superposition of such waves using loudspeakers at any point $p \in \partial\mathbb{D}$, and using all possible frequencies $\xi \in \mathbb{R}$.

**Theorem 9.58 (Fourier inversion on** $\mathrm{d}$**).** *Let* $f \in C_c^\infty(\mathrm{d})$ *and define the abbreviation*

$$\langle f, \chi_{p,\xi}\rangle = \int_{\mathrm{d}} f \overline{\chi_{p,\xi}}\,\mathrm{dvol}$$

*for* $p \in \partial\mathrm{d}$ *and* $\xi \in \mathbb{R}$*. Then*

$$f(z) = \frac{1}{16\pi}\int_{\mathbb{R}}\int_{\partial\mathrm{d}} \langle f, \chi_{p,\xi}\rangle \chi_{p,\xi}(z)\,\mathrm{d}p\,\xi\tanh\big(\tfrac{\pi\xi}{2}\big)\,\mathrm{d}\xi$$

*for all* $z \in \mathrm{d}$*, where* $\mathrm{d}p$ *denotes the normalized Lebesgue measure on* $\partial\mathrm{d}$*.*

### 9.4.7 The Unitary Isomorphism

We now show that the hyperbolic Fourier transform is in fact a unitary isomorphism between the Koopman representation $\pi^{\mathbb{H}}$ and an integral of all even principal series representations as in Definition 9.38.

**Theorem 9.59 (Unitary isomorphism).** *The hyperbolic Fourier transform satisfies the identity*

$$\|f\|^2_{L^2(\mathbb{H})} = \tfrac{1}{8\pi}\int_{[0,\infty)} \big\|\widehat{f}^{\,\mathrm{hyp}}(\cdot,\xi)\big\|^2_{\mathcal{H}_\xi}\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi$$

*for every $f \in C_c(\mathbb{H})$. Moreover, it extends to an equivariant unitary isomorphism between $\pi^{\mathbb{H}}$ and*

$$\pi^{\mu,\mathrm{e}} = \int_{[0,\infty)} \pi^{\xi,\mathrm{e}}\,\mathrm{d}\mu(\xi),$$

*where*

$$\mathrm{d}\mu(\xi) = \tfrac{1}{8\pi}\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi,$$

*and $\mathrm{d}\xi$ denotes the Lebesgue measure on $[0,\infty)$.*

The proof of the theorem above will be split into two parts. We start the proof of the isometric property on p. 456, and surjectivity will be established on p. 462. We start with some preparatory material for the isometric property.

We note that the left regular representation and right convolutions commute, and that $\widehat{f}^{\,\mathrm{hyp}}(\cdot,\xi)$ belongs to the irreducible space $\mathcal{H}_\xi^{\mathrm{even}}$ for any functions $f \in C_c(\mathbb{H})$ and $\xi \in \mathbb{R}$ (see Lemma 9.48 and Theorem 9.31). Together with the equivariance properties of the hyperbolic Fourier transform, we might expect for a spherical function $\psi \in C_c(\mathbb{H})$ that $\widehat{f*\psi}^{\,\mathrm{hyp}}(\cdot,\xi)$ to be a multiple of $\widehat{f}^{\,\mathrm{hyp}}(\cdot,\xi)$ by Schur's lemma (Theorem 1.29). We refer to Figure 9.9 for the geometric meaning of $f*\psi$ for spherical $\psi$.

**Lemma 9.60 (Right convolution by spherical functions).** *Assume that $f \in C_c(\mathbb{H})$ and $\psi \in C(\mathbb{H})$. Then $f*\psi$ is again right $K$-invariant and so can be considered a function in $C(\mathbb{H})$.*

**Exercise 9.61.** Show that if $\psi \in C_c(\mathbb{H})$ is spherical, then

$$\widehat{f*\psi}^{\,\mathrm{hyp}}(h,\xi) = \widehat{\psi}^{\,\mathrm{hyp}}(\xi)\widehat{f}^{\,\mathrm{hyp}}(h,\xi)$$

for all $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$.

Proof of Lemma 9.60. By definition of convolution and the identification of right $K$-invariant functions on $\mathrm{SL}_2(\mathbb{R})$ and functions on $\mathbb{H}$, we have

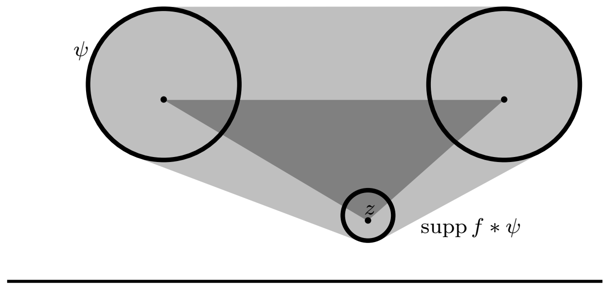


**Fig. 9.9:** Suppose $\psi$ is the normalized characteristic function of a ball around the point $\mathrm{i} \in \mathbb{H}$ (or a continuous approximation of it). The geometric meaning of the value $f * \psi(z)$ is, in this case, that the ball is moved to $z \in \mathbb{H}$ and $f$ is averaged over it. This creates a blurred-out version of $f$ with support in a neighbourhood (drawn in light grey) of the original support (drawn in dark grey).

$$f * \psi(g) = \int_G f(h \bullet \mathrm{i}) \psi(h^{-1} g \bullet \mathrm{i}) \, \mathrm{d}m(h)$$

for $g \in \mathrm{SL}_2(\mathbb{R})$. It follows that $f * \psi$ is also right $K$-invariant. This, together with Exercise 1.46, gives the first part of the lemma. We note that we may now also write

$$f * \psi(z) = \int_G f(h \bullet \mathrm{i}) \psi(h^{-1} \bullet z) \, \mathrm{d}m(h)$$

by using $z = g \bullet \mathrm{i} \in \mathbb{H}$ as the argument instead of $g \in \mathrm{SL}_2(\mathbb{R})$. □

**Lemma 9.62 (Symmetry on $C_c(\mathbb{H})$).** *Let $f \in C_c(\mathbb{H})$ and $\xi \in \mathbb{R}$. Then*

$$\int_K \left| \widehat{f}^{\mathrm{hyp}}(k, \xi) \right|^2 \mathrm{d}m_K(k) = \int_{\mathbb{H}} f * \phi_\xi \overline{f} \, \mathrm{dvol} = \int_K \left| \widehat{f}^{\mathrm{hyp}}(k, -\xi) \right|^2 \mathrm{d}m_K(k).$$

Proof. By Lemma 9.49 we have the convolution formula $\widehat{f}^{\mathrm{hyp}}(\cdot, \xi) = f * F_\xi$. Combining this with Fubini's theorem, we obtain for $\int_K |\widehat{f}^{\mathrm{hyp}}(k, \xi)|^2 \, \mathrm{d}m_K(k)$ the formula

$$
\begin{aligned}
\int_K\int_G f(h_1{\bullet}\mathrm{i})F_\xi(h_1^{-1}k)\,\mathrm{d}m(h_1)&\int_G \overline{f(h_2{\bullet}\mathrm{i})F_\xi(h_2^{-1}k)}\,\mathrm{d}m(h_2)\,\mathrm{d}m_K(k)\\
&=\int_G\int_G f(h_1{\bullet}\mathrm{i})\underbrace{\int_K F_\xi(h_1^{-1}k)\overline{F_\xi(h_2^{-1}k)}\,\mathrm{d}m_K(k)}_{=\langle\pi^\xi_{h_1}F_\xi,\pi^\xi_{h_2}F_\xi\rangle_{\mathcal{H}_\xi}}\,\mathrm{d}m(h_1)\overline{f(h_2{\bullet}\mathrm{i})}\,\mathrm{d}m(h_2)\\
&=\int_G\int_G f(h_1{\bullet}\mathrm{i})\phi_\xi(h_2^{-1}h_1)\,\mathrm{d}m(h_1)\overline{f(h_2{\bullet}\mathrm{i})}\,\mathrm{d}m(h_2)\\
&=\int_G\int_G f(h_1{\bullet}\mathrm{i})\phi_\xi(h_1^{-1}h_2)\,\mathrm{d}m(h_1)\overline{f(h_2{\bullet}\mathrm{i})}\,\mathrm{d}m(h_2)\\
&=\int_G f*\phi_\xi\overline{f}\,\mathrm{d}m,
\end{aligned}
$$

where we used the fact that

$$
\phi_\xi(g)=\overline{\phi_\xi(g^{-1})}=\phi_\xi(g^{-1})
$$

is real-valued (see Lemma 9.18). Since $\phi_\xi=\phi_{-\xi}$ by Lemma 9.53, the lemma follows. □

Proof of isometry formula in Theorem 9.59. Let $f\in C_c^\infty(\mathbb{H})$. Applying the hyperbolic Fourier inversion formula (Theorem 9.51) to $f$ and Fubini's theorem, we see that

$$
\begin{aligned}
\|f\|^2_{L^2(\mathbb{H})}&=\int_{\mathbb{H}} f(z)\overline{f(z)}\,\mathrm{dvol}(z)\\
&=\frac{1}{16\pi}\int_{\mathbb{H}}\int_{\mathbb{R}}\int_K\widehat{f}^{\mathrm{hyp}}(k,\xi)\chi_{\infty,\xi}(k^{-1}{\bullet}z)\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}m_K(k)\,\mathrm{d}\xi\overline{f(z)}\,\mathrm{dvol}(z)\\
&=\frac{1}{16\pi}\int_{\mathbb{R}}\int_K\widehat{f}^{\mathrm{hyp}}(k,\xi)\underbrace{\int_{\mathbb{H}}\chi_{\infty,\xi}(k^{-1}{\bullet}z)\overline{f(z)}\,\mathrm{dvol}(z)}_{=\overline{\widehat{f}^{\mathrm{hyp}}(k,\xi)}}\mathrm{d}m_K(k)\xi\tanh(\tfrac{\pi\xi}{2})\mathrm{d}\xi\\
&=\frac{1}{16\pi}\int_{\mathbb{R}}\|\widehat{f}^{\mathrm{hyp}}(\cdot,\xi)\|^2_{\mathcal{H}_\xi}\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi.
\end{aligned}
$$

Applying Lemma 9.62, we can also write this in the form

$$
\|f\|^2_{L^2(\mathbb{H})}=\frac{1}{8\pi}\int_{[0,\infty)}\int_K\left|\widehat{f}^{\mathrm{hyp}}(k,\xi)\right|^2\,\mathrm{d}m_K(k)\xi\tanh(\tfrac{\pi\xi}{2})\,\mathrm{d}\xi.
$$

It remains to show that this formula not only holds for all $f \in C_c^\infty(\mathbb{H})$ but also for $f \in C_c(\mathbb{H})$.

For this we let $(B_n)$ be a decreasing sequence of compact neighbourhoods of $I \in \mathrm{SL}_2(\mathbb{R})$ that form a basis of neighbourhoods of $I$. Using compactness of $K$ and continuity of conjugation, we may also suppose that $B_n$ is invariant under conjugation by all $k \in K$, meaning that

$$B_n = \{khk^{-1} \mid k \in K, h \in B_n\}.$$

Let $(\psi_n)$ be an approximate identity in $C_c^\infty(G)$ as in Proposition 1.43, with $\operatorname{supp}\psi_n \subseteq B_n$ for all $n \geqslant 1$. Replacing $\psi_n$ by the function

$$\mathrm{SL}_2(\mathbb{R}) \ni g \longmapsto \int_K \psi_n\bigl(\ell g \ell^{-1}\bigr)\,\mathrm{d}m_K(\ell)$$

if necessary, we may also suppose that

$$\psi_n\bigl(kgk^{-1}\bigr) = \psi_n(g) \tag{9.50}$$

for all $k \in K$, $g \in \mathrm{SL}_2(\mathbb{R})$, and $n \in \mathbb{N}$.

Now let $f \in C_c(\mathbb{H})$ and define

$$f_n = f * \psi_n \in C_c^\infty(G)$$

for $n \in \mathbb{N}$. Note that by continuity of $f$ we have

$$\begin{gathered} f_n \to f \text{ as } n \to \infty; \\ \operatorname{supp} f_n \subseteq (\operatorname{supp} f)B_1 \text{ for } n \in \mathbb{N}. \end{gathered} \tag{9.51}$$

We claim that $f_n$ is again right $K$-invariant, and so can be thought of as a function on $\mathbb{H}$. To see this, let $g \in \mathrm{SL}_2(\mathbb{R})$, let $k \in K$, and combine (9.50) with the substitution $\widetilde{h} = hk^{-1}$ to see that

$$\begin{aligned} f_n(gk) = f * \psi_n(gk) &= \int_G f(h) \underbrace{\psi_n(h^{-1}gk)}_{=\psi_n(kh^{-1}g)} \,\mathrm{d}m(h) \\ &= \int_G \underbrace{f(\widetilde{h}k)}_{=f(\widetilde{h})} \psi_n(\widetilde{h}^{-1}g)\,\mathrm{d}m(\widetilde{h}) \\ &= f * \psi(g) = f_n(g). \end{aligned}$$

For the smooth functions $f_n \in C_c^\infty(\mathbb{H})$, we already established the isometry formula. Moreover, (9.51) shows that $f_n \to f$ as $n \to \infty$ in $L^2(\mathbb{H})$ by dominated convergence. Together, these show that $\widehat{f_n}^{\mathrm{hyp}}|_{K\times\mathbb{R}}$, considered as an element of $\mathcal{H}_\mu^{\mathrm{even}} \cong L^2(K \times [0,\infty), \mu)$ forms a Cauchy sequence, which will have an $L^2$ limit $F$ with

$$\begin{aligned}\|F\|_{L^2(K\times\mathbb{R},\mu)} &= \lim_{n\to\infty}\|\widehat{f_n}^{\mathrm{hyp}}|_{K\times[0,\infty)}\|_{L^2(K\times[0,\infty),\mu)}\\ &= \lim_{n\to\infty}\|f_n\|_{L^2(\mathbb{H},\mathrm{vol})} = \|f\|_{L^2(\mathbb{H},\mathrm{vol})}.\end{aligned}$$

This implies that along a subsequence $\widehat{f_n}^{\mathrm{hyp}}|_{K\times\mathbb{R}}$ converges to $F$, which, together with (**??**), implies that $F=\widehat{f}^{\mathrm{hyp}}$ and hence the isometry formula

$$\|\widehat{f}^{\mathrm{hyp}}|_{K\times[0,\infty)}\|_{L^2(K\times[0,\infty),\mu)} = \|f\|_{L^2(\mathbb{H},\mathrm{vol})}.$$

□

**Exercise 9.63.** Show that $f\in C_c(\mathbb{H})$ implies that $f*m_K=f$.

It follows from the now established isometry formula for the $L^2$-norms of $f\in C_c(\mathbb{H})$ and $\widehat{f}^{\mathrm{hyp}}|_{K\times[0,\infty)}$ and Lemma 9.48 that the hyperbolic Fourier transform can be extended uniquely to an equivariant isometry from $L^2(\mathbb{H})$ into $\mathcal{H}_\mu^{\mathrm{even}}$. Moreover, the image $\mathcal{V}$ is then a closed $\pi^\mu$-invariant subspace of $\mathcal{H}_\mu^{\mathrm{even}}$. This brings up the question of whether such subspaces can be classified. We answer this in a slightly more general case in the following proposition.

**Proposition 9.64 (Invariant subspaces).** *Let $\mu$ be a $\sigma$-finite measure on $[0,\infty)$ and define $\mathcal{H}_\mu^{\mathrm{even}}$ as in Definition 9.38. Then, for any closed $\pi^{\mu,\mathrm{e}}$-invariant subspace $\mathcal{V}\subseteq\mathcal{H}_\mu^{\mathrm{even}}$, there exists a measurable set $S_\mathcal{V}\subseteq[0,\infty)$, that may be thought of as the 'support' of the subspace, so that*

$$\mathcal{V}=\bigl\{F\in\mathcal{H}_\mu^{\mathrm{even}}\mid F(\cdot,\xi)=0 \text{ for } \mu\text{-almost every } \xi\in[0,\infty)\smallsetminus S_\mathcal{V}\bigr\}. \quad (9.52)$$

**Exercise 9.65.** Let $\mu$ be as above, and let $\mathcal{V}\subseteq\mathcal{H}_\mu^{\mathrm{even}}$ be a closed subspace. Show that a measurable subset $S_\mathcal{V}\subseteq[0,\infty)$ satisfying (9.52) is uniquely determined up to a null set by this property, assuming it exists.

Before starting the formal proof, we outline the structure of the argument. For simplicity, we write $\pi=\pi^{\mu,\mathrm{e}}$ and $\mathcal{H}_\pi=\mathcal{H}_\mu^{\mathrm{even}}$.

We will show, in turn, the following statements.

(a) The operator $T_\pi=\pi_\partial(\Omega)$ from Corollary 9.12 is given by

$$T_\pi=M_{-\mathrm{id}^2} \quad (9.53)$$

where $M_{-\mathrm{id}^2}$ is the multiplication operator defined by

$$M_{-\mathrm{id}^2}(F)(h,\xi)=-\xi^2F(h,\xi)$$

for all $(h,\xi)\in\mathrm{SL}_2(\mathbb{R})\times[0,\infty)$ and $F$ in the domain

$$D_{M_{-\mathrm{id}^2}}=\Bigl\{F\in\mathcal{H}_\pi\mid\int_0^\infty\int_K|\xi^2F(k,\xi)|^2\,\mathrm{d}m_K(k)\,\mathrm{d}\mu(\xi)<\infty\Bigr\}.$$

(b) We define

$$f_0\colon [0,\infty) \ni \xi \longmapsto (1+\xi^2)^{-1}$$

and obtain from the above that $M_{f_0} = (I - T_\pi)^{-1}$. We claim that this multiplication operator $M_{f_0}$ commutes with the centralizer $C(\pi)$ of $\pi$.

(c) Using the functional calculus of $M_{f_0}$, we will prove that all multiplication operators $M_f$ for $f \in L^\infty_\mu([0,\infty))$ commute with $C(\pi)$.

(d) Applying this to the orthogonal projection operator $P_{\mathcal{V}}$ of a $\pi$-invariant subspace $\mathcal{V} \subseteq \mathcal{H}_\pi$, we then obtain that $\mathcal{V}$ has $L^\infty_\mu([0,\infty))\mathcal{V} \subseteq \mathcal{V}$, which will allow us to conclude the proof.

We now discuss these four steps in detail.

Proof that $T_\pi = \pi_\partial(\Omega) = M_{\mathrm{id}^2}$ as claimed in (a). Let $T > 0$ and assume that $f$ is a function in $L^2_\mu([0,\infty))$ with $f(\xi) = 0$ for $\mu$-almost every $\xi > T$. For an integer $n \in 2\mathbb{Z}$ we now define

$$F_n(h,\xi) = f(\xi) F_{\xi,n}(h)$$

for $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times [0,\infty)$, where $F_{\xi,n} \in \mathcal{H}^{\mathrm{even}}_\xi$ is as defined in Lemma 9.33. By (9.20), we have that $F_{\xi,n}(h)$ depends smoothly on $h \in \mathrm{SL}_2(\mathbb{R})$ for any value of $\xi \in [0,\infty)$. Moreover, for a fixed $n$ and for $\xi \in [0,T]$ the derivatives are uniformly bounded. Applying the mean value theorem, dominated convergence, and (9.21), it follows that

$$\pi^\xi_\partial(\mathbf{a}) F_n(h,\xi) = \frac{n+1+\mathrm{i}\xi}{2} F_{n+2}(h,\xi) + \frac{-n+1+\mathrm{i}\xi}{2} F_{n-2}(h,\xi). \tag{9.54}$$

As in the proof of Lemma 9.33, we can now use Proposition 9.13 to conclude that the first summand is equal to $\pi_\partial(\mathbf{r}^+)F_n$ and second is equal to $\pi_\partial(\mathbf{r}^-)F_n$. Using the fact that $\mathbf{k}, \mathbf{r}^+, \mathbf{r}^-$ span $\mathfrak{sl}_2(\mathbb{C})$, it follows that $F_n$ is a smooth vector for $\pi$, and using the formula for $\Omega$ in (9.13), we also obtain

$$\pi_\partial(\Omega) F_n(h,\xi) = -\xi^2 F_n(h,\xi)$$

for $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times [0,\infty)$. Equivalently, the closed self-adjoint operator $T_\pi$ satisfies (9.53) for the function $F_n$ as above.

We now extend (9.53) to other functions $F \in \mathcal{H}_\pi$. To begin with, we may vary $n \in 2\mathbb{Z}$ over a finite set (using different functions $f_n \in L^2_\mu([0,\infty))$ with support in $[0,T]$). Note that $M_{-\mathrm{id}^2}$ is bounded on

$$\mathcal{H}_{\pi,\leqslant T} = \bigl\{F \in \mathcal{H}_\pi \mid F(\cdot,\xi) = 0 \text{ for } \mu\text{-almost every } \xi \in (T,\infty)\bigr\},$$

and these finite sums are precisely the $K$-finite vectors in $\mathcal{H}_{\pi,\leqslant T}$. Hence, it follows by continuity of $M_{-\mathrm{id}^2}$ and closedness of $T_\pi$ that (9.53) also holds for all $F \in \mathcal{H}_{\pi,\leqslant T}$.

Next let $F \in D_{M_{-\mathrm{id}^2}}$ and define

$$F_{\leqslant T} = \mathbb{1}_{[0,T]} F \in \mathcal{H}_{\pi,\leqslant T}$$

so that

$$T_\pi(F_{\leqslant T}) = M_{-\mathrm{id}^2}(F_{\leqslant T}).$$

For $T \to \infty$ we have $F_{\leqslant T} \to F$ and $M_{-\mathrm{id}^2}(F_{\leqslant T}) \to M_{-\mathrm{id}^2}(F)$ by dominated convergence. Since $T_\pi$ is a closed operator, we see that (9.53) holds for all functions $F \in D_{M_{-\mathrm{id}^2}}$. Equivalently, we have $M_{-\mathrm{id}^2} \subseteq T_\pi$. It is easy to see that $M_{-\mathrm{id}^2}$ is a self-adjoint (and, depending on $\operatorname{supp} \mu$, possibly unbounded) operator. By Corollary 9.12 the same is true for $T_\pi$. For two self-adjoint operators, the inclusion $M_{-\mathrm{id}^2} \subseteq T_\pi$ actually implies equality of the operators. Indeed, for $v \in D_{M_{-\mathrm{id}^2}}$ and $w \in D_{T_\pi}$, we have

$$\langle M_{-\mathrm{id}^2} v, w\rangle = \langle T_\pi v, w\rangle = \langle v, T_\pi w\rangle.$$

This shows that

$$D_{M_{-\mathrm{id}^2}} \ni v \longmapsto \langle M_{-\mathrm{id}^2} v, w\rangle$$

is a bounded linear function, which implies that $w$ belongs to the domain of $M^*_{-\mathrm{id}^2} = M_{-\mathrm{id}^2}$, and $M_{-\mathrm{id}^2} w = T_\pi w$.

To summarize, we have shown that $T_\pi = M_{-\mathrm{id}^2}$ as claimed in (a). □

Proof that $M_{f_0}$ commutes with $C(\pi)$ as claimed in (b). We define $f_0(\xi) = (1+\xi^2)^{-1}$ for $\xi \in [0,\infty)$, and first show that (a) implies that

$$M_{f_0} = (I - T_\pi)^{-1}.$$

Indeed,

$$I - T_\pi = M_{(1+\mathrm{id}^2)}$$

is injective on its domain, maps onto $\mathcal{H}_\pi$, and has $M_{f_0}$ as its (bounded) inverse operator. We now show that $M_{f_0}$ commutes with every equivariant bounded operator $B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$. For this, we first note that $B$ maps every smooth vector $v$ to a smooth vector $Bv$, and we have

$$\pi_\partial(\Omega) Bv = B\pi_\partial(\Omega)v.$$

If now $v \in D_{T_\pi}$, then there exists a sequence of smooth vectors $(v_n)$ with $v_n \to v$ and $\pi_\partial(\Omega)v_n \to T_\pi v$ as $n \to \infty$. However, this implies $Bv_n \to Bv$ and $\pi_\partial(\Omega)Bv_n \to BT_\pi v$ as $n \to \infty$, and hence also $T_\pi B \supseteq BT_\pi$. Using $M_{f_0} = (I - T_\pi)^{-1}$, we now obtain the claim. Indeed, let $v \in \mathcal{H}_\pi$ and $(I - T_\pi)^{-1} v = w$ so that $v = (I - T_\pi)w$. Then $Bv = (I - T_\pi)Bw$, which implies that

$$(I - T_\pi)^{-1}Bv = Bw = B(I - T_\pi)^{-1}v$$

as claimed in (b). □

Proof that $M_f$ commutes with $C(\pi)$ as claimed in (c). Since

$$(I - T_\pi)^{-1} = M_{f_0}$$

is already realized as a multiplication operator and $f_0$ is injective, the following are now relatively easy claims to prove. The measurable functional calculus for $M_{f_0}$ (see [24, Sec. 12.6]) gives rise to other multiplication operators. By the injectivity of $f_0$, every multiplication operator $M_f$ for $f \in L^\infty_\mu([0,\infty))$ can be obtained from the measurable functional calculus of $M_{f_0}$. Together with the previous claim and the properties of the measurable functional calculus (see [24, Prop. 12.68]), this implies that every equivariant bounded operator $B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ commutes with $M_f$ for $f \in L^\infty_\mu([0,\infty))$. □

Conclusion of the proof of Proposition 9.64, as outlined in (d). Now let $\mathcal{V} \subseteq \mathcal{H}_\pi$ be a closed $\pi$-invariant subspace, and let $B = P_{\mathcal{V}}$ be the orthogonal projection onto $\mathcal{V}$. By invariance of $\mathcal{V}$, the projection is equivariant and, by (c) above, we have $M_f\mathcal{V} \subseteq \mathcal{V}$ for all $f \in L^\infty_\mu([0,\infty))$.

Now let $F \in \mathcal{V}$ and define

$$S_F = \{\xi \in [0,\infty) \mid F(\cdot,\xi) \neq 0\}.$$

Using invariance of $\mathcal{V}$ under $K$, we can split $F$ into a sum of $K$-eigenfunctions as

$$F = \sum_{n\in 2\mathbb{Z}} F_n$$

satisfying $F_n \in \mathcal{V}_n$ for all $n \in 2\mathbb{Z}$, and

$$S_F = \bigcup_{n\in 2\mathbb{Z}} S_{F_n}.$$

Since $\mathcal{H}_\xi$ contains (up to scalars) only one $K$-eigenfunction of weight $n \in 2\mathbb{Z}$ (namely $F_{\xi,n}$ in (9.20)) there exists some $f_n \in L^2_\mu([0,\infty))$ such that

$$F_n(h,\xi) = f_n(\xi)F_{\xi,n}(h)$$

for $(h,\xi) \in \mathrm{SL}_2(\mathbb{R}) \times [0,\infty)$. In particular,

$$S_{F_n} = \{\xi \in [0,\infty) \mid f_n(\xi) \neq 0\}.$$

Using $M_f\mathcal{V} \subseteq \mathcal{V}$ for all $f \in L^\infty_\mu([0,\infty))$, the fact that $F_n \in \mathcal{V}$, and dominated convergence, it follows that the function

$$\mathrm{SL}_2(\mathbb{R}) \times [0,\infty) \ni (h,\xi) \longmapsto f(\xi)F_{\xi,n}(h)$$

belongs to $\mathcal{V}$ for any $f \in L^2_\mu([0,\infty))$ with $\{\xi \mid f(\xi) \neq 0\} \subseteq S_{F_n}$. For $T > 0$ we define

$$S_{n,T} = S_{F_n} \cap [0,T].$$

By (9.54) and the argument directly following it, we see that $\pi_\partial(\mathbf{r}^\pm)$ are bounded operators on

$$\{F \in \mathcal{H}_{\pi,\leqslant T} \mid F \text{ has } K\text{-weight } n\}.$$

Since $\mathcal{V}$ is invariant under $\pi_\partial(\mathbf{r}^\pm)$ (where it is defined) it follows that the function

$$\mathrm{SL}_2(\mathbb{R}) \times [0,\infty) \ni (h,\xi) \longmapsto f(\xi)F_{\xi,m}(h)$$

belongs to $\mathcal{V}$ for any $f \in L^2_\mu([0,\infty))$ with $\{\xi \mid f(\xi) \neq 0\} \subseteq S_{F_n}$, where $m,n$ lie in $2\mathbb{Z}$ are arbitrary. Varying $m,n \in 2\mathbb{Z}$ and the functions $f \in L^2_\mu([0,\infty))$, we can write any $F \in \mathcal{H}_\pi$ with $\{\xi \mid F(\cdot,\xi) \neq 0\} \subseteq S_F$ as a convergent sum of elements of $\mathcal{V}$ and obtain $F \in \mathcal{V}$.

Since $\mathcal{V}$ is separable, we can find a dense set $\{F_{(k)} \mid k \in \mathbb{N}\}$ of vectors, apply the above argument to each $F_{(k)}$ and obtain the same statement for

$$B = \bigcup_{k\in\mathbb{N}} B_{F_{(k)}} = \{\xi \in [0,\infty) \mid \text{there exists a } k \in \mathbb{N} \text{ with } F_{(k)}(\cdot,\xi) \neq 0\}.$$

This proves the proposition. □

**Exercise 9.66.** Let $\mu$ be a $\sigma$-finite measure on $[0,\infty)$ as in Proposition 9.64. Show that the centralizer of $\pi^{\mu,\mathrm{e}}$ is given by all multiplication operators $M_f$ with $f \in L^\infty_\mu([0,\infty))$.

The following finishes our discussions of the hyperbolic Fourier transform.

Concluding the proof of Theorem 9.59. Let $\mu$ be the measure on $[0,\infty)$ defined by $\frac{1}{8\pi}\xi\tanh\xi\,\mathrm{d}\xi$. By the first part of the proof on p. 456, we know that

$$C_c(\mathbb{H}) \ni f \longmapsto \widehat{f}^{\,\mathrm{hyp}} \in \mathcal{H}^{\mathrm{even}}_\mu$$

is an equivariant isometry between the Koopman representation $\pi^{\mathbb{H}}$ and the integral $\pi^{\mu,\mathrm{e}}$ of the even principal series representation. Hence it can be extended to an equivariant isometry from $L^2(\mathbb{H})$ to a closed $\pi^{\mu,\mathrm{e}}$-invariant subspace $\mathcal{V} \subseteq \mathcal{H}^{\mathrm{even}}_\mu$. By Proposition 9.64, this subspace can be defined by a measurable subset $S_{\mathcal{V}} \subseteq [0,\infty)$ and the formula (9.52). We show that $S_{\mathcal{V}} = [0,\infty)$ (up to null sets) by finding a sequence $(f_n)$ in $C_c(\mathbb{H})$, so that for every $\xi \in [0,\infty)$ there exists some $n \in \mathbb{N}$ with $\widehat{f_n}^{\,\mathrm{hyp}}(\cdot,\xi) \neq 0$.

In fact we let $(f_n)$ be a sequence of spherical functions in $C_c(\mathbb{H})$ with

$$\int_{\mathbb{H}} f_n \,\mathrm{dvol} = 1$$

for all $n \in \mathbb{N}$, so that $\operatorname{supp} f_n$ is a shrinking neighbourhood of $\mathrm{i} \in \mathbb{H}$. For every $\xi \in [0,\infty)$ it now follows that

$$\widehat{f_n}^{\,\mathrm{hyp}}(k,\xi) = \int_{\mathbb{H}} f_n(z)\overline{\chi_{\infty,\xi}(z)}\,\mathrm{dvol}(z) \longrightarrow 1$$

as $n \to \infty$ and for all $k \in K$ by $K$-invariance of $f_n$ and continuity of the function $z \mapsto \chi_{\infty,\xi}(z) = \Im(z)^{\frac{1}{2}-\frac{\mathrm{i}}{2}\xi}$.

Hence we have $S_{\mathcal{V}} = [0, \infty)$ (up to null sets), which gives $\mathcal{V} = \mathcal{H}_\mu^{\text{even}}$ by (9.52), and Theorem 9.59 follows. □

## 9.5 The Complementary Series Representation

We recall from Section 9.3 that for $\xi \in \mathbb{R}$ the principal series representation $\pi^{\xi,\mathrm{e}}$ is constructed from the unitary character $\chi_\xi$ defined by

$$\chi_\xi(a_t u_x) = \mathrm{e}^{\mathrm{i}\xi t}$$

for $a_t u_x \in B$. Moreover, $\pi^{\xi,\mathrm{e}}$ then turned out to be an irreducible unitary representation with Casimir eigenvalue $\alpha_{\pi^{\xi,\mathrm{e}}} = -\xi^2$. According to the proof of Theorem 9.22 in Section 9.2.5, there could (and, according to the statement of the theorem, there should) be another type $\gamma^s$ of even irreducible unitary representation with Casimir eigenvalues $\alpha_{\gamma^s} = s^2$ for $s \in (0,1)$ that we have not yet seen. To construct $\gamma^s$ we try to mimic the construction of $\pi^{\xi,\mathrm{e}}$ in Definition 9.30 while attempting to 'replace $\mathrm{i}\xi$ by $s$'.

**Definition 9.67 (Complementary series representation).** For $s \in (0,1)$ we define the non-unitary character $\chi_{(s)}$ on $B = \{a_t u_x \mid t, x \in \mathbb{R}\}$ by

$$\chi_{(s)}(a_t u_x) = \mathrm{e}^{st}$$

for $a_t u_x \in B$. The *complementary series representation* $\gamma^s$ of $G = \mathrm{SL}_2(\mathbb{R})$ is initially defined as the left-regular representation on the space $\mathcal{V}_s$ of those functions $f\colon G \to \mathbb{C}$ with the following properties:

(1) $f$ is smooth, and
(2) $f$ is even, and $f(gb) = \chi_{(s)}(b)^{-1} \Delta_B(b)^{\frac{1}{2}} f(g)$ for all $g \in G$ and $b \in B$.

The main difference between this and the construction of the principal series representation is, of course, that we are using here non-unitary characters, which means that the $L^2$-norm on $K$ will not be preserved under the left-regular representation (as was the case for the principal series representation). Instead, we will have to define a new norm and inner product on $\mathcal{V}_s$.

**Theorem 9.68 (Complementary series representations).** *Let $s \in (0,1)$. The regular representation on the completion $\mathcal{H}_{(s)}$ of $\mathcal{V}_s$ defines a non-tempered irreducible unitary representation $\gamma^s$ with Casimir eigenvalue $s^2$, called the* complementary series representation.

### 9.5.1 The Space and its Inner Product

In the following, let $M = \{\pm I\}$ be the centre of $\mathrm{SL}_2(\mathbb{R})$. We will always identify functions on $K/M$ with even functions on $K$.

**Lemma 9.69 (Smooth functions on $K/M$).** *Let $s \in (0,1)$. Then $\mathcal{V}_s$ is isomorphic to $C^\infty(K/M)$. In fact every element $f \in \mathcal{V}_s$ is uniquely determined by its restriction $f|_K \in C^\infty(K/M)$ and every even smooth function $f_K$ on $K$ can be extended via*

$$f(ka_t u_x) = f_K(k)\mathrm{e}^{-(s+1)t} \tag{9.55}$$

*for $ka_t u_x \in KAU$ to an element of $\mathcal{V}_s$. In particular, for every $n \in 2\mathbb{Z}$ the function $F_{s,n}$ defined by*

$$F_{s,n}(k_\psi a_t u_x) = \mathrm{e}^{-\mathrm{i}n\psi-(s+1)t} \tag{9.56}$$

*for $k_\psi a_t u_x \in KAU$ belongs to $\mathcal{V}_s$ and is a $K$-eigenfunction with $K$-weight $n$.*

Proof. We note that Definition 9.67(2) implies that any function $f \in \mathcal{V}_s$ is uniquely determined by $f|_K$. Clearly the map

$$\varPhi\colon K \times A \times U \ni (k_\psi, a_t, u_x) \longmapsto g = k_\psi a_t u_x \in \mathrm{SL}_2(\mathbb{R})$$

is smooth, and hence $f|_K \in C^\infty(K/M)$ for all $f \in \mathcal{V}_s$. The inverse of $\varPhi$ is also smooth, since it maps $g \in \mathrm{SL}_2(\mathbb{R})$ first to the polar coordinates $(\psi, r)$ of $ge_1$, and then to $k_\psi \in K$, $a_{\log r} \in A$, and $a_{\log r}^{-1} k_\psi^{-1} g = u_x \in U$. This shows that (9.55) defines a smooth function on $G$ for any $f_K \in C^\infty(K/M)$. Moreover, $g = ka_t u_x \in KAU$ and $b = a_{t_0} u_{x_0} \in B = AU$ implies that

$$\begin{aligned} f(gb) &= f(ka_{t+t_0} u_{\mathrm{e}^{-2t_0}x+x_0}) \\ &= f_K(k)\mathrm{e}^{-(s+1)(t+t_0)} \\ &= f_K(k)\mathrm{e}^{-(s+1)t}\mathrm{e}^{-st_0}\mathrm{e}^{-t_0} \\ &= f(ka_t u_x)\chi_{(s)}(b)^{-1}\varDelta_B(b)^{\frac{1}{2}}, \end{aligned}$$

which shows that $f \in \mathcal{V}_s$, as claimed in the lemma.

Applying the above to the character $\chi_{-n}$ on $K$ for some $n \in 2\mathbb{Z}$ defines the function $F_{s,n} \in \mathcal{V}_s$ in (9.56). By definition,

$$\big(\gamma^s_{k_\theta}(F_{s,n})\big)(k_\psi) = F_{s,n}\big(k_\theta^{-1}k_\psi\big) = \mathrm{e}^{-\mathrm{i}n(\psi-\theta)} = \mathrm{e}^{\mathrm{i}n\theta}F_{s,n}(k_\psi)$$

for all $k_\theta, k_\psi \in K$. By the first part of the proof, this shows that $F_{s,n}$ has $K$-weight $n$ for $\gamma^s$. □

**Definition 9.70 (The inner product on $\mathcal{V}_s$).** For $s \in (0,1)$ we define

$$\langle f_1, f_2\rangle_{\mathcal{V}_s} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{f_1(k_{\theta_1})\overline{f_2(k_{\theta_2})}}{|\sin(\theta_1-\theta_2)|^{1-s}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \tag{9.57}$$

for $f_1, f_2 \in \mathcal{V}_s$.

For now, (9.57) simply falls from the sky. We will, however, give additional meaning to it after we have established the fundamental properties of this inner product.

**Lemma 9.71 (The inner product on $\mathcal{V}_s$).** *For any $s \in (0,1)$, the form* (9.57) *defines an inner product on $\mathcal{V}_s$.*

We note that the proof of this lemma will require both $s > 0$ and $1-s > 0$.

Proof of Lemma 9.71. We note that $\mathbb{R} \ni \theta \mapsto |\sin\theta|$ has period $\pi$. This gives us

$$\frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{|\sin(\theta_1-\theta_2)|^{1-s}} = \frac{1}{\pi^2}\int\limits_{\mathbb{R}/\mathbb{Z}\pi}\int\limits_{\mathbb{R}/\mathbb{Z}\pi} \frac{\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{|\sin(\underbrace{\theta_1-\theta_2}_{=\theta})|^{1-s}}$$

$$= \frac{1}{\pi}\int\limits_{\mathbb{R}/\mathbb{Z}\pi} \frac{\mathrm{d}\theta}{|\sin\theta|^{1-s}} < \infty$$

since $|\sin\theta| \asymp |\theta|$ as $\theta \to 0$ and $s > 0$ implies that $\int_0^1 \frac{\mathrm{d}\theta}{\theta^{1-s}} < \infty$. Therefore, the function $(\theta_1,\theta_2) \mapsto \frac{1}{|\sin(\theta_1-\theta_2)|^{1-s}}$ lies in $L^1([0,\pi]^2)$, and the integral in (9.57) converges for all $f_1, f_2 \in \mathcal{V}_s$.

Sesqui-linearity of $\langle\cdot,\cdot\rangle_{\mathcal{V}_s}$ follows directly from the definition in (9.57). Thus it remains to show that $\langle f, f\rangle_{\mathcal{V}_s} > 0$ for all $f \in \mathcal{V}_s \smallsetminus \{0\}$. The Fourier expansion of $f|_K$ allows us to write

$$f = \sum_{n\in 2\mathbb{Z}} c_n F_{s,n}$$

for some sequence of coefficients $(c_n) \in \ell^1(2\mathbb{Z})$. Using sesqui-linearity, we obtain

$$\langle f, f\rangle_{\mathcal{V}_s} = \sum_{m,n\in 2\mathbb{Z}} c_m\overline{c_n}\langle F_{s,m}, F_{s,n}\rangle_{\mathcal{V}_s}.$$

We claim that $m, n \in 2\mathbb{Z}$ with $m \neq n$ implies that $\langle F_{s,m}, F_{s,n}\rangle_{\mathcal{V}_s} = 0$ and that $\langle F_{s,n}, F_{s,n}\rangle_{\mathcal{V}_s} > 0$ for all $n \in 2\mathbb{Z}$. Together, these show that $\langle f, f\rangle_{\mathcal{V}_s} > 0$ for all $f \in \mathcal{V}_s \smallsetminus \{0\}$.

Suppose first that $m, n \in 2\mathbb{Z}$ with $m \neq n$. Then

$$
\begin{aligned}
\langle F_{s,m}, F_{s,n}\rangle_{\mathcal{V}_s} &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{\mathrm{e}^{-\mathrm{i}m\theta_1+\mathrm{i}n\theta_2}}{|\sin(\theta_1-\theta_2)|^{1-s}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\
&= \frac{1}{\pi^2}\int\limits_{\mathbb{R}/\mathbb{Z}\pi}\ \int\limits_{\mathbb{R}/\mathbb{Z}\pi} \frac{\mathrm{e}^{\mathrm{i}(n(\theta_2'+\psi)-m(\theta_1'+\psi))}}{|\sin(\theta_1'-\theta_2')|^{1-s}}\,\mathrm{d}\theta_1'\,\mathrm{d}\theta_2' \\
&= \mathrm{e}^{\mathrm{i}(n-m)\psi}\langle F_{s,m}, F_{s,n}\rangle_{\mathcal{V}_s}
\end{aligned}
$$

by using the substitutions $\theta_1 = \theta_1' + \psi$ and $\theta_2 = \theta_2' + \psi$ for some $\psi \in \mathbb{R}$. This implies that $\langle F_{s,m}, F_{s,n}\rangle_{\mathcal{V}_s} = 0$ for $m \neq n$.

For $n \in 2\mathbb{Z}$, we define

$$
\begin{aligned}
I_n = \pi\langle F_{s,n}, F_{s,n}\rangle_{\mathcal{V}_s} &= \frac{1}{\pi}\int\limits_{\mathbb{R}/\mathbb{Z}\pi}\ \int\limits_{\mathbb{R}/\mathbb{Z}\pi} \frac{\mathrm{e}^{\mathrm{i}n(\theta_2-\theta_1)}}{|\sin(\theta_1-\theta_2)|^{1-s}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\
&= \int\limits_{\mathbb{R}/\mathbb{Z}\pi} \frac{\mathrm{e}^{\mathrm{i}n\theta}}{|\sin\theta|^{1-s}}\,\mathrm{d}\theta.
\end{aligned}
$$

Note that

$$
\overline{I_n} = \int_{\mathbb{R}/\mathbb{Z}\pi} \frac{\mathrm{e}^{-\mathrm{i}n\theta}}{|\sin\theta|^{1-s}}\,\mathrm{d}\theta = I_n
$$

via the substitution $\theta' = -\theta$. Thus

$$
I_n = \int_0^\pi \frac{\cos(n\theta)}{(\sin\theta)^{1-s}}\,\mathrm{d}\theta,
$$

and so we wish to show that $I_n > 0$ for all $n \in 2\mathbb{Z}$. Since

$$
I_0 = \int_0^\pi \frac{1}{(\sin\theta)^{1-s}}\,\mathrm{d}\theta > 0
$$

and $I_{-n} = I_n$ for all $n \in 2\mathbb{Z}$, it remains to show that $I_n > 0$ for all $n \in 2\mathbb{N}$. For $I_2$ we have, using integration by parts,

$$
\begin{aligned}
I_2 &= \int_0^\pi \cos 2\theta(\sin\theta)^{s-1}\,\mathrm{d}\theta \\
&= \left[\frac{\sin 2\theta}{2}(\sin\theta)^{s-1}\right]_0^\pi - \int_0^\pi \frac{\sin 2\theta}{2}(s-1)(\sin\theta)^{s-2}\cos\theta\,\mathrm{d}\theta \\
&= \underbrace{[\cos\theta(\sin\theta)^s]_0^\pi}_{=0} + (1-s)\int_0^\pi \cos^2\theta(\sin\theta)^{s-1}\,\mathrm{d}\theta \\
&= (1-s)\int_0^\pi \frac{1+\cos 2\theta}{2}(\sin\theta)^{s-1}\,\mathrm{d}\theta \\
&= \frac{1-s}{2}\left(I_0 + I_2\right),
\end{aligned}
$$

where the boundary terms vanish since $s > 0$. We now solve this equation for $I_2$, to obtain

$$I_2 = \left(\frac{1-s}{1+s}\right) I_0 > 0$$

since $s < 1$.

For a general $n \in 2\mathbb{N}$, we again use integration by parts to see that

$$\begin{aligned} I_n &= \int_0^\pi \cos(n\theta)(\sin\theta)^{s-1}\,\mathrm{d}\theta \\ &= \underbrace{\left[\frac{\sin(n\theta)}{n}(\sin\theta)^{s-1}\right]_0^\pi}_{=0} - \int_0^\pi \frac{\sin n\theta}{n}(s-1)(\sin\theta)^{s-2}\cos\theta\,\mathrm{d}\theta \\ &= \frac{(1-s)}{n}\int_0^\pi \frac{\sin(n\theta)\cos\theta}{\sin\theta}(\sin\theta)^{s-1}\,\mathrm{d}\theta. \end{aligned}$$

We again wish to relate $I_n$ to earlier values of the sequence, and hence use the fact that $n \in 2\mathbb{N}$ to calculate that

$$\begin{aligned} \frac{\sin(n\theta)\cos\theta}{\sin\theta} &= \frac{1}{2}\frac{(\mathrm{e}^{in\theta} - \mathrm{e}^{-in\theta})(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})}{(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta})} \\ &= \frac{1}{2}\Big(\mathrm{e}^{\mathrm{i}(n-1)\theta} + \mathrm{e}^{\mathrm{i}(n-3)\theta} + \cdots + \mathrm{e}^{-\mathrm{i}(n-3)\theta} + \mathrm{e}^{-\mathrm{i}(n-1)\theta}\Big)(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) \\ &= \frac{1}{2}\left(\mathrm{e}^{in\theta} + 2\mathrm{e}^{\mathrm{i}(n-2)\theta} + \cdots + 2\mathrm{e}^{-\mathrm{i}(n-2)\theta} + \mathrm{e}^{-\mathrm{i}n\theta}\right) \\ &= \cos(n\theta) + 2\cos\left((n-2)\theta\right) + \cdots + 2\cos(2\theta) + 1. \end{aligned}$$

Putting this into the above formula for $I_n$ gives

$$I_n = \frac{1-s}{n}\left(I_n + 2I_{n-2} + \cdots + 2I_2 + I_0\right),$$

which may be solved for $I_n$ to give the recursion formula

$$I_n = \frac{1-s}{n-1+s}\left(2I_{n-2} + \cdots + 2I_2 + I_0\right).$$

Using the fact that $s < 1$, this implies once more that $I_n > 0$ by induction on $n \in 2\mathbb{N}$. □

**Definition 9.72 (The space $\mathcal{H}_{(s)}$).** Let $s \in (0,1)$. We define $\mathcal{H}_{(s)}$ to be the completion of $\mathcal{V}_s$ with respect to the norm induced by $\langle\cdot,\cdot\rangle_{\mathcal{V}_s}$.

### 9.5.2 Unitarity of the Complementary Series

Proof of unitarity in Theorem 9.68. Recall from Example 1.6 that the group $\mathrm{SL}_2(\mathbb{R})$ acts on $v \in \mathbb{S}^1$ via $g \cdot v = \frac{1}{\|gv\|} gv$ for $g \in \mathrm{SL}_2(\mathbb{R})$, and that the Radon–Nikodym derivative for the normalized length measure $m$ on $\mathbb{S}^1$ satisfies

$$\frac{\mathrm{d}g_* m}{\mathrm{d}m}(v) = \|g^{-1}v\|^{-2}. \tag{9.58}$$

Also recall that we used this set-up on p. 421 to discuss the principal series representation $\pi^\xi$ on $\mathcal{H}_\xi \cong L^2(K) \cong L^2_m(\mathbb{S}^1)$.

For $u_1, u_2 \in \mathbb{R}^2 \smallsetminus \{(0,0)\}$ we define the function

$$\mathcal{D}\big(u_1, u_2\big) = |\det(u_1, u_2)|,$$

and note that $\mathcal{D}(r_1 u_1, r_2 u_2) = |r_1||r_2| \mathcal{D}(u_1, u_2)$ for all $r_1, r_2 \in \mathbb{R}^\times$. Moreover, we also have

$$\mathcal{D}\big(g^{-1}u_1, g^{-1}u_2\big) = |\det(g^{-1}(u_1, u_2))| = \mathcal{D}\big(u_1, u_2\big) \tag{9.59}$$

for $g \in \mathrm{SL}_2(\mathbb{R})$, by multiplicativity of the determinant. To understand the connection between $\mathcal{D}(\cdot,\cdot)$ and $\langle \cdot, \cdot \rangle_{\mathcal{V}_s}$, we let $\theta_1, \theta_2 \in [0, \pi)$, set

$$v_{\theta_j} = k_{\theta_j} e_1 = \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}$$

for $j = 1, 2$ and calculate

$$\mathcal{D}\big(v_{\theta_1}, v_{\theta_2}\big) = \left|\det \begin{pmatrix} \cos\theta_1 & \cos\theta_2 \\ \sin\theta_1 & \sin\theta_2 \end{pmatrix}\right| = |\sin(\theta_1 - \theta_2)|. \tag{9.60}$$

Because of these formulas, it will be convenient to identify $k_\theta \in K$ with

$$v_\theta = k_\theta e_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \in \mathbb{S}^1$$

for $\theta \in [0, 2\pi)$. In this notation, and since $f \in \mathcal{V}_s$ is even, we may use (9.60) to express the norm in the form

$$\|f\|^2_{\mathcal{V}_s} = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{f(k_{\theta_1})\overline{f(k_{\theta_2})}}{\mathcal{D}(v_{\theta_1}, v_{\theta_2})^{1-s}} \,\mathrm{d}m(v_{\theta_1}) \,\mathrm{d}m(v_{\theta_2}),$$

where $m$ denotes the normalized arc length measure on $\mathbb{S}^1$.

Fix some $f \in \mathcal{V}_s$ and $g \in \mathrm{SL}_2(\mathbb{R})$. Then, by definition, we have

$$\|\gamma_g^s f\|^2_{\mathcal{V}_s} = \iint\limits_{\mathbb{S}^1\,\mathbb{S}^1} \frac{f(g^{-1}k_{\theta_1})\overline{f(g^{-1}k_{\theta_2})}}{\mathcal{D}(v_{\theta_1}, v_{\theta_2})^{1-s}}\,\mathrm{d}m(v_{\theta_1})\,\mathrm{d}m(v_{\theta_2}).$$

Let us write $g^{-1}k_{\theta_j} = k_{\psi_j} a_{t_j} u_{x_j}$ for $j = 1, 2$ for the Iwasawa decomposition of these products, so that $\psi_j$, $t_j$, and $x_j$ are functions of $v_{\theta_j} \in \mathbb{S}^1$ for $j = 1, 2$. Then

$$g^{-1}v_{\theta_j} = g^{-1}k_{\theta_j} e_1 = k_{\psi_j} a_{t_j} e_1 = \mathrm{e}^{t_j} v_{\psi_j},$$

in particular

$$\|g^{-1}v_{\theta_j}\| = \|a_{t_j} e_1\| = \mathrm{e}^{t_j}, \tag{9.61}$$

and

$$g^{-1}\cdot v_{\theta_j} = \|g^{-1}v_{\theta_j}\|^{-1} g^{-1}v_{\theta_j} = v_{\psi_j} \tag{9.62}$$

for $j = 1, 2$. Combining the definition of $\mathcal{V}_s$ with (9.61), we obtain

$$f(g^{-1}k_{\theta_j}) = f(k_{\psi_j} a_{t_j} u_{x_j}) = \mathrm{e}^{-t_j(s+1)} f(k_{\psi_j}) = \|g^{-1}v_{\theta_j}\|^{-(s+1)} f(k_{\psi_j})$$

for $f \in \mathcal{V}_s$. Using (9.59) and (9.62), we also have

$$\mathcal{D}(v_{\theta_1}, v_{\theta_2}) = \mathcal{D}(g^{-1}v_{\theta_1}, g^{-1}v_{\theta_2}) = \|g^{-1}v_{\theta_1}\|\|g^{-1}v_{\theta_2}\|\mathcal{D}(v_{\psi_1}, v_{\psi_2}).$$

For the norm of $\gamma_g^s f$, this leads to

$$\begin{aligned}
\|\gamma_g^s f\|^2_{\mathcal{V}_s} &= \iint\limits_{\mathbb{S}^1\mathbb{S}^1} \frac{\|g^{-1}v_{\theta_1}\|^{-(s+1)} f(k_{\psi_1})\|g^{-1}v_{\theta_2}\|^{-(s+1)}\overline{f(k_{\psi_2})}}{\|g^{-1}v_{\theta_1}\|^{1-s}\|g^{-1}v_{\theta_2}\|^{1-s}\mathcal{D}(v_{\psi_1}, v_{\psi_2})^{1-s}}\,\mathrm{d}m(v_{\theta_1})\mathrm{d}m(v_{\theta_2}) \\
&= \iint\limits_{\mathbb{S}^1\mathbb{S}^1} \frac{f(k_{\psi_1})\overline{f(k_{\psi_2})}}{\mathcal{D}(v_{\psi_1}, v_{\psi_2})^{1-s}} \|g^{-1}v_{\theta_1}\|^{-2}\|g^{-1}v_{\theta_2}\|^{-2}\,\mathrm{d}m(v_{\theta_1})\mathrm{d}m(v_{\theta_2}) \\
&= \iint\limits_{\mathbb{S}^1\mathbb{S}^1} \frac{f(k_{\psi_1})\overline{f(k_{\psi_2})}}{\mathcal{D}(v_{\psi_1}, v_{\psi_2})^{1-s}}\,\mathrm{d}g_* m(v_{\theta_1})\mathrm{d}g_* m(v_{\theta_2}). \qquad \text{(by (9.58))}
\end{aligned}$$

However, by (9.62) this double integral now has the form

$$\begin{aligned}
&\iint\limits_{\mathbb{S}^1\mathbb{S}^1} F\big(g^{-1}\cdot v_{\theta_1}, g^{-1}\cdot v_{\theta_2}\big)\,\mathrm{d}g_* m(v_{\theta_1})\,\mathrm{d}g_* m(v_{\theta_2}) \\
&\qquad\qquad = \iint\limits_{\mathbb{S}^1\mathbb{S}^1} F(v_{\theta_1}, v_{\theta_2})\,\mathrm{d}m(v_{\theta_1})\,\mathrm{d}m(v_{\theta_2}).
\end{aligned}$$

Therefore the last expression for $\|\gamma_g^s f\|^2_{\mathcal{V}_s}$ turns into $\|f\|^2_{\mathcal{V}_s}$. This shows that $\gamma_g^s$ is unitary on $\mathcal{V}_s$, and that it extends by continuity to a unitary operator on $\mathcal{H}_{(s)}$.

Finally, we verify the continuity of the complementary series representation $\gamma^s$. To see this, let $f_1, f_2 \in \mathcal{V}_s$ and note that

$$\langle \gamma^s_g f_1, f_2 \rangle_{\mathcal{V}_s} = \int\!\!\int_{\mathbb{S}^1 \mathbb{S}^1} \frac{f_1(g^{-1}k_{\theta_1}) f_2(k_{\theta_2})}{\mathcal{D}(v_{\theta_1}, v_{\theta_2})^{1-s}} \, \mathrm{d}m(v_{\theta_1}) \, \mathrm{d}m(v_{\theta_2})$$

depends continuously on $g \in \mathrm{SL}_2(\mathbb{R})$ by dominated convergence. This implies continuity of $g \mapsto \gamma^s_g f$ by simply expanding $\|\gamma^s_g f - \gamma^s_{g_0} f\|^2_{\mathcal{V}_s}$ into a sum of inner products. Hence Lemma 1.11 gives continuity of the unitary representation $\gamma^s$. □

### 9.5.3 Irreducibility of the Complementary Series

**Lemma 9.73 (Casimir eigenvalue).** *Let $s \in (0,1)$. For the complementary series representation, the Casimir element $\gamma^s_\partial(\Omega)$ is multiplication by $s^2$ on $\mathcal{V}_s$. Moreover,*

$$\gamma^s_\partial(\mathbf{r}^+) F_{s,n} = \frac{n+1+s}{2} F_{s,n+2} \quad \textit{and}$$
$$\gamma^s_\partial(\mathbf{r}^-) F_{s,n} = \frac{-n+1+s}{2} F_{s,n-2}$$

*for all $n \in 2\mathbb{Z}$.*

Proof. We reuse the calculation in the proof of Lemma 9.33. In fact we proved (9.21) by calculating a pointwise derivative, and this part of the argument would apply for any $\xi \in \mathbb{C}$. Using $\xi = -\mathrm{i}s$ replaces $\mathrm{i}\xi$ by $s$, and (9.21) takes the form

$$\gamma^s_\partial(\mathbf{a}) F_{s,n} = \frac{n+1+s}{2} F_{s,n+2} + \frac{-n+1+s}{2} F_{s,n-2} \tag{9.63}$$

for $n \in 2\mathbb{Z}$ (see Exercise 9.74 below). We recall that $\mathbf{a} = \mathbf{r}^+ + \mathbf{r}^-$ and apply Proposition 9.13 for $\gamma^s$. From this the formulas for $\gamma^s_\partial(\mathbf{r}^+) F_{s,n}$ in the lemma follow.

We now apply $\Omega$ in the form (9.13) and obtain

$$\begin{aligned} \gamma^s_\partial(\Omega) F_{s,n} &= \gamma^s_\partial\big(4\mathbf{r}^+ \circ \mathbf{r}^- + (\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})^{\circ 2}\big) F_{s,n} \\ &= (-n+1+s)\gamma^s_\partial(2\mathbf{r}^+) F_{s,n-2} + \gamma^s_\partial(\mathbb{1}_{\mathfrak{E}} + \mathrm{i}\mathbf{k})^{\circ 2} F_{s,n} \\ &= (-n+1+s)(n-1+s) F_{s,n} + (1-n)^2 F_{s,n} = s^2 F_{s,n} \end{aligned}$$

for all $n \in 2\mathbb{Z}$. Using Fourier series for a smooth function on $K/M$, this extends to all smooth $f \in \mathcal{V}_s$. □

**Exercise 9.74.** Prove (9.63) as a partial derivative within $\mathcal{H}_{(s)}$.

Proof of irreducibility in Theorem 9.68. Let $s \in (0,1)$. We recall that $\mathcal{V}_s$ contains the orthogonal basis vectors $F_{s,n}$ for $n \in 2\mathbb{Z}$ of $\mathcal{H}_{(s)}$. Hence the completion $\mathcal{H}_{(s)}$ has, for every $n \in 2\mathbb{Z}$, a one-dimensional weight space spanned by $F_{s,n}$ and no eigenvector for odd weights. Moreover, by Lemma 9.73, the Casimir operator acts by multiplication by $s^2$. This is all one needs to know in order to obtain the irreducibility of $\gamma^s$ using arguments we have used many times before.

Indeed, if $\mathcal{V} \subseteq \mathcal{H}_{(s)}$ is a non-trivial closed $\gamma^s$-invariant subspace, then as $K$ is compact and abelian there must exist a $K$-eigenvector $F \in \mathcal{V}$. By the above, $F$ is a multiple of $F_{s,n_0}$ for some $n_0 \in 2\mathbb{Z}$. Using the raising and lowering operators and the fact that they do not map $F_{s,n_0}$ to zero (by Corollary 9.14 or Lemma 9.73) we obtain from the fact that $F_{s,n_0} \in \mathcal{V}$ that $F_{s,n_0 \pm 2} \in \mathcal{V}$ also. Iterating we see that $\mathcal{V}$ contains $F_{s,n} \in \mathcal{V}$ for all $n \in 2\mathbb{Z}$ and hence $\mathcal{V} = \mathcal{H}_{(s)}$. □

### 9.5.4 Decay and Integrability Properties

The following shows, in particular, the remaining claim in Theorem 9.68 that the complementary series is not tempered. However, the precise information regarding the decay properties of the matrix coefficients of $F_{s,0} \in \mathcal{H}_{(s)}$ will be useful in the next section.

**Lemma 9.75 (Matrix coefficient of $F_{s,0}$).** *Let $s \in (0,1)$. The matrix coefficient $\phi_{(s)} = \varphi^{\gamma^s}_{F_{s,0}}$ is bi-$K$-invariant, satisfies the asymptotics*

$$\phi_{(s)}(g) \asymp_s \|g\|_{\mathrm{HS}}^{s-1}$$

*for $g \in \mathrm{SL}_2(\mathbb{R})$, and belongs to $L^p(G)$ if and only if $p > \frac{2}{1-s}$. Moreover, $\phi_{(s)}$ converges for $s \nearrow 1$, uniformly on compact subsets of $G$, to the constant function 1.*

Proof. As $F_{s,0}$ has $K$-weight 0, it is clear that $\phi_{(s)}(g) = \langle \gamma^s_g F_{s,0}, F_{s,0} \rangle$ is bi-$K$-invariant. For that reason it suffices to consider $g = a_t$ for $t \geqslant 0$ in the proof of the asymptotics of $\phi_{(s)}$. For the matrix coefficient, this gives

$$\begin{aligned}\phi_{(s)}(a_t) &= \langle \gamma^s_{a_t} F_{s,0}, F_{s,0}\rangle\\ &= \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \frac{F_{s,0}(a_t^{-1}k_{\theta_1})F_{s,0}(k_{\theta_2})}{|\sin(\theta_1-\theta_2)|^{1-s}}\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\\ &= \frac{1}{\pi}\int_0^\pi F_{s,0}(a_t^{-1}k_{\theta_1})\underbrace{\frac{1}{\pi}\int_0^\pi \frac{1}{|\sin(\theta_1-\theta_2)|^{1-s}}\,\mathrm{d}\theta_2}_{C_s}\,\mathrm{d}\theta_1,\end{aligned} \tag{9.64}$$

where the inner integral contributes a constant $C_s$ only depending on $s$ (because we may use the substitution $\psi = \theta_1 - \theta_2$ for $\theta_2$). As in the proof of Proposition 8.39, we apply the Iwasawa decomposition to $a_t^{-1}k_\theta = k_\psi a_{t_0} u_{x_0}$ for varying $k_\theta \in K$, which determines $k_\psi \in K$, $a_{t_0} \in A$, and $u_{x_0} \in U$. For $t_0$ this gives

$$\mathrm{e}^{2t_0} = \|k_\psi a_{t_0} u_{x_0} e_1\|^2 = \|a_t^{-1}k_\theta e_1\|^2 = \mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta.$$

Using the definition of $F_{s,0}$ in Lemma 9.69, we have

$$F_{s,0}(a_t^{-1}k_\theta) = \mathrm{e}^{-(s+1)t_0} = \left(\sqrt{\mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta}\right)^{-(s+1)} \tag{9.65}$$

We note that this implies that $F_{s,0}(a_t^{-1}k_{\theta_1})$ is unchanged if we replace $\theta_1$ by $\pi-\theta_1$. Hence we may also replace the outer normalized integral over $[0,\pi]$ in (9.64) by the normalized integral over $[0,\frac{\pi}{2}]$, which gives

$$\phi_{(s)}(a_t) = \frac{2C_s}{\pi}\int_0^{\frac{\pi}{2}} F_{s,0}(a_t^{-1}k_\theta)\,\mathrm{d}\theta. \tag{9.66}$$

To study the asymptotics of the matrix coefficient $\phi(s)$ of $F_{s,0}$, we first note that (9.65) implies that

$$F_{s,0}(a_t^{-1}k_\theta) \asymp \max\left(\mathrm{e}^{-t}|\cos\theta|, \mathrm{e}^{t}|\sin\theta|\right)^{-(s+1)}.$$

We now replace $F_{s,0}(a_t^{-1}k_{\theta_1})$ in the integral (9.66) by this maximum. The latter is given by $\mathrm{e}^t\sin\theta_1$ unless $\theta_1$ is very close to 0—specifically, unless $\tan\theta_1 < \mathrm{e}^{-2t}$. Therefore

$$\begin{aligned}\phi_{(s)}(a_t) &\asymp_s \int_0^{\arctan \mathrm{e}^{-2t}} \big(\mathrm{e}^{-t}\underbrace{\cos\theta}_{\asymp 1}\big)^{-(s+1)}\,\mathrm{d}\theta + \int_{\arctan \mathrm{e}^{-2t}}^{\frac{\pi}{2}} \big(\mathrm{e}^{t}\underbrace{\sin\theta}_{\asymp\theta}\big)^{-(s+1)}\,\mathrm{d}\theta \\ &\asymp_s \mathrm{e}^{(s+1)t}\arctan \mathrm{e}^{-2t} + \mathrm{e}^{-(s+1)t}\frac{1}{-s}\theta^{-s}\Big|_{\arctan \mathrm{e}^{-2t}}^{\frac{\pi}{2}} \\ &\asymp_s \mathrm{e}^{(s+1)t}\mathrm{e}^{-2t} + \frac{1}{s}\mathrm{e}^{-(s+1)t}\mathrm{e}^{2st} \\ &\asymp_s \mathrm{e}^{(s-1)t} + \frac{1}{s}\mathrm{e}^{(s-1)t} \asymp_s \mathrm{e}^{(s-1)t},\end{aligned}$$

which gives the claimed asymptotic.

Now let $p > 0$. Using the asymptotics and the decomposition of the Haar measure in (8.11), we obtain

$$\int_{\mathrm{SL}_2(\mathbb{R})} \phi_{(s)}(g)^p\,\mathrm{d}m(g) \asymp \int_0^\infty \phi_{(s)}(a_t)^p \sinh 2t\,\mathrm{d}t.$$

Since we are only interested in whether this integral converges, we restrict the integral to $[1,\infty)$, use the estimate $\sinh 2t \asymp \mathrm{e}^{2t}$ for $t \in [1,\infty)$, and the asymptotics for $\phi_s(a_t)$ to see that

$$\int_1^\infty \phi_{(s)}(a_t)^p \sinh 2t\,\mathrm{d}t \asymp_s \int_1^\infty \mathrm{e}^{t(s-1)p}\mathrm{e}^{2t}\,\mathrm{d}t.$$

Notice that the exponent

$$t(s-1)p + 2t = (2-(1-s)p)t$$

of the integrand has a negative coefficient if and only if $p > \frac{2}{1-s}$, and that this characterizes finiteness of the integral.

It remains to prove the final claim in the lemma concerning the behaviour of $\phi_{(s)}$ as $s \nearrow 1$. For this, first note that $C_s$ as in (9.64) depends continuously on $s \in (0,1)$, and that

$$C_s = \frac{1}{\pi}\int_0^\pi \frac{1}{|\sin\theta|^{1-s}}\,\mathrm{d}\theta \longrightarrow C_1 = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta = 1$$

for $s \nearrow 1$ by dominated convergence. Next note that $F_{s,0}(a_t^{-1}k_\theta)$ as in (9.65) also makes sense for $s \in [\frac{1}{2},1]$ and depends continuously on

$$(\theta,s,t) \in [0,\tfrac{\pi}{2}] \times [\tfrac{1}{2},1] \times \mathbb{R}.$$

On restricting $t$ to a compact interval $I$, uniform continuity implies that the function $\phi_{(s)}(a_t)$ as defined in (9.66) makes sense and depends continuously on $(s,t) \in [\frac{1}{2},1] \times I$. Therefore by uniform continuity again and (9.65) we have that $\phi_{(s)}(a_t)$ converges to the function

$$I \ni t \longmapsto \phi_{(1)}(a_t) = \frac{2C_1}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta}\,\mathrm{d}\theta.$$

uniformly on $I$ as $s \nearrow 1$. Now note that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\arctan\bigl(\mathrm{e}^{2t}\tan\theta\bigr)\right) = \frac{1}{1+\mathrm{e}^{4t}\tan^2\theta}\mathrm{e}^{2t}\frac{1}{\cos^2\theta} = \frac{1}{\mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta},$$

which together with $C_1 = 1$ gives

$$\phi_{(1)}(a_t) = \frac{2}{\pi}\lim_{b\nearrow\frac{\pi}{2}}\left[\arctan\bigl(\mathrm{e}^{2t}\tan\theta\bigr)\right]_0^b = 1$$

as required. □

### 9.5.5 A Sobolev Space of the Projective Line*

Recall that the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R}^2\smallsetminus\{0\}/\sim$ is defined as the quotient space of $\mathbb{R}^2\smallsetminus\{0\}$ modulo the equivalence relation $u_1 \sim u_2$ if $u_1$ and $u_2$ are scalar multiples of each other. Note that $\mathbb{P}^1(\mathbb{R}) = \mathbb{S}^1/\sim$ can also be obtained from the circle by identifying opposite points. Moreover, we may also identify the equivalence class $[k_\theta e_1]_\sim \in \mathbb{P}^1(\mathbb{R})$ for some $k_\theta \in K$ with $k_\theta M \in K/M$. Consequently functions on $\mathbb{P}^1(\mathbb{R})$ correspond to even functions on $K$. In this sense, Lemma 9.69 shows that $\mathcal{V}_s$ can be identified with $C^\infty(\mathbb{P}^1(\mathbb{R}))$. For the completion $\mathcal{H}_{(s)}$ of $\mathcal{V}_s$ this leads to the following result.

**Proposition 9.76 (A Sobolev space).** *Let $s \in (0,1)$. The norm on $\mathcal{V}_s$ is equivalent to the $L^2$-Sobolev norm with $-\frac{s}{2}$ derivatives. Hence $\mathcal{H}_{(s)}$ is the $L^2$-Sobolev space $\mathcal{W}^{-\frac{s}{2},2}(\mathbb{P}^1(\mathbb{R}))$ with $-\frac{s}{2}$ derivatives.*

Proof. We recall that the $L^2$-Sobolev space $\mathcal{W}^{-\frac{s}{2},2}(\mathbb{T})$ with $-\frac{s}{2}$ derivatives is defined as the completion of $C^\infty(\mathbb{T})$ with respect to the norm defined by

$$\|f\|^2_{-\frac{s}{2},2} = \sum_{n\in\mathbb{Z}} |c_n|^2|n|^{-s}$$

for $f = \sum_{n\in\mathbb{Z}} c_n\chi_n$. In the case of $\mathbb{P}^1(\mathbb{R}) = \mathbb{S}^1/\sim \,\cong K/M \cong \mathbb{T}/\langle\frac{1}{2}+\mathbb{Z}\rangle$, we simply restrict to even functions and $n \in 2\mathbb{Z}$. Due to the orthogonality relations satisfied by $F_{s,n}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{V}_s}$ and $\langle\cdot,\cdot\rangle_{-\frac{s}{2},2} = \langle\cdot,\cdot\rangle_{\mathcal{W}^{-\frac{s}{2},2}}$ for $n \in 2\mathbb{Z}$, the proposition is equivalent to the statement that

$$|n|^{-s} \ll \|F_{s,n}\|^2_{\mathcal{V}_s} \ll \|F_{s,n}\|^2_{-\frac{s}{2},2} = |n|^{-s} \tag{9.67}$$

for all $n \in 2\mathbb{Z}$. For this, recall that $\|F_{s,-n}\|^2_{\mathcal{V}_s} = \|F_{s,n}\|^2_{\mathcal{V}_s}$ for all $n \in \mathbb{N}$. Hence to prove (9.67) it suffices to calculate the asymptotics of $\|F_{s,n}\|^2_{\mathcal{V}_s}$ as $n \to \infty$, which will follow by combining Corollary 9.14, Lemma 9.73, and Stirling's formula for the gamma function.

Let $n \in \mathbb{Z}$ and apply Lemma 9.73 to $F_{s,2n}$ to obtain

$$\gamma^s_\partial(\mathbf{r}^+)F_{s,2n} = \frac{2n+1+s}{2}F_{s,2n+2}. \tag{9.68}$$

On the other hand, Corollary 9.14 gives

$$\|\gamma^s_\partial(\mathbf{r}^+)F_{s,2n}\|^2 = \tfrac{1}{4}\big((2n+1)^2 - s^2\big)\|F_{s,2n}\|^2. \tag{9.69}$$

Together, we obtain the recursion formula

$$\begin{aligned}
\|F_{s,2n+2}\|^2 &= \frac{4}{(2n+1+s)^2}\|\gamma^s_\partial(\mathbf{r}^+)F_{s,2n}\|^2 && \text{(by (9.68))}\\
&= \frac{(2n+1)^2 - s^2}{(2n+1+s)^2}\|F_{s,2n}\|^2 && \text{(by (9.69))}\\
&= \frac{2n+1-s}{2n+1+s}\|F_{s,2n}\|^2 = \frac{n+\frac{1-s}{2}}{n+\frac{1+s}{2}}\|F_{s,2n}\|^2
\end{aligned}$$

for the norms.

Now recall the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

for $x > 0$, and that integration by parts shows that $\Gamma(x+1) = x\Gamma(x)$. We define $c = c_s > 0$ by the formula

$$\|F_{s,2}\|^2 = \frac{\Gamma\big(1+\frac{1-s}{2}\big)}{\Gamma\big(1+\frac{1+s}{2}\big)}c,$$

and prove by induction on $n \in \mathbb{N}$ that

$$\|F_{s,2n}\|^2 = \frac{\Gamma\big(n+\frac{1-s}{2}\big)}{\Gamma\big(n+\frac{1+s}{2}\big)}c. \tag{9.70}$$

Indeed the definition of $c$ is the start of the induction, and the recursion formula gives the inductive step

$$\|F_{s,2n+2}\|^2 = \frac{n+\frac{1-s}{2}}{n+\frac{1+s}{2}}\|F_{s,2n}\|^2 = \frac{\left(n+\frac{1-s}{2}\right)\Gamma\left(n+\frac{1-s}{2}\right)}{\left(n+\frac{1+s}{2}\right)\Gamma\left(n+\frac{1+s}{2}\right)}c$$
$$= \frac{\Gamma\left(n+1+\frac{1-s}{2}\right)}{\Gamma\left(n+1+\frac{1+s}{2}\right)}c$$

for $n \in \mathbb{N}$.

Next we recall Stirling's formula for the gamma function, which states that

$$\Gamma(x) \sim \sqrt{\frac{2\pi}{x}}\left(\frac{x}{\mathrm{e}}\right)^x,$$

where as usual $\sim$ means that the ratio of the left-hand and right-hand side converges to 1 as $x \to \infty$ (see Section B.2). We will also write $\approx$ to mean that the ratio converges to a positive constant depending on $s \in (0,1)$. Using Stirling's formula on (9.70) gives

$$\|F_{s,2n}\|^2 = \frac{\Gamma\left(n+\frac{1-s}{2}\right)}{\Gamma\left(n+\frac{1+s}{2}\right)}c \sim \underbrace{\sqrt{\frac{n+\frac{1+s}{2}}{n+\frac{1-s}{2}}}}_{\sim 1}\frac{\left(\frac{n+\frac{1-s}{2}}{\mathrm{e}}\right)^{n+\frac{1-s}{2}}}{\left(\frac{n+\frac{1+s}{2}}{\mathrm{e}}\right)^{n+\frac{1+s}{2}}}c$$
$$\approx \underbrace{\left(\frac{n+\frac{1-s}{2}}{n+\frac{1+s}{2}}\right)^{n+\frac{1-s}{2}}}_{\approx 1}\left(n+\frac{1+s}{2}\right)^{\frac{1-s}{2}-\frac{1+s}{2}}$$
$$\approx n^{-s}.$$

Taking the square root gives the desired asymptotic in (9.67) for $\|F_{s,2n}\|_{\mathcal{V}_s}$ as $n \to \infty$. □

## 9.6 Spectral Gap, Decay, and Integrability Exponents

Using the complete description of $\widehat{\mathrm{SL}_2(\mathbb{R})}$, and in particular the complementary series representation, we can upgrade the results concerning integrability and decay exponents ($p_\pi$ and $\kappa_\pi$, respectively) from Section 8.7. Moreover, we will relate these to the notion of spectral gap defined in Section 4.2.1 and, in addition, to the following quantity.

**Definition 9.77 (Complementary series parameter).** Let $\pi$ be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. Then the *complementary series parameter* of $\pi$ is defined by

$$s_\pi = \sup\{s \in [0,1) \mid s = 0 \text{ or } \gamma^s \prec \pi\}.$$

**Theorem 9.78.** *Let $\pi$ be a unitary representation of* $\mathrm{SL}_2(\mathbb{R})$*. Then*

$$\min(1, \kappa_\pi) = \frac{2}{\max(2, p_\pi)} = 1 - s_\pi. \tag{9.71}$$

*For $\kappa < \min(1, \kappa_\pi)$ and any $g \in G$ we have*

$$\left|\langle \pi_g v, w\rangle\right| \ll_{\kappa_\pi, \kappa} \|v\| \|w\| \|g\|_{\mathrm{HS}}^{-\kappa} \tag{9.72}$$

*for any $K$-eigenvectors $v, w \in \mathcal{H}_\pi$, and*

$$\left|\langle \pi_g v, w\rangle\right| \ll_{\kappa_\pi, \kappa} \mathcal{S}(v)\,\mathcal{S}(w) \|g\|_{\mathrm{HS}}^{-\kappa} \tag{9.73}$$

*for any $C^1$-smooth vectors $v, w \in \mathcal{H}_\pi$. Moreover, $\kappa_\pi > 0$, $p_\pi < \infty$, and $s_\pi < 1$ are all equivalent to $\pi$ having spectral gap.*

We note that, even for the proof of the first equality concerning $\kappa_\pi$ and $p_\pi$, knowledge about the complementary series will be useful. We again write $G$ for $\mathrm{SL}_2(\mathbb{R})$.

Proof of (9.72)–(9.73) and that $\min(1, \kappa_\pi) = \frac{2}{\max(2, p_\pi)}$. Suppose that $\pi$ is a unitary representation of $\mathrm{SL}_2(\mathbb{R})$. Comparing the definition of integrability and decay exponents in Definitions 7.23 and 8.43, we may assume that $\pi$ has no fixed vectors. Suppose first that $p_\pi < 2$. Then, by definition, $\pi$ is almost square integrable, and hence tempered by Theorem 8.5, and satisfies (9.72) for all $\kappa < 1$ by Theorem 8.41. Hence $\kappa_\pi \geqslant 1$ and $\min(1, \kappa_\pi) = 1 = \frac{2}{\max(2, p_\pi)}$, as claimed.

We suppose now that $p_\pi \geqslant 2$. By Lemma 8.45 we have $p_\pi \leqslant \frac{2}{\kappa_\pi}$. With $p_\pi \geqslant 2$ we obtain from this that $\kappa_\pi \leqslant 1$, and hence

$$\min(1, \kappa_\pi) = \kappa_\pi \leqslant \frac{2}{p_\pi} = \frac{2}{\max(2, p_\pi)}. \tag{9.74}$$

To prove the opposite inequality, suppose that $p > 0$ is such that $\pi$ is $p$-integrable. By definition, this means that there exists a dense set of vectors $\mathcal{V} \subseteq \mathcal{H}_\pi$ so that $\varphi^\pi_{v,w} \in L^p(G)$ for all $v, w \in \mathcal{V}$. Let $\varepsilon > 0$, fix a positive $\widetilde{s} \in \left(\frac{2}{p} - \varepsilon, \frac{2}{p}\right)$, and note that $\widetilde{s} \in (0,1)$ since $p \geqslant p_\pi \geqslant 2$. We claim that this makes the inner tensor product $\pi \otimes \gamma^{\widetilde{s}}$ tempered.

Assuming the claim for now, we let $v, w \in \mathcal{H}_\pi$ be $K$-eigenvectors and let $F_{\widetilde{s},0}$ be the spherical function as in Lemma 9.69. Then both $v \otimes F_{\widetilde{s},0}$ and $w \otimes F_{\widetilde{s},0}$ are also $K$-eigenvectors. Therefore temperedness of $\pi \otimes \gamma^{\widetilde{s}}$ and Theorem 8.41(2), together with the estimate for the Harish-Chandra spherical function in Theorem 8.31, give

$$\begin{aligned} \left|\langle \pi_g v, w\rangle\right| \phi_{(\widetilde{s})}(g) &= \left|\langle (\pi \otimes \gamma^{\widetilde{s}})_g v \otimes F_{\widetilde{s},0}, w \otimes F_{\widetilde{s},0}\rangle\right| \\ &\ll_\varepsilon \|v \otimes F_{\widetilde{s},0}\| \|w \otimes F_{\widetilde{s},0}\| \|g\|_{\mathrm{HS}}^{-1+\varepsilon} \\ &\ll_{\varepsilon, \widetilde{s}} \|v\| \|w\| \|g\|_{\mathrm{HS}}^{-1+\varepsilon} \end{aligned}$$

for all $g \in G$, where $\phi_{(\widetilde{s})}$ is the matrix coefficient of $F_{\widetilde{s},0}$. Together with the lower bound for $\phi_{(\widetilde{s})}$ in Lemma 9.75, we obtain after dividing by $\phi_{(s)}$ the estimate

$$|\langle \pi_g v, w\rangle| \ll_{\varepsilon,\widetilde{s}} \|v\|\|w\|\|g\|_{\mathrm{HS}}^{-\widetilde{s}+\varepsilon}.$$

Recalling the assumption $\widetilde{s} \in \left(\frac{2}{p} - \varepsilon, \frac{2}{p}\right)$ we also obtain

$$|\langle \pi_g v, w\rangle| \ll_{p,\varepsilon} \|v\|\|w\|\|g\|_{\mathrm{HS}}^{-\frac{2}{p}+2\varepsilon}$$

for all $K$-eigenvectors $v, w \in \mathcal{H}_\pi$. By Proposition 7.27 this upgrades automatically to all $C^1$-smooth vectors $v, w \in \mathcal{H}_\pi$ if we replace the norm of $v, w$ by the degree-one Sobolev norms of $v, w$ (and multiply the implicit constant by an absolute constant). Hence $\kappa = \frac{2}{p} - 2\varepsilon$ is a decay exponent for $\pi$ satisfying (9.72) and (9.73). Recalling that $\varepsilon > 0$ and $p > 0$ with $\pi$ being $p$-integrable were arbitrary, we see that the claim implies that $\kappa_\pi \geqslant \frac{2}{p_\pi}$. Together with (9.74), this gives the desired equality.

Turning to the claim that $\pi \otimes \gamma^{\widetilde{s}}$ is tempered, notice first that the linear hull $\langle \gamma^{\widetilde{s}}(G) F_{\widetilde{s},0}\rangle$ of the $G$-orbit of $F_{\widetilde{s},0}$ is dense in $\mathcal{H}_{(\widetilde{s})}$ by irreducibility of the complementary series representation. Therefore $\langle \mathcal{V}\rangle \otimes_{\mathrm{la}} \langle \gamma_G^{\widetilde{s}} F_{\widetilde{s},0}\rangle$ is dense in $\mathcal{H}_\pi \otimes \mathcal{H}_{(\widetilde{s})}$ and, by sesquilinearity of matrix coefficients, it suffices to consider the matrix coefficient $\phi$ of $v \otimes \gamma_{g_1}^{\widetilde{s}} F_{\widetilde{s},0}$ and $w \otimes \gamma_{g_2}^{\widetilde{s}} F_{\widetilde{s},0}$. This gives

$$\phi(g) = \langle \pi_g v, w\rangle \langle \gamma_{gg_1}^{\widetilde{s}} F_{\widetilde{s},0}, \gamma_{g_2}^{\widetilde{s}} F_{\widetilde{s},0}\rangle = \varphi_{v,w}^{\pi}(g) \lambda_{g_2} \rho_{g_1} \phi_{(\widetilde{s})}(g).$$

By assumption, $\varphi_{v,w}^{\pi} \in L^p(G)$ and, by Lemma 9.75, $\phi_{(s)} \in L^{\frac{2}{1-\frac{2}{p}}}$ since $\widetilde{s} < \frac{2}{p}$ implies $\frac{2}{1-\widetilde{s}} < \frac{2}{1-\frac{2}{p}}$. Using (B.1), this implies that $\phi$ belongs to $L^q(G)$ for

$$q = \frac{p\frac{2}{1-\frac{2}{p}}}{p + \frac{2}{1-\frac{2}{p}}} = \frac{2p}{p(1-\frac{2}{p}) + 2} = 2,$$

which proves the claim. □

**Corollary 9.79.** *Let $s \in (0,1)$. Then the complementary series $\gamma^s$ has almost decay exponent*

$$\kappa_{\gamma^s} = 1 - s.$$

Proof. Applying the already established first part of Theorem 9.78 to $\gamma^s$ and the vector $F_{s,0} \in \mathcal{H}_s$, we obtain the upper bound in

$$\|g\|_{\mathrm{HS}}^{s-1} \ll_s |\phi_{(s)}(g)| \ll_{s,\kappa} \|g\|_{\mathrm{HS}}^{-\kappa} \tag{9.75}$$

for all $g \in G$ and any $\kappa < \min(1, \kappa_{\gamma^s})$. The lower bound in (9.75) comes from Lemma 9.75. Together we obtain $s - 1 \leqslant -\kappa$ or, equivalently, $\kappa \leqslant 1 - s$

by letting $g \to \infty$. Since the choice of $\kappa < \min(1, \kappa_{\gamma^s})$ was arbitrary, this gives $\kappa_{\gamma^s} \leqslant 1 - s < 1$.

By Lemma 9.75, $\phi_{(s)} = \varphi^{\gamma^s}_{F_{s,0}}$ belongs to $L^p(G)$ for all $p > \frac{2}{1-s}$. By Exercise 8.44 and irreducibility of $\gamma^{(s)}$, this implies that $\gamma^s$ is $p$-integrable for all $p > \frac{2}{1-s}$ and so

$$p_{\gamma^s} \leqslant \frac{2}{1-s}.$$

Together with the first part of Theorem 9.78 again, we obtain from this

$$\kappa_{\gamma^s} = \frac{2}{p_{\gamma^s}} \geqslant 1 - s,$$

and hence the corollary. □

Concluding the proof of Theorem 9.78. We start by proving the inequality $\min(1, \kappa_\pi) \leqslant 1 - s_\pi$. For this we suppose that $\gamma^s \prec \pi$ for some $s \in (0, 1)$ and $\kappa < \min(1, \kappa_\pi)$. By the first part of the theorem, we know that $\kappa$ satisfies (9.72) for all $K$-eigenvectors $v, w \in \mathcal{H}_\pi$. Using Lemma 8.42 just as in the proof of (1) $\Longrightarrow$ (2) in Theorem 8.41 on p. 367, it follows that $\kappa$ is also a decay exponent for $\gamma^s$. By Corollary 9.79 this implies $\kappa \leqslant \kappa_{\gamma^s} = 1-s$.

As $\kappa < \min(1, \kappa_\pi)$ and $s \in (0, 1)$ with $\gamma^s \prec \pi$ were arbitrary, we obtain

$$\min(1, \kappa_\pi) \leqslant 1 - s_\pi. \tag{9.76}$$

For this, we also note that (9.76) holds trivially if there is no complementary series $\gamma^s$ weakly contained in $\pi$, since in this case $s_\pi = 0$.

Note that if $s_\pi = 1$ then (9.76) shows that $\kappa_\pi = 0$ and so there is equality in (9.76). If $s_\pi = 0$ (that is, if no complementary series are weakly contained in $\pi$) then $\pi$ is tempered and so $\kappa_\pi = 1$. This follows, for example, from the argument in Corollary 9.29 but also from the discussion below. So we now suppose that $s_\pi \in [0, 1)$ and claim that there exists a countable set $S \subseteq (0, s_\pi]$ (with $S = \emptyset$ if $s_\pi = 0$) so that

$$\pi \prec \lambda \oplus \bigoplus_{s \in S} \gamma^s. \tag{9.77}$$

Let $\kappa \in (0, 1 - s_\pi)$ so that $\kappa < 1 - s$ for all $s \in S$. By Corollary 9.79 this shows that $\kappa$ is a decay exponent for $\gamma^s$ for all $s \in S$. In fact, we have

$$|\langle \gamma^s v, w \rangle| \ll \|v\| \|w\| \|g\|_{\mathrm{HS}}^{-\kappa} \tag{9.78}$$

for all $s \in S$ and $K$-eigenfunctions $v, w \in \mathcal{H}_{(s)}$. As $\kappa < 1 - s_\pi \leqslant 1$ the estimate (9.78) holds similarly for $\lambda$. This allows us to prove the same estimate for $\lambda \oplus \bigoplus_{s \in S} \gamma^s$. Indeed, let $v, w \in L^2(G) \oplus \bigoplus_{s \in S} \mathcal{H}_{(s)}$ be two $K$-eigenfunctions, and let us write $v_0, w_0 \in L^2(G)$ for the components of $w, w$ corresponding to $\lambda$, and $v_s, w_s \in \mathcal{H}_{(s)}$ for the components corresponding to $s \in S$. Then

$$\left|\left\langle \left(\lambda \oplus \bigoplus_{s\in S} \gamma^s\right)(g)v, w \right\rangle\right| = \left|\left\langle \lambda_g v_0, w_0 \right\rangle\right| + \sum_{s\in S} \left|\left\langle \gamma_g^s v_s, w_s \right\rangle\right|$$
$$\ll \left( \|v_0\|\|w_0\| + \sum_{s\in S} \|v_s\|\|w_s\| \right) \|g\|_{\mathrm{HS}}^{-\kappa}.$$

Applying the Cauchy–Schwarz in the Hilbert space $\mathbb{C}^{|S|+1}$, we may estimate the parenthesis on the right by the product

$$\sqrt{\|v_0\|^2 + \sum_{s\in S} \|v_s\|^2} \sqrt{\|w_0\|^2 + \sum_{s\in S} \|w_s\|^2} = \|v\|\|w\|.$$

Using (9.77), applying Lemma 8.42, and using the argument for (1) $\Longrightarrow$ (2) in the proof of Theorem 8.41 on p. 367, we obtain

$$\left|\left\langle \pi_g v, w \right\rangle\right| \ll \|v\|\|w\|\|g\|_{\mathrm{HS}}^{-\kappa}$$

for any $K$-eigenvectors $v, w \in \mathcal{H}_\pi$ (without changing the implicit constant), which once more implies (9.73) for $C^1$-smooth vectors. We therefore see that any $\kappa < 1 - s_\pi$ is a decay exponent, which gives $\kappa_\pi \geqslant 1 - s_\pi$ for the supremum, and hence equality in (9.76).

To prove the claim (9.77) let $v \in \mathcal{H}_\pi$ be a unit vector, $Q \subseteq G$ a compact set, and $\varepsilon > 0$. Applying Proposition 4.36 we can find finitely many irreducible representations $\pi_j \prec \pi$ and vectors $v_j \in \mathcal{H}_{\pi_j}$ for $j = 1, \ldots, n$ so that $\sum_{j=1}^n \|v_j\|^2 = 1$ and $\varphi_v^\pi$is equal to $\sum_{j=1}^n \varphi_{v_j}^{\pi_j}$ on $Q$ up to $\mathrm{O}(\varepsilon)$. In Theorem 9.22 we found all irreducible representations of $G = \mathrm{SL}_2(\mathbb{R})$, and by Table 9.1 (see Theorem 8.23, Theorem 8.30, and Theorem 9.31) we have $\pi_j \prec \lambda$ or $\pi_j = \gamma^s$ for some $s \in (0, s_\pi]$. In the former case we may apply the definition of weak containment $\pi_j \prec \lambda$ and replace $\varphi_{v_j}^{\pi_j}$ by a sum of matrix coefficients for the regular representation. In other words, we may assume instead that $\pi_j = \lambda$ or $\pi_j = \gamma^s$ for some $s \in (0, s_\pi]$.

We now vary $v$ within a dense countable subset of the unit sphere in $\mathcal{H}_\pi$, set $Q = B_n^{\|\cdot\|_{\mathrm{HS}}}$ and $\varepsilon = \frac{1}{n}$ for $n \in \mathbb{N}$. This way we obtain a subset $S \subseteq (0, s_\pi]$ that is at most countable so that (9.77) holds by the definition of weak containment in Definition 4.1.

The argument above completes the proof of (9.71) except for a tiny detail that we have intentionally kept hidden under the rug until now. To prove that $\kappa \in (0, 1 - s_\pi)$ is a decay exponent for $\pi$, we used the estimate (9.78) but did not discuss the dependency of the implicit multiplicative constant on $s$ in $S$. (Note that Corollary 9.79 makes no claim concerning this.) Assuming that we can choose the implicit constant so that it does not depend on $s \in S$ but only on $\kappa$ and $s_\pi$, the above argument applies as explained.

To see that (9.78) holds with an implicit constant that only depends on $\kappa$ and $s_\pi$ we will review the proof of Corollary 9.79. So let $s \in S$. By Lemma 9.75 we have that $|\phi(s)|$ belongs to $L^p(G)$ if and only if $p > \frac{2}{1-s}$, which implies that $\gamma^s$ is $p$-integrable for $p > \frac{2}{1-s}$ by Exercise 8.44. Note that $\kappa < 1 - s_\pi$ and $s \leqslant s_\pi$ imply $\frac{2}{1-s} \leqslant \frac{2}{1-s_\pi} < \frac{2}{\kappa}$, which allows us to fix some $p \in \left(\frac{2}{1-s_\pi}, \frac{2}{\kappa}\right)$ so that $\gamma^s$ is $p$-integrable for all $s \in S$. Since $p < \frac{2}{\kappa}$ we may also fix some $\widetilde{s}$ in $\left(\kappa, \frac{2}{p}\right)$. By the first part of the proof of Theorem 9.78 we have that $\gamma^s \otimes \gamma^{\widetilde{s}}$ is tempered. which gives

$$\left|\left\langle \gamma_g^s v, w\right\rangle \phi_{(\widetilde{s})}(g)\right| \ll_\varepsilon \|v\|\|w\|\|g\|_{\mathrm{HS}}^{-1+\varepsilon}$$

for any $K$-eigenvectors $v, w \in \mathcal{H}_s$. Dividing by $\left|\phi_{(\widetilde{s})}(g)\right| \asymp_{\widetilde{s}} \|g\|_{\mathrm{HS}}^{\widetilde{s}-1}$ we obtain

$$\left|\left\langle \gamma_g^s v, w\right\rangle\right| \ll_{\widetilde{s},\varepsilon} \|v\|\|w\|\|g\|_{\mathrm{HS}}^{-\widetilde{s}+\varepsilon}.$$

As $\widetilde{s} > \kappa$ we may set $\varepsilon = \widetilde{s} - \kappa$ and obtain (9.78) with an implicit constant that only depends on $\kappa$ and $s_\pi$.

Finally we note that $\kappa_\pi > 0$ (and so, equivalently, $p_\pi < \infty$ or $s_\pi < 1$) implies that $\pi$ has spectral gap by Proposition 7.25. Assume for the converse that $\kappa_\pi = 0$ and so, equivalently, that $s_\pi = 1$. However, this means by definition that there exists a sequence $s_n \nearrow 1$ with $\gamma^{s_n} \prec \pi$. By the definition of weak containment and Lemma 9.75 this shows that $\mathbb{1}_G \prec \pi$. Using (for example) Proposition 4.24 and the condition $(\prec_{\mathrm{op}})$ in Theorem 4.30, this shows that $\pi$ cannot have spectral gap. □

**Exercise 9.80.** Suppose the unitary representation $\pi$ of $\mathrm{SL}_2(\mathbb{R})$ is a countable direct sum of irreducible representations. Define

$$s_\pi^\oplus = \sup\{s \in [0,1) \mid s = 0 \text{ or } \gamma^s \text{ is one of the summands of } \pi\}.$$

Show that in this case Theorem 9.78 also holds for $s_\pi^\oplus$ instead of $s_\pi$.

## 9.7 Compact Quotients of SL(2)

We will study in this section the Koopman representation on compact quotients $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ by uniform lattices.

### 9.7.1 Effective Decay of Matrix Coefficients

**Corollary 9.81.** *Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a uniform lattice and $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$. Then the Koopman representation $\pi^X$ of $\mathrm{SL}_2(\mathbb{R})$ has effective decay of matrix coefficients.*

Proof. By Proposition 4.28 the Koopman representation $\pi^X$ has spectral gap. By Theorem 9.78 this implies that $\pi^X$ also has effective decay of matrix coefficients. □

We note that the above result also holds more generally for finite volume quotients $X = \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ by any lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})$. Proving this requires a different argument, for example using an analysis of the so-called Eisenstein series and the cuspidal spectrum. We will not pursue this further here.

### 9.7.2 Complete Decomposability

The following result is special to compact quotients.

**Theorem 9.82.** *Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a uniform lattice and $X = \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$. Then the Koopman representation $\pi^X$ is a countable direct sum of irreducible unitary representations.*

Proof. For the proof it will be useful to first refine part of Proposition 4.28. Indeed we claim that for $\psi \in C_c^\infty(G)$ the convolution operator $\pi_*(\psi)$ maps $L^2(X)$ into $C^\infty(X)$.

To see the claim, let $\psi \in C_c^\infty(X)$ and $h = \exp(t\mathbf{m})$ for some $t \in \mathbb{R}$ and $\mathbf{m} \in \mathfrak{sl}_2(\mathbb{R})$. For $f \in L^2(X)$ we than have

$$\pi_*(\psi)f(xh) = \int_G \psi(g)f(xhg)\,\mathrm{d}m_G(g) = \int_G \psi(h^{-1}g)f(xg)\,\mathrm{d}m_G(g).$$

Using this together with (4.16) we have for $t \in [-1,1]\smallsetminus\{0\}$ that

$$\begin{aligned}\left\|\frac{1}{t}\left((\pi_*(\psi)f)(\cdot\exp(t\mathbf{m})) - (\pi_*(\psi)f)(\cdot)\right) - \pi_*\left(\lambda_\partial(\mathbf{m})\psi\right)f\right\|_\infty \\ = \left\|\pi_*\left(\frac{1}{t}\left(\lambda_{\exp(t\mathbf{m})}\psi - \psi\right) - \lambda_\partial(\mathbf{m})\psi\right)\right\|_\infty \\ \ll \left\|\frac{1}{t}\left(\lambda_{\exp(t\mathbf{m})}\psi - \psi\right) - \lambda_\partial(\mathbf{m})\psi\right\|_\infty \|f\|_2\end{aligned}$$

with the implicit constant depending on $\operatorname{supp}\psi$, $\mathbf{m} \in \mathfrak{sl}_2(\mathbb{R})$, and $X$ only. However, as $\psi \in C_c^\infty(G)$ the final supremum norm converges to 0 as $t \to \infty$. This implies that the derivative of $\pi_*(\psi)f$ in the direction of $\mathbf{m}$ is equal to $\pi_*\left(\lambda_\partial(\mathbf{m})\psi\right)f$. Iterating this statement shows that $\pi_*(\psi)f \in C^\infty(X)$ as claimed.

We now show that any non-trivial invariant subspace $\mathcal{V} \subseteq L^2(X)$ contains an irreducible closed subspace. For this let $f_0 \in \mathcal{V}$ be a unit vector and choose from a suitable approximate identity $(\psi_n)$ an element $\psi \in C_c^\infty(G)$ satisfying $\psi \geqslant 0$, $\psi^* = \psi$, $\|\psi\|_1 = 1$ and $\|\pi_*(\psi)f_0 - f_0\| < 1$. This shows that $\pi_*(\psi)|_{\mathcal{V}} \neq 0$. By Proposition 4.28 and invariance of $\mathcal{V}$, we have that $\pi_*(\psi)|_{\mathcal{V}} \colon \mathcal{V} \to \mathcal{V}$ is a compact self-adjoint operator. Let $\mu \neq 0$ be an eigenvalue of $\pi_*(\psi)|_{\mathcal{V}}$ and $\mathcal{V}_\mu$ its associated finite-dimensional eigenspace. By the above claim we have that $\mathcal{V}_\mu = \pi_*(\psi)\mathcal{V}_\mu \subseteq C^\infty(X)$. In particular, $\pi_\partial(\Omega)$ is defined on $\mathcal{V}_\mu$. For $f \in \mathcal{V}_\mu$ we also have

$$\pi_* \pi_\partial(\Omega) f = \pi_\partial(\Omega)\pi_*(\psi) f = \pi_\partial(\Omega)\mu f = \mu\pi_\partial(\Omega) f,$$

which shows that $\pi_\partial(\Omega)\mathcal{V}_\mu \subseteq \mathcal{V}_\mu$. Therefore the restriction of $\pi_\partial(\Omega)$ to the finite-dimensional subspace $\mathcal{V}_\mu$ has a smooth eigenfunction $g \in \mathcal{V}_\mu \subseteq \mathcal{V}$.

Next we decompose $g = \sum_{n\in\mathbb{Z}} g_n$ into a sum of $K$-eigenfunctions and choose $n \in \mathbb{Z}$ so that $g_n \neq 0$. Note that

$$F = g_n = \left(\pi^X|_K\right)_* (\overline{\chi_n})\, g \in \mathcal{V}$$

as $\mathcal{V}$ is invariant, and that

$$F(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\theta} \left(\pi^X_{k_\theta} g\right)(x)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\theta} g(xk_\theta)\,\mathrm{d}\theta$$

can be defined by a parameter integral for $x \in X$. As $g$ is smooth we may differentiate under the integral sign, which when applied to the central Casimir element $\Omega$ shows that $F \in \mathcal{V}$ is also an eigenfunction for $\pi^X_\partial(\Omega)$.

To summarise, we have shown that any closed invariant subspace $\mathcal{V}$ contains a non-zero $K$-eigenvector with $K$-weight $n \in \mathbb{Z}$ so that $F$ is also an eigenvector for $\pi^X_\partial(\Omega)$ and eigenvalue $\lambda \in \mathbb{R}$. By Corollary 9.23 this implies that the restriction of $\pi^X$ to the cyclic subspace $\langle F\rangle_{\pi^X} \subseteq \mathcal{V}$ is irreducible.

The theorem now follows from a simple application of Zorn's lemma. Let

$$\mathcal{C} = \{S \mid S \text{ is a set of pairwise orthogonal irreducible subspaces of } L^2(X)\}$$

ordered by inclusion. It is straightforward to see that any linearly ordered chain in $\mathcal{C}$ has an upper bound, namely the union of the chain. Hence there exists a maximal element in $\mathcal{C}$. That is, there exists a maximal set $S_0$ of pairwise orthogonal subspaces in $L^2(X)$. As $L^2(X)$ is separable, $S_0$ is at most countable. Let $\mathcal{W}$ be the direct sum of the subspaces in $S_0$ and let $\mathcal{V} = \mathcal{W}^\perp$. If $\mathcal{V} \neq 0$ we can apply the above argument to find an irreducible subspace of $\mathcal{V}$, which contradicts maximality of $S_0$. Hence $\mathcal{V} = 0$ and $\mathcal{W} = L^2(X)$ is a direct sum of the irreducible subspaces contained in $S$. Note that each irreducible subspace satisfies that the space of $K$-invariant vectors is at most one-dimensional. As $L^2(X)^K = L^2(X/K) = L^2(\Gamma\backslash\mathbb{H})$ is infinite-dimensional, it follows that $|S_0| = \infty$. □

### 9.7.3 The First Non-trivial Eigenvalue

We conclude our excursion into hyperbolic surfaces by establishing a link between effective decay of matrix coefficients and the first non-trivial eigenvalue for the Laplace–Beltrami operator on the surface. Following the choices made in Section 8.3 concerning $\mathbb{H}$ we define the Laplace–Beltrami operator $\Delta_{\mathrm{hyp}}$ on $\mathbb{H}$ by

$$\Delta_{\mathrm{hyp}} f(z) = y^2 \left(\partial_x^2 f(z) + \partial_y^2 f(z)\right)$$

for $f \in C^\infty(\mathbb{H})$ and $z \in \mathbb{H}$. It can be verified directly that the action of an element $g \in \mathrm{SL}_2(\mathbb{R})$ satisfies

$$(\Delta_{\mathrm{hyp}} f) \circ g = \Delta_{\mathrm{hyp}}(f \circ g)$$

for $f \in C^\infty(\mathbb{H})$ (see Exercise 9.85). This also shows that $\Delta_{\mathrm{hyp}}$ descends to a well-defined operator on $C^\infty(\Gamma\backslash\mathbb{H}$ for any discrete subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ (which we will also obtain in Lemma 9.84 by a different argument). To avoid technicalities concerning cone points of $\Gamma\backslash\mathbb{H}$ we will assume in the following that $\Gamma/\{\pm I\}$ is torsion-free, which implies in particular that no $\gamma \in \Gamma$ other than $\pm I$ has a fixed point in $\mathbb{H}$. Indeed, if $\gamma \cdot z = z$ for some $z \in \mathbb{H}$ we find some $g \in \mathrm{SL}_2(\mathbb{R})$ with $z = g \cdot \mathrm{i}$. This gives $g^{-1}\gamma g \cdot \mathrm{i} = \mathrm{i}$ and hence $g^{-1}\gamma g$ generates a discrete subgroup of $\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\mathrm{i}) = \mathrm{SO}_2(\mathbb{R})$, and so must be a torsion element.

For a compact surface $M = \Gamma\backslash\mathbb{H}$ defined by a uniform lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})$ with no non-central torsion elements the Laplace–Beltrami operator $\Delta_{\mathrm{hyp}}$ has a satisfying spectral theory. Indeed, there exists a sequence of eigenvalues

$$\lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

with $\lambda_n \to \infty$ for $n \to \infty$, and a sequence of eigenfunctions

$$f_0 = \mathbb{1}_M, f_1, f_2, \dots \in C^\infty(M)$$

so that

$$\Delta_{\mathrm{hyp}} f_n = -\lambda_n f_n$$

for all $n \in \mathbb{N}_0$.

The first non-trivial eigenvalue $\lambda_1 > 0$ measures in a sense the amount of connectivity of the surface $M$. For us it is of interest because of the following result.

**Corollary 9.83 (First eigenvalue and almost decay exponent).** *Let $\Gamma$ be a torsion-free uniform lattice in $\mathrm{SL}_2(\mathbb{R})$, let $X = \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$, and let*

$$M = \Gamma\backslash\mathbb{H} \cong X/K.$$

*Then the Koopman representation $\pi^X$ has almost decay exponent*

$$\kappa_{\pi^X} = \begin{cases} 1 & \text{if } \lambda_1 \geqslant \frac{1}{4}, \text{ and} \\ 1 - \sqrt{1 - 4\lambda_1} & \text{if } \lambda_1 < \frac{1}{4}. \end{cases}$$

For the proof we first need to establish a link between $\Delta_{\text{hyp}}$ and the Casimir operator

$$\Omega = \mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{d}^{\circ 2} - \mathbf{k}^{\circ 2}$$

considered so often in this chapter.

**Lemma 9.84.** *Let $\Gamma$, $X$, and $M$ be as in Corollary 9.83. Let $f \in C^\infty(M)$, which we may identify with a smooth $K$-invariant function on $X$. Then*

$$\pi_\partial^X(\Omega) f = f + 4\Delta_{hyp} f.$$

PROOF. We identify $f \in C^\infty(M)$ with the smooth functions $\mathbb{H} \ni z \mapsto f(\Gamma z)$ and $\mathrm{SL}_2(\mathbb{R}) \ni g \mapsto f(\Gamma g \boldsymbol{\cdot} \mathrm{i})$. In order to prove the lemma, we have to first calculate $\pi_\partial^X(\mathbf{a}) f$ and $\pi_\partial^X(\mathbf{d}) f$ as functions of $g \in \mathrm{SL}_2(\mathbb{R})$.

By definition we have

$$\pi_\partial^X(\mathbf{a}) f(g) = \partial_t|_{t=0} f(g a_t \boldsymbol{\cdot} \mathrm{i}) \tag{9.79}$$

for all $g \in \mathrm{SL}_2(\mathbb{R})$ as $a_t = \exp(t\mathbf{a})$. To calculate (9.79) we use the chain rule for differentiation, while always expressing total derivatives using the standard basis of $\mathbb{R}^2$. Hence the total derivative of $f$ at $z = g \boldsymbol{\cdot} \mathrm{i} = x + \mathrm{i}y \in \mathbb{H}$ is simply

$$\left(\partial_x f(z), \partial_y f(z)\right).$$

Next we write $g = u_x \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} k_\theta$ for some $k_\theta \in K$ and apply (8.5) for the Möbius transformation corresponding to $u_x$, $\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}$, and $k_\theta$ respectively, to see that their derivatives are the matrices representing multiplication by the complex numbers 1, $y$, and $\frac{1}{(\sin\theta \mathrm{i} + \cos\theta)^2} = \mathrm{e}^{-2\theta\mathrm{i}}$ respectively. Finally we note that the total derivative of $a_t \boldsymbol{\cdot} \mathrm{i} = \mathrm{e}^{2t}\mathrm{i}$ at $t = 0$ is simply $2\mathrm{i}$, which we identify with $2e_2$. Putting these together, we obtain

$$\begin{aligned} \left(\pi_\partial^X(\mathbf{a}) f\right)(g) &= \left(\partial_x f(z), \partial_y f(z)\right) \cdot y \cdot \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} 2e_2 \\ &= 2y \left(\sin(2\theta)\partial_x f(z) + \cos(2\theta)\partial_y f(z)\right) \end{aligned}$$

where $g = u_x \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} k_\theta$ and $z = g \boldsymbol{\cdot} \mathrm{i}$.

For $\pi_\partial^X(\mathbf{d})(f)$ we only have to change the last step of the calculation. Indeed a simple calculation reveals that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\exp(t\mathbf{d})\boldsymbol{\cdot}\mathrm{i}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}\boldsymbol{\cdot}\mathrm{i}\right) = 2,$$

and so we simply have to replace $2e_2$ as above by $2e_1$. This gives

$$\pi_\partial^X(\mathbf{d})f(g) = 2y\left(\cos(2\theta)\partial_x f(z) - \sin(2\theta)\partial_y f(z)\right).$$

Fix some $z_0 \in \mathbb{H}$. As $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is discrete and has no non-central torsion elements, it follows that there exists some $r > 0$ so that

$$\Psi\colon B_r^{\mathbb{H}}(z_0) \ni z \longmapsto \Gamma z \in M$$

is injective. We let $O = B_r^{\mathbb{H}}(z_0)$ and choose some $F \in C_c^\infty(O)$. Using the chart map $\Psi$ we may also consider $F$ as a function on $M$. We now calculate

$$\begin{aligned}\left\langle \pi_\partial^X(\Omega)f, F\right\rangle &= \left\langle \pi_\partial^X\left(\mathbb{1}_{\mathfrak{E}} + \mathbf{a}^{\circ 2} + \mathbf{d}^{\circ 2} - \mathbf{k}^{\circ 2}\right)f, F\right\rangle\\ &= \langle f, F\rangle - \left\langle \pi_\partial^X(\mathbf{a})f, \pi_\partial^X(\mathbf{a})F\right\rangle - \left\langle \pi_\partial^X(\mathbf{d})f, \pi_\partial^X(\mathbf{d})F\right\rangle.\end{aligned}$$

Using our preparations above for $f$ and $F$ we have that $\pi_\partial^X(\mathbf{a})f\overline{\pi_\partial^X(\mathbf{a})F}$ is equal to

$$2y\Big(\sin(2\theta)\partial_x f(z) + \cos(2\theta)\partial_y f(z)\Big)2y\Big(\sin(2\theta)\partial_x\overline{F(z)} + \cos(2\theta)\partial_y\overline{F(z)}\Big).$$

Next we use the fact that $m = \frac{1}{\pi}\,\mathrm{d}\theta\,\mathrm{dvol} = \frac{1}{\pi}\,\mathrm{d}\theta\frac{1}{y^2}\,\mathrm{d}x\,\mathrm{d}y$. Integrating over $\theta$, we see that $\left\langle \pi_\partial^X(\mathbf{a})f, \pi_\partial^X(\mathbf{a})F\right\rangle$ is equal to

$$2\int_O\Big(\partial_x f(x+\mathrm{i}y)\partial_x\overline{F(x+\mathrm{i}y)} + \partial_y f(x+\mathrm{i}y)\overline{\partial_y F(x+\mathrm{i}y)}\Big)\mathrm{d}x\,\mathrm{d}y.$$

Finally, we use the fact that $F \in C_c(O)$ and apply integration by parts along $x$ and along $y$ separately, which leads to

$$\begin{aligned}-2\int_O\Big(\partial_x^2 f(x+\mathrm{i}y)\overline{F(x+\mathrm{i}y)} + \partial_y^2 f(x+\mathrm{i}y)\,\overline{F(x+\mathrm{i}y)}\Big)\mathrm{d}x\,\mathrm{d}y&\\ = -2\int_O\left(\Delta_{\mathrm{hyp}}f\right)\overline{F}\,\mathrm{dvol}(z).&\end{aligned}$$

The expression $\left\langle \pi_\partial^X(\mathbf{d})f, \pi_\partial^X(\mathbf{d})F\right\rangle$ gives the same result, which shows that

$$\left\langle \pi_\partial^X(\Omega)f, F\right\rangle = \left\langle f + 4\Delta_{\mathrm{hyp}}f, F\right\rangle.$$

As $F \in C_c^\infty(O)$ was arbitrary, we see that $\pi_\partial^X(\Omega)f$ is equal to $f + 4\Delta_{\mathrm{hyp}}f$ on the image of $O$. Varying $z_0 \in \mathbb{H}$ proves the lemma. □

Proof of Corollary 9.83. By Theorem 9.82 we have

$$L^2(X) \cong \bigoplus_{j=0}^{\infty} \mathcal{V}_j \tag{9.80}$$

for countably many irreducible subspaces $\mathcal{V}_j < L^2(X)$. We may assume that $\mathcal{V}_0 = \mathbb{C}\mathbb{1}$, which is the only trivial representation (by transitivity of the action of $\mathrm{SL}_2(\mathbb{R})$ on $X$). For $j \geqslant 1$ and a non-spherical $\mathcal{V}_j$ we apply Corollary 9.29 to see that the restriction of $\pi^X$ to $\mathcal{V}_j$ is tempered and has decay exponent $1 - \varepsilon$ for all $\varepsilon > 0$.

Suppose now $j \geqslant 1$ and $\mathcal{V}_j$ is spherical, and let $f \in \mathcal{V}_j$ be a non-zero $K$-invariant function. Let $\alpha_j$ be the eigenvalue of $\pi_\partial^X(\Omega)|_{\mathcal{V}_j}$ so $\pi_\partial^X(\Omega)f = \alpha_j f$. By Lemma 9.84 this shows that $f$ is also an eigenfunction of $\Delta_{\mathrm{hyp}}$; that is, $\Delta_{\mathrm{hyp}}(f) = -\lambda_n f$ and, moreover,

$$\alpha_j = 1 - 4\lambda_n \leqslant 1 - 4\lambda_1. \tag{9.81}$$

If now $\lambda_1 \geqslant \frac{1}{4}$, then $\alpha_j \leqslant 0$ for all $j \in \mathbb{N}$. However, this implies by Theorem 9.31 that $\mathcal{V}_j$ is isomorphic to a principal series representation and is tempered with almost decay exponent 1. Hence in this case all direct summands of (9.80) with $j \geqslant 1$ are tempered.

Suppose now that $\lambda < \frac{1}{4}$. If $\mathcal{V}_j$ is isomorphic to the complementary series representation $\gamma^{s_j}$, then (9.81) shows that

$$\alpha_j = s_j^2 \leqslant 1 - 4\lambda_1.$$

Hence the complementary series parameter satisfies

$$s_{\pi^X} \leqslant \sqrt{1 - 4\lambda_1}.$$

On the other hand the eigenfunction for the first non-trivial eigenvalue generates an irreducible representation by Corollary 9.23, which must be a complementary series representation for parameter $\sqrt{1 - 4\lambda_1}$. Therefore

$$s_{\pi^X} = \sqrt{1 - 4\lambda_1}.$$

Theorem 9.78 and Exercise 9.80 show that $\pi^X$ has almost decay exponent

$$\kappa_{\pi^X} = 1 - \sqrt{1 - 4\lambda_1}.$$

□

**Exercise 9.85.** Prove that $(\Delta_{\mathrm{hyp}} f) \circ g = \Delta_{\mathrm{hyp}}(f \circ g)$ for $f \in C^\infty(\mathbb{H})$ and $g \in \mathrm{SL}_2(\mathbb{R})$.