Chapter 1
Unitary Representations

In this chapter we develop the initial theory of unitary representations, and discuss the principal examples and constructions.

1.1 Unitary Representations

1.1.1 Why Study Unitary Representations?

We start by offering several different answers to the question raised in the title of the subsection, in an order influenced by the interests of the authors — and with many important details ignored for the moment.

(1) One of the central objects of study in ergodic theory is a measure-preserving action $(g, x) \mapsto g{\boldsymbol\cdot}x$ of a topological group G on a measure space (X, μ). One of the main tools used to study these systems is the associated linear action π of G on the space of functions $f\colon X \to \mathbb{C}$ defined by

$$\big(\pi_g(f)\big)(x) = f(g^{-1}{\boldsymbol\cdot}x).$$

The assumption that the action[†] preserves the measure μ implies that

$$\|\pi_g(f)\|_2 = \|f\|_2$$

for all $f \in L^2_\mu(X)$, so that π defines a unitary representation of G on the Hilbert space $\mathcal{H}_\pi = L^2_\mu(X)$.

[†] The reader is aware that the symbol π has a more illustrious conventional meaning, as *quantitas in quam cum multiflicetur diameter, proveniet circumferencia* (the quantity which, when the diameter is multiplied by it, gives the circumference). We will have to use it in both meanings, often in close proximity. Both uses are so standard that we can only apologise, and hope the context provides clarity.

(2) Modern number theory uses unitary representations heavily, as many problems in number theory have a naturally arising large group of symmetries (this group of symmetries may be intrinsic to the problem but not immediately evident). The unitary representations arising then often become associated to certain L-functions. The most basic examples of these are Dirichlet L-functions, which are associated to Dirichlet characters. At the same time, a character $\chi\colon G \to \mathbb{S}^1$ on a topological group G gives rise to the most basic unitary representation of G on the Hilbert space $\mathcal{H}_\chi = \mathbb{C}$, defined by

$$v \overset{g}{\longmapsto} \chi(g)v$$

for all $g \in G$ and $v \in \mathbb{C}$.

(3) Given a problem in $\mathbb{R}^d$ with rotational symmetry, for example a partial differential equation involving the Laplace operator on a sphere or a ball, the unitary representation theory of the group $\mathrm{SO}_d(\mathbb{R})$ may help to reduce the problem to a potentially easier lower-dimensional sub-problem.

(4) In physics many different groups of symmetries appear naturally, also in surprising places. For instance, the representation theory of Lie groups may be used to explain data concerning the hydrogen atom and subatomic particles.

(5) A beautiful setting that combines all of the topics above (and others) is the Quantum Unique Ergodicity (QUE) conjecture of Rudnick and Sarnak (we refer to the original paper [58] and a survey of Sarnak [60]). The uncertainty principle in quantum physics states that a quantum particle whose position is known very precisely must have a large uncertainty in its momentum, and *vice-versa*. The QUE conjecture, when specialised to the setting of a hyperbolic universe M of finite volume, goes much further. It states that if the energy of a quantum particle is known and very large, then the particle's position and direction of movement are not simply highly uncertain but in fact are nearly equidistributed in the unit tangent bundle of M (that is, nearly equidistributed in both position and direction). In mathematical terms, the conjecture concerns the eigenfunctions $\phi_1, \phi_2, \dots$ of the Laplacian Δ on M with corresponding eigenvalues $\lambda_1, \lambda_2, \dots$ with $|\lambda_j| \to \infty$ as $j \to \infty$. If vol denotes the volume on M, and each ϕ_j is normalized to have $\|\phi_j\|_2 = 1$, then the conjecture states in particular that $|\phi_j|^2 \,\mathrm{dvol} \to \,\mathrm{dvol}$ in the weak* topology as $j \to \infty$. The only case of the conjecture known to hold concerns arithmetically defined hyperbolic surfaces, in which case the conjecture is called the Arithmetic Quantum Unique Ergodicity (AQUE) conjecture. The arithmetic nature of the space here gives rise to additional symmetries, and in these cases the proof of equidistribution by Lindenstrauss [46] and Soundararajan [64] combines the theory of unitary representations for $\mathrm{SL}_2(\mathbb{R})$, number theory, and ergodic theory.

1.1.2 Standing Assumptions

We begin our formal discussion by recalling that a *topological group* is a group G that carries a topology with respect to which the maps $(g,h) \mapsto gh$ and $\imath\colon g \mapsto g^{-1}$ are continuous as maps $G \times G \to G$ and $G \to G$ respectively. A *locally compact* (*compact*, *connected*, and so on) group is a topological group for which the topological space is locally compact (compact, connected, and so on). We recall from the preface that, throughout, any compact or locally compact space is assumed to be Hausdorff. We similarly extend algebraic properties to topological groups.

Let us make the following simple observation in the setting of topological groups. If H, G are topological groups and $\Psi\colon H \to G$ is a homomorphism, then Ψ is continuous if and only if Ψ is continuous at the identity $e \in H$. In fact, if Ψ is continuous at the identity the composition $H \to H \to G \to G$ defined by

$$h \longmapsto hh_0^{-1} \longmapsto \Psi(hh_0^{-1}) \longmapsto \Psi(hh_0^{-1})\Psi(h_0)$$

is continuous at $h_0 \in H$. However, this map equals Ψ, and we obtain continuity of Ψ at any $h_0 \in H$, as claimed.

In pursuit of our interests in Lie-theoretic, dynamical, and number-theoretic applications of representation theory, we limit the generality to what is necessary as it simplifies some discussions and reduces the prerequisites. For instance, we will use the following assumptions and notation:

- G (sometimes H) denotes a locally compact, σ-compact, metric group equipped with a left-invariant Haar[†] measure denoted m_G (or m if there is only one group involved). As a cryptic reminder of the standing assumptions we often speak of 'the group G'. The identity of a multiplicative group will be denoted e, except in the case of explicitly presented matrix groups where the identity will be denoted I. For an abelian group G we will often use the conventional additive notation with identity denoted 0. Elements of the groups will be denoted by g, h and occasionally also by k, ℓ, or t. The function spaces $L^p_m(G)$ will be denoted $L^p(G)$ when the measure is clear, and we write $\int f \,\mathrm{d}m$ as shorthand for $\int_G f(g)\,\mathrm{d}m(g)$ when the domain of integration is clear.
- X (sometimes Y) denotes a locally compact, σ-compact, metric space often carrying a locally finite Borel measure μ (or ν). Measurability of sets or maps is always understood with respect to the Borel σ-algebra $\mathcal{B}_X$ on X. We will again write $\int f\,\mathrm{d}\mu$ when the domain of integration is clear from the context.
- For a continuous G-action on X, we denote the action of $g \in G$ by

$$X \ni x \longmapsto g\raisebox{0.2ex}{\scriptsize$\bullet$}x \in X.$$

[†] We refer to [21, Sec. 10.1] for a discussion of Haar measures.

As usual 'action' means that $e\cdot x = x$ for all $x \in X$ and $(gh)\cdot x = g\cdot(h\cdot x)$ for all $g, h \in G$ and $x \in X$, and 'continuous' means that the map

$$G \times X \ni (g, x) \longmapsto g\cdot x \in X$$

is continuous.

- $\mathcal{H}$ denotes a separable complex Hilbert space, with elements written u, v, or w (and occasionally a, b), and $\mathrm{B}(\mathcal{H})$ denotes the space of bounded linear operators $\mathcal{H} \to \mathcal{H}$. We will write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the inner product and norm unless we want to emphasise the Hilbert space, in which case we write $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$.
- We will also use j, k, ℓ, m, n as indices.

1.1.3 Definition of Unitary Representations

Definition 1.1 (Unitary representation). A *unitary representation* π of a topological group G on a complex Hilbert space $\mathcal{H}_\pi$ is a map $\pi\colon G \to \mathrm{B}(\mathcal{H}_\pi)$ such that:

(1) π is a *representation*, meaning that $\pi(e) = I$ is the identity on $\mathcal{H}_\pi$ and $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$;
(2) π is *unitary*, meaning that $\pi(g)^* = \pi(g)^{-1} = \pi(g^{-1})$ for all $g \in G$, where $\pi(g)^*$ denotes the adjoint of $\pi(g)$; and
(3) π is *continuous* with respect to the strong operator[†] topology on $\mathrm{B}(\mathcal{H}_\pi)$, meaning that for any $v \in \mathcal{H}_\pi$ the map $G \ni g \mapsto \pi(g)v \in \mathcal{H}_\pi$ is continuous.

As in the definition above, we will decorate the Hilbert space with the letter denoting the representation as a subscript, so that it is clear that π is a unitary representation on $\mathcal{H}_\pi$, ρ is a unitary representation on $\mathcal{H}_\rho$, and so on. We will also write $\pi_g = \pi(g)$ for $g \in G$ as it reduces the number of parenthesis needed. To avoid trivialities we will always assume that $\mathcal{H}_\pi \neq \{0\}$.

One of the fundamental classes of unitary representations comes from the following notion. A *unitary character on* G is a continuous group homomorphism

$$\chi\colon G \longrightarrow \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\},$$

which defines a unitary representation of G on $\mathcal{H}_\chi = \mathbb{C}$ by defining the action of $g \in G$ to be the map $v \mapsto \chi_g v = \chi(g)v$ for all $v \in \mathcal{H}_\chi$. We will also encounter more general[‡] *characters*, which are continuous group homomorphisms from G to the multiplicative group $\mathbb{C}^\times = \{z \in \mathbb{C} \mid z \neq 0\}$.

[†] The uniform (that is, operator) norm topology cannot be used in this context as many natural examples would not satisfy this requirement.

[‡] In Chapters 5, 6 and 9 other generalizations will be important.

As we will discuss in Section 2.1, unitary characters are important and exist in great abundance for abelian groups. However, for a non-abelian group G it may happen[†] that the only unitary character of G is the *trivial character* $\mathbb{1}_G(g) = \mathbb{1}(g) = 1$ for all $g \in G$ associated to the *trivial representation* $\mathbb{1}_G$ defined by

$$\mathbb{C} \ni v \stackrel{g}{\longmapsto} \mathbb{1}_G(g)v = v$$

for all $g \in G$. More generally, we call a unitary representation π *trivial* if it satisfies $\pi_g = I$ for every $g \in G$.

We finish this section with a few more basic notions. Given a unitary representation π of a topological group G, if a subspace $\mathcal{V} < \mathcal{H}_\pi$ is π_g-invariant for each $g \in G$, then we will simply say that $\mathcal{V}$ is *π-invariant.* It is clear that if $\mathcal{V}$ is invariant, then its closure $\overline{\mathcal{V}}$ is also invariant (as π_g is a continuous operator for every $g \in G$). This is a rather simple notion, but is fundamental for many of the following discussions.

We also say that a vector $v \in \mathcal{H}_\pi$ is *π-invariant* if $\pi_g v = v$ for all $g \in G$. We denote the subspace of invariant vectors by $\mathcal{H}_\pi^G$; it is the maximal subspace of $\mathcal{H}_\pi$ with the property that the restriction of π to $\mathcal{H}_\pi^G$ is trivial.

1.2 First Examples and Results

1.2.1 Continuous Actions and the Regular Representation

As indicated in Section 1.1.1, one important motivation for the study of unitary representations is the following set-up.

Definition 1.2. Suppose the group G acts continuously[‡] on the space X. We say that G *preserves a measure μ on X*, or equivalently that μ is an *invariant measure* if $\mu(g^{-1}\bullet B) = \mu(B)$ for all measurable $B \subseteq X$ and $g \in G$.

Proposition 1.3 (Koopman representation). *Suppose that G acts continuously on X preserving a locally finite measure μ on X. Then*

$$\bigl(\pi_g^X(f)\bigr)(x) = f(g^{-1}\bullet x)$$

for $x \in X$, $f \in L^2_\mu(X)$, and $g \in G$, defines a unitary representation π^X of G on $\mathcal{H}_{\pi^X} = L^2_\mu(X)$.

The reader is invited to work out the proof of this proposition (see below for a generalization). An important special case of this is given in the following definition.

[†] In particular, because the kernel of any character is a normal subgroup of G, and G may be simple (see also Exercise A.1 as well as Exercises 1.76, 1.79, and 1.81).

[‡] As explained in Section 1.1.2, we assume implicitly that G and X are locally compact, σ-compact, and metric.

Definition 1.4 (Regular representation). The (left) regular representation $\lambda = \lambda^G$ of G is defined by

$$\lambda_{g_0}(f)(g) = f(g_0^{-1}g)$$

for all $f \in \mathcal{H}_\lambda = L^2_m(G)$ and $g_0, g \in G$.

Notice that the (left) regular representation is indeed a special case of the Koopman representation by letting G act on itself by left multiplication. If G is unimodular (that is, if the left Haar measure is also right-invariant) then we can also use right multiplication to define similarly the right-regular representation. However, in general right multiplication only preserves the measure class of the left Haar measure but not the measure itself. We recall that measures μ and ν on X define the *same measure class* when $\mu(A) = 0$ if and only if $\nu(A) = 0$ for all measurable $A \subseteq X$. Hence the right-regular representation, and many other examples, motivate a generalization of Proposition 1.3.

For this let us first recall that for a measurable map $T\colon X \to Y$ and a measure μ on X the formula

$$T_*\mu(B) = \mu(T^{-1}B),$$

for all Borel measurable $B \in \mathcal{B}_Y$, defines the *push-forward measure* $T_*\mu$ on Y. Using monotone convergence, the substitution rule

$$\int_Y f(y)\,\mathrm{d}T_*\mu(y) = \int_X f(T(x))\,\mathrm{d}\mu(x) \tag{1.1}$$

for any measurable $f \geqslant 0$ follows quickly from the definition. We note that if Y is equipped with a measure ν, then T is called *measure-preserving* if $T_*\mu = \nu$.

Moreover, let us recall that a measure ν on X is *absolutely continuous* with respect to another measure μ on X if $\mu(N) = 0$ implies $\nu(N) = 0$ for any $N \in \mathcal{B}_X$. For σ-finite measures (and hence, in particular, for locally finite measures on the σ-compact X) this is equivalent to the existence of a measurable non-negative function on X, the *Radon–Nikodym derivative* $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, with the property that

$$\int f \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu = \int f\,\mathrm{d}\nu$$

for any measurable function $f \geqslant 0$ (see for instance [21, Prop. 3.29]). The Radon–Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is in these settings also uniquely determined μ-almost surely.

Finally, a measurable map $T\colon X \to X$ *preserves the measure class* of μ if $T_*\mu$ and μ define the same measure class.

Proposition 1.5 (Unitarily normalized representation). *Let G act continuously on X, and let μ be a locally finite measure on X. Suppose that for every $g \in G$ the measure $g_*\mu$ is absolutely continuous with respect to μ.*

Suppose also that $c\colon G \times X \to \mathbb{C}^\times$ *is continuous and satisfies the* cocycle equation

$$c(g_1 g_2, x) = c(g_1, x) c(g_2, g_1^{-1} \bullet x), \tag{1.2}$$

and its squared absolute value equals the Radon–Nikodym derivative

$$\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu} = |c(g, \cdot)|^2. \tag{1.3}$$

Then the normalized *(and possibly* twisted*) representation* π^c *defined by*

$$\left(\pi_g^c f\right)(x) = c(g, x) f(g^{-1} \bullet x) \tag{1.4}$$

for $f \in \mathcal{H}_{\pi^c} = L^2_\mu(X)$*,* $g \in G$*, and* $x \in X$*, is a unitary representation.*

As the proof will show, building the square root

$$|c(g, x)| = \left(\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x)\right)^{\frac{1}{2}}$$

of the Radon–Nikodym derivative into the definition (1.4) is necessary to obtain a unitary representation, and we will refer to this as the (*unitarily*) *normalized Koopman representation.* Allowing the function c to be complex-valued gives us more flexibility in constructing unitary representations. We will refer to the case where c is indeed complex-valued as a *twisted normalized representation.* We again note that we do not use the minimal assumptions if some stronger assumptions (in this case, continuity of c) are more convenient and sufficient for our purposes (see also Exercise 1.8).

Proof of Proposition 1.5. For simplicity, we write π for π^c. For a function f on X, $g \in G$, and $x \in X$, we define $\pi_g(f)(x) = c(g, x) f(g^{-1} \bullet x)$ as in the proposition. Now calculate

$$\begin{aligned} \pi_{g_1}\left(\pi_{g_2}(f)\right)(x) &= c(g_1, x)\pi_{g_2}(f)(g_1^{-1} \bullet x) \\ &= c(g_1, x) c(g_2, g_1^{-1} \bullet x) f(g_2^{-1} g_1^{-1} \bullet x) \\ &= c(g_1 g_2, x) f((g_1 g_2)^{-1} \bullet x) = \pi_{g_1 g_2}(f)(x) \end{aligned}$$

for $g_1, g_2 \in G$, by (1.2). If $\|f\|_2 < \infty$ and $g \in G$, then we also have

$$\begin{aligned} \|\pi_g(f)\|_2 &= \int_X |c(g, x)|^2 |f(g^{-1} \bullet x)|^2 \,\mathrm{d}\mu(x) \\ &= \int_X |f(g^{-1} \bullet x)|^2 \,\mathrm{d}g_*\mu(x) = \int_X |f(x)|^2 \,\mathrm{d}\mu(x) \end{aligned}$$

by (1.3) and the substitution rule (1.1). This shows that $\pi\colon G \to \mathrm{B}\big(L^2_\mu(X)\big)$ is a homomorphism and that π_g is unitary for every $g \in G$.

It remains to show the continuity requirement. Fix $f_0 \in L^2_\mu(X)$, $\varepsilon > 0$, and $g_0 \in G$. By [21, Prop. 2.51] there exists some $f \in C_c(X)$ with

$$\|f - f_0\|_2 < \varepsilon.$$

Let U be a compact neighbourhood of $g_0 \in G$ and define $K = U\bullet \operatorname{supp} f$. Notice that

$$U \times K \ni (g,x) \longmapsto c(g,x)f(g^{-1}\bullet x)$$

is uniformly continuous and that $f(g^{-1}\bullet x) = 0$ for $g \in U$ and $x \in X\smallsetminus K$. Suppose now that $g \in U$ is sufficiently close to g_0 to ensure that

$$\left|c(g,x)f(g^{-1}\bullet x) - c(g_0,x)f(g_0^{-1}\bullet x)\right| < \varepsilon\mu(K)^{-\frac{1}{2}}$$

for every $x \in K$. Integrating the square of this inequality over K gives

$$\left\|\pi_g(f) - \pi_{g_0}(f)\right\| < \varepsilon$$

for all $g \in U$. Together we obtain

$$\|\pi_g(f_0)-\pi_{g_0}(f_0)\|_2 \leqslant \|\pi_g(f_0-f)\|_2+\|\pi_g(f)-\pi_{g_0}(f)\|_2+\|\pi_{g_0}(f-f_0)\|_2 < 3\varepsilon$$

for all $g \in U$, which gives the proposition. □

Example 1.6. Let $\mathbb{S}^1$ be the one-dimensional unit circle in $\mathbb{R}^2$, endowed with the normalized length measure m. For $g \in \mathrm{SL}_2(\mathbb{R})$ and $v \in \mathbb{S}^1$ we define the action $g\bullet v = \frac{1}{\|gv\|}gv$. We claim that every $g \in \mathrm{SL}_2(\mathbb{R})$ preserves the measure class of m, and that

$$\frac{\mathrm{d}g_*m}{\mathrm{d}m}(v) = \|g^{-1}v\|^{-2} \tag{1.5}$$

is a continuous version of the Radon–Nikodym derivative for $g \in \mathrm{SL}_2(\mathbb{R})$ and $v \in \mathbb{S}^1$.

To see the claim, let $m_{\mathbb{R}^2}$ denote the two-dimensional Lebesgue measure on $\mathbb{R}^2$, and use polar coordinates to see that

$$m(B) = \tfrac{1}{\pi}m_{\mathbb{R}^2}\bigl((0,1)B\bigr)$$

for all measurable $B \subseteq \mathbb{S}^1$. For $g \in \mathrm{SL}_2(\mathbb{R})$, this gives

$$\begin{aligned} g_*m(B) = m(g^{-1}\bullet B) &= \tfrac{1}{\pi}m_{\mathbb{R}^2}\bigl((0,1)(g^{-1}\bullet B)\bigr)\\ &= \tfrac{1}{\pi}m_{\mathbb{R}^2}\bigl(\{tv_1 \mid g\bullet v_1 \in B, t \in (0,1)\}\bigr)\\ &= \tfrac{1}{\pi}m_{\mathbb{R}^2}\bigl(\{v \in \mathbb{R}^2\smallsetminus\{0\} \mid \|v\| \leqslant 1 \text{ and } \tfrac{1}{\|gv\|}gv \in B\}\bigr). \end{aligned}$$

Next we use the fact that $g \in \mathrm{SL}_2(\mathbb{R})$ preserves $m_{\mathbb{R}^2}$, together with the substitution $w = g\bullet v$, to obtain

$$g_*m(B) = \tfrac{1}{\pi} m_{\mathbb{R}^2}\big(\{w \in \mathbb{R}^2\smallsetminus\{0\} \mid \|g^{-1}w\| \leqslant 1 \text{ and } \tfrac{1}{\|w\|}w \in B\}\big).$$

Finally, we set $u = \frac{1}{\|w\|}$ and $r = \|w\|$ so that $\|g^{-1}\cdot w\| \leqslant 1$ is equivalent to $r\|g^{-1}\cdot u\| \leqslant 1$. Using polar coordinates, this leads to

$$g_*m(B) = 2\int_B \int_0^{\|g^{-1}u\|^{-1}} r \,\mathrm{d}r \,\mathrm{d}m(u) = \int_B \|g^{-1}u\|^{-2} \,\mathrm{d}m(u),$$

which proves the claim. We note that, together with Proposition 1.5, this can be used to define unitary representations of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{S}^1)$ that will be studied in greater detail in Chapter 9.

Throughout the text we will give a number of exercises, some of which will have hints in an appendix starting on p. 473; those flagged as 'essential' exercises will be used in a crucial way in the ensuing discussion.

Exercise 1.7. Prove that the function in (1.5) satisfies the cocycle equation (1.2) for any $g_1, g_2 \in \mathrm{SL}_2(\mathbb{R})$ and $v \in \mathbb{S}^1$.

Exercise 1.8. Let G act continuously on X, and let μ be a locally finite measure on X. Suppose that every $g \in G$ preserves the measure class of μ. Show that the Radon–Nikodym derivative $(g, x) \mapsto \frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x)$ can be chosen to be measurable and satisfies the cocycle equation (1.2) for any $g_1, g_2 \in G$, for almost every $x \in X$.

We will upgrade the statement in the last exercise in Exercise 1.11 in Section 1.2.2 to show that the continuity requirement for a unitary representation holds even though c may not be continuous.

1.2.2 The Continuity Requirement

Proving the continuity requirement for a unitary representation is usually not difficult. To simplify this process in the many examples that follow, we extract an abstract principle from the proof of Proposition 1.5. Let us write $\langle S\rangle_{\mathbb{C}}$ for the complex linear hull of a subset S of a given complex vector space.

Lemma 1.9 (Criteria for continuity). *Let $\pi\colon G \to \mathrm{B}(\mathcal{H}_\pi)$ be a representation on a Hilbert space $\mathcal{H}_\pi$ taking unitary values, so that $\pi_{g_1g_2} = \pi_{g_1}\pi_{g_2}$ and $\pi_g^* = \pi_g^{-1} = \pi_{g^{-1}}$ for all $g_1, g_2, g \in G$. To prove the continuity requirement in Definition* 1.1(3) *it suffices to find a subset $D \subseteq \mathcal{H}_\pi$ satisfying one of the following properties:*

(1) *The linear hull $\langle \pi_G D\rangle_{\mathbb{C}}$ is dense in $\mathcal{H}_\pi$ and $G \ni g \mapsto \pi_g(v)$ is continuous for every $v \in D$.*
(2) *The linear hull $\langle D\rangle_{\mathbb{C}}$ is dense and $G \ni g \mapsto \pi_g(v)$ is continuous at the identity $e \in G$ for every $v \in D$.*

Proof. Suppose first $D \subseteq \mathcal{H}_\pi$ is dense and $G \ni g \mapsto \pi_g(v)$ is continuous for all $v \in D$. For a given $w \in \mathcal{H}_\pi$, we can then choose a sequence $v_n \in D$ with $v_n \to w$ as $n \to \infty$. Now notice that by unitarity of π_g for $g \in G$ the functions $G \ni g \mapsto \pi_g(v_n)$ converge uniformly to $G \ni g \mapsto \pi_g(w)$. In fact, we have

$$\sup_{g \in G} \|\pi_g(v_n) - \pi_g(w)\| = \|v_n - w\| \to 0$$

as $n \to \infty$. However, as uniform convergence of a sequence of continuous functions ($G \ni g \mapsto \pi_g v_n$ with $v_n \in D$) implies that the limit ($g \mapsto \pi_g w$ for some $w \in \mathcal{H}_\pi$) is continuous too, we obtain that the representation is continuous.

If D satisfies the assumptions in (1), then continuity of group multiplication in G implies that $G \ni g \mapsto \pi_{gg_0} v = \pi_g(\pi_{g_0} v)$ is continuous for every $g_0 \in G$ and $v \in D$. Continuity of the vector space operations now implies that $G \ni g \mapsto \pi_g(\widetilde{v})$ is continuous for every $\widetilde{v} \in \widetilde{D} = \langle \pi_G D \rangle_{\mathbb{C}}$. Since this set is dense, the above argument applies.

So suppose now that $G \ni g \mapsto \pi_g v$ is continuous at $e \in G$ for every $v \in D$ as in (2). We can again take linear combinations and apply uniform convergence as above to see that this extends to all elements of $\overline{\langle D \rangle_{\mathbb{C}}} = \mathcal{H}_\pi$. It remains to extend the continuity to all of G. So suppose that $w \in \mathcal{H}$ and $g_n \to g \in G$ as $n \to \infty$. Then

$$\pi_{g_n} w - \pi_g w = \pi_g(\pi_{g^{-1} g_n} w - w) \to 0$$

since $g^{-1} g_n \to e$ as $n \to \infty$ and π_g is unitary (and hence continuous). □

Exercise 1.10. Show that the continuity requirement in Definition 1.1(3) can equivalently be expressed in the following way: π is *weakly continuous*, that is continuous with respect to the weak operator topology on $\mathrm{B}(\mathcal{H})$, meaning that for any $u, v \in \mathcal{H}$ the map

$$G \ni g \longmapsto \langle \pi_g u, v \rangle \in \mathbb{C}$$

is continuous.

Exercise 1.11. (a) Suppose for a homomorphism π from G into the group of unitary operators on a separable† Hilbert space $\mathcal{H}$ that for every $v \in \mathcal{H}$ the map $G \ni g \mapsto \langle \pi_g v, v \rangle$ is measurable. Show that this implies that π is a unitary representation (satisfying the continuity requirement in Definition 1.1).
(b) We suppose now as in Exercise 1.8 that G acts continuously on X and preserves the measure class of a locally finite measure μ on X. Show that the unitarily normalized Koopman representation π is a unitary representation by applying part (a).

† We emphasize this standing assumption as it is crucial for the problem. For example, if $\mathcal{H}$ is defined to be the space of L^2 functions on $\mathbb{R}$ with respect to the counting measure on $\mathbb{R}$, then the assumed measurability holds even though the representation is not continuous.

1.2.3 An Aside on Haar Measures

We wish to discuss the right-regular representation in greater detail. However, for this we have to recall a few more facts about Haar measures (see [21, Sec. 10.1] for the details).

One fundamental property is that the left Haar measure $m = m_G$ is unique up to a scalar multiple. If now $\theta\colon G \to G$ is an automorphism of G (as a topological group) then $\theta_* m$ is again a left Haar measure, since:

- $(\theta_* m)(K) = m\left(\theta^{-1}K\right) < \infty$ for every compact set $K \subseteq G$;
- $(\theta_* m)(O) = m\left(\theta^{-1}(O)\right) > 0$ for every non-empty open set $O \subseteq G$; and
- $(\theta_* m)(gB) = m\left(\theta^{-1}(gB)\right) = m\left(\theta^{-1}(g)\theta^{-1}B\right) = m\left(\theta^{-1}B\right) = \theta_* m(B)$ for every $g \in G$ and Borel measurable $B \subseteq G$.

Hence, by the uniqueness of Haar measure, we have $\theta_* m = \mathrm{mod}(\theta) m$ for some scalar $\mathrm{mod}(\theta) > 0$.

Applying this in the case of an inner automorphism $\theta_{g_0}\colon G \to G$ defined by $\theta_{g_0}(g) = g_0 g g_0^{-1}$, we obtain the *modular character*, a function

$$\Delta_G\colon G \longrightarrow \mathbb{R}_{>0}$$

defined by $\Delta_G(g_0) = \mathrm{mod}(\theta_{g_0})$ for all $g_0 \in G$. When the group is clear from the context we will usually write $\Delta = \Delta_G$. Also note that G is unimodular if and only if Δ is equal to the constant function $\mathbb{1}_G$.

Lemma 1.12 (Modular character). *The modular character is a continuous group homomorphism $\Delta = \Delta_G\colon G \to \mathbb{R}_{>0}$ satisfying*

$$\int f(g g_0^{-1})\,\mathrm{d}m(g) = \Delta(g_0) \int f\,\mathrm{d}m$$

or, equivalently,

$$\Delta(g_0) \int f(g g_0)\,\mathrm{d}m(g) = \int f\,\mathrm{d}m$$

for any non-negative measurable or integrable function f on G.

Proof. Let $g_0, g_1 \in G$. Then $\theta_{g_0 g_1}(g) = g_0 g_1 g (g_0 g_1)^{-1} = \theta_{g_0}\big(\theta_{g_1}(g)\big)$ for all $g \in G$, and hence

$$\begin{aligned}\Delta(g_0 g_1) m &= \left(\theta_{g_0 g_1}\right)_* m = \left(\theta_{g_0}\right)_* \left(\theta_{g_1}\right)_* m \\ &= \left(\theta_{g_0}\right)_* \left(\Delta(g_1) m\right) = \Delta(g_0)\Delta(g_1) m.\end{aligned}$$

To prove continuity of Δ, fix some compact neighbourhood K of $e \in G$ and some positive ε. By regularity of m, there exists some open set $U \supseteq K$ with

$$m(U) < (1+\varepsilon)m(K).$$

By continuity of the group operations and compactness of K, there exists some neighbourhood $V = V^{-1}$ of $e \in G$ such that $VKV \subseteq U$. This now implies that for any $g_0 \in V$ we have

$$\Delta(g_0) = \frac{\left(\theta_{g_0}\right)_* m(K)}{m(K)} = \frac{m(g_0^{-1}Kg_0)}{m(K)} \leqslant \frac{m(U)}{m(K)} < 1+\varepsilon,$$

and by symmetry of V we also have $\Delta(g_0) > (1+\varepsilon)^{-1}$. As $\varepsilon > 0$ was arbitrary, this gives continuity of Δ at $e \in G$, and hence continuity by the argument on p. 3.

Now let f be a non-negative measurable or integrable function on G. Then

$$\begin{aligned}\int f(gg_0^{-1})\,\mathrm{d}m(g) &= \int f(g_0gg_0^{-1})\,\mathrm{d}m(g) = \int f(\theta_{g_0}g)\,\mathrm{d}m(g)\\ &= \int f\,\mathrm{d}(\theta_{g_0})_*m = \Delta(g_0)\int f\,\mathrm{d}m,\end{aligned}$$

which implies the lemma. □

Sometimes we will need to use the right Haar measure on G, which is related to the left Haar measure via the following lemma.

Lemma 1.13 (Right Haar measure). *The right Haar measure $m_G^{(\mathrm{r})}$ on G (denoted by $m^{(\mathrm{r})}$ when the group G is clear from the context) can be defined by*

$$\mathrm{d}m_G^{(\mathrm{r})} = \mathrm{d}\imath_*m_G = \Delta^{-1}\,\mathrm{d}m_G,$$

where $\imath\colon G \to G$ is the inversion map sending g to g^{-1}. Using this definition we have, equivalently,

$$\int f\,\mathrm{d}m^{(\mathrm{r})} = \int f(g^{-1})\,\mathrm{d}m(g) = \int f(g)\Delta(g)^{-1}\,\mathrm{d}m(g)$$

and

$$\int f(g)\,\mathrm{d}m(g) = \int f(g^{-1})\Delta(g)^{-1}\,\mathrm{d}m(g) \tag{1.6}$$

for any measurable non-negative function f on G.

Proof. We begin by noting that $\imath_*m$ is a right Haar measure, since

$$(\imath_*m)\,(Bg) = m(\imath^{-1}(Bg)) = m(g^{-1}\imath^{-1}(B)) = \imath_*m(B)$$

for any measurable $B \subseteq G$ and $g \in G$. Furthermore, $\Delta^{-1}\,\mathrm{d}m$ defines another right Haar measure. Indeed, for a measurable non-negative function f we have

$$\int f(gg_0)\Delta(g)^{-1}\,\mathrm{d}m(g) = \Delta(g_0)\int f(gg_0)\Delta(gg_0)^{-1}\,\mathrm{d}m(g)$$
$$= \int f(g)\Delta(g)^{-1}\,\mathrm{d}m$$

by Lemma 1.12 applied to $f\Delta^{-1}$, which gives the claim.

By uniqueness of right Haar measure, it follows that the measure $\Delta^{-1}\,\mathrm{d}m$ must be a scalar multiple of the measure $\imath_*m$, that is there exists some $\alpha > 0$ so that $\Delta^{-1}\,\mathrm{d}m = \alpha\imath_*m$. We claim that $\alpha = 1$. By continuity of Δ there exists a compact neighbourhood $V = V^{-1}$ of $e \in G$ such that $|\Delta(g)^{-1} - 1| < \varepsilon$ for all $g \in V$. Hence

$$|1-\alpha|m(V) = \left|m(V) - \int_V \Delta^{-1}\,\mathrm{d}m\right| = \left|\int_V (1-\Delta^{-1})\,\mathrm{d}m\right| < \varepsilon m(V).$$

Dividing by $m(V)$, we see that $\alpha = 1$ as claimed.

Applying the above to the function $G \ni g \mapsto f(g^{-1})$ for a non-negative measurable f on G also gives the last formula in the lemma, and so concludes the proof. □

Exercise 1.14. (a) Show that every compact group is unimodular.
(b) Show that if G has finite (left or right) Haar measure, then G is compact.

Example 1.15 (Affine group in one dimension). As an example of a non-unimodular group, we consider the group defined by

$$G = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}.$$

We note that G is called the affine group in one dimension or the '$ax + b$' group because of the natural action of the elements of G on $x \in \mathbb{R}$ via

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ 1 \end{pmatrix}.$$

We will study the unitary representations of G in detail in Chapter 3.

To describe the (left) Haar measure on G, we use the coordinates $a > 0$ and $b \in \mathbb{R}$ for the element

$$g_{a,b} = \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \in G.$$

With these coordinates, the Haar measure m can be defined by the formula

$$\mathrm{d}m(g_{a,b}) = \frac{\mathrm{d}a}{a^2}\,\mathrm{d}b, \tag{1.7}$$

or, equivalently, by

$$\int_G f\,\mathrm{d}m = \int_0^\infty \int_{-\infty}^\infty f\left(\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\right) \mathrm{d}b\,\frac{\mathrm{d}a}{a^2}$$

for any measurable function $f \geqslant 0$. To see this, let $g_{a_0,b_0} \in G$ and calculate

$$\begin{aligned}\int_G f(g_{a_0,b_0}g)\,\mathrm{d}m(g) &= \int_0^\infty\int_{-\infty}^\infty f\left(\begin{pmatrix} a_0 & b_0 \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\right) \mathrm{d}b\,\frac{\mathrm{d}a}{a^2} \\ &= \int_0^\infty\int_{-\infty}^\infty f\left(\begin{pmatrix} a_0a & a_0b+b_0 \\ & 1 \end{pmatrix}\right) \underbrace{\mathrm{d}b}_{=\frac{\mathrm{d}b'}{a_0}}\ \underbrace{\frac{\mathrm{d}a}{a^2}}_{=\frac{a_0'\,\mathrm{d}a'}{(a')^2}} = \int f\,\mathrm{d}m\end{aligned}$$

by using the substitutions $b' = a_0b + b_0$ and $a' = a_0a$. As this holds for all measurable functions $f \geqslant 0$, it follows that (1.7) indeed defines the Haar measure on G.

Next we prove that

$$\Delta(g_{a_0,b_0}) = a_0^{-1} \tag{1.8}$$

for $g_{a_0,b_0} \in G$. For this we calculate $g_{a_0,b_0}^{-1} = \begin{pmatrix} a_0^{-1} & -a_0^{-1}b \\ & 1 \end{pmatrix}$ and consider the integral

$$\begin{aligned}\int f(gg_{a_0,b_0}^{-1})\,\mathrm{d}m(g) &= \int_0^\infty\int_{-\infty}^\infty f\left(\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\begin{pmatrix} a_0^{-1} & -a_0^{-1}b_0 \\ & 1 \end{pmatrix}\right) \mathrm{d}b\,\frac{\mathrm{d}a}{a^2} \\ &= \int_0^\infty\int_{-\infty}^\infty f\left(\begin{pmatrix} aa_0^{-1} & -aa_0^{-1}b_0+b \\ & 1 \end{pmatrix}\right) \underbrace{\mathrm{d}b}_{=\mathrm{d}b'}\ \underbrace{\frac{\mathrm{d}a}{a}}_{=\frac{a_0\,\mathrm{d}a'}{(a'a_0)^2}} = a_0^{-1}\int f\,\mathrm{d}m\end{aligned}$$

with the substitution $b' = -aa_0^{-1}b_0 + b$ and $a' = aa_0^{-1}$. Together with Lemma 1.12, this calculation shows (1.8).

Exercise 1.16. (a) Calculate the left and the right Haar measures of the group

$$G = \left\{\begin{pmatrix} a & b \\ & 1 \end{pmatrix} \mid a \in \mathbb{C}^\times, b \in \mathbb{C}\right\}$$

and its modular character.
(b) Repeat (a) for the group

$$G = \left\{\begin{pmatrix} a_1 & x & z \\ & a_2 & y \\ & & a_3 \end{pmatrix} \mid a_1, a_2, a_3 > 0 \text{ and } x, y, z \in \mathbb{R}\right\}.$$

1.2.4 The Right-Regular Representation

We are now ready to discuss the right-regular representation $\rho = \rho^G$. The action of $g_0 \in G$ on G by right multiplication is defined by

$$R_{g_0} \colon G \ni g \mapsto g g_0^{-1} \in G,$$

and so the *normalized right-regular representation* $\rho_{g_0} \colon L^2(G) \to L^2(G)$ is defined by

$$\rho_{g_0}(f)(g) = \Delta(g_0)^{\frac{1}{2}} f(g g_0). \tag{1.9}$$

Indeed, Lemma 1.12 and uniqueness of the Radon–Nikodym derivative shows that $\Delta(g_0)$ is the (constant) Radon–Nikodym derivative $\frac{\mathrm{d}(R_{g_0})_* m}{\mathrm{d}m}$ for every $g_0 \in G$. This makes the definition above a special case of Proposition 1.5.

As we will discuss later, the right-regular representation is, from an abstract point of view, not all that different from the left-regular representation even when $\Delta \not\equiv 1$.

1.2.5 Isomorphisms and Equivariant Maps

Definition 1.17 (Isomorphism and equivariance). Let π and ρ be unitary representations of G. A bounded operator $B \colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ is *equivariant* if $B \circ \pi_g = \rho_g \circ B$ for all $g \in G$. An equivariant unitary isomorphism $U \colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ is called a *unitary isomorphism* between π and ρ. When such an isomorphism exists, we say that π and ρ are *isomorphic*.

The requirement that U is unitary instead of just a bounded isomorphism is not a severe restriction, due to the following lemma.

Lemma 1.18 (Obtaining equivariant isometries). *Suppose that π and ρ are two unitary representations of G and $B \colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ is bounded, equivariant and injective. Then there is a unitary isomorphism*

$$U \colon \mathcal{H}_\pi \longrightarrow \operatorname{Im} U = \overline{\operatorname{Im} B}$$

between π and the restriction of ρ to the ρ-invariant closed subspace $\operatorname{Im} U$. *In particular, if B has dense image then π and ρ are isomorphic.*

As we will see, this is a consequence of the polar decomposition of bounded operators, which we will essentially prove together with the lemma using the functional calculus for bounded self-adjoint operators.

Proof of Lemma 1.18. Taking the adjoint of the equation defining equivariance gives

$$\pi_{g^{-1}} \circ B^* = B^* \circ \rho_{g^{-1}}$$

for all $g \in G$, since the representations are themselves unitary. In other words B^* is equivariant between ρ and π, which in turn shows that

$$B^*B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$$

is equivariant too, since

$$B^*B\pi_g = B^*\rho_g B = \pi_g B^*B \tag{1.10}$$

for all $g \in G$.

Next notice that $\langle B^*Bv, v\rangle = \langle Bv, Bv\rangle > 0$ for all non-zero $v \in \mathcal{H}_\pi$, since B is assumed to be injective. Thus B^*B is also injective. Moreover, B^*B is self-adjoint.

Finally we claim that B^*B has dense image. Indeed if $v \in \bigl(\operatorname{Im} B^*B\bigr)^\perp$, then $\langle B^*Bv, w\rangle = \langle v, B^*Bw\rangle = 0$ for all $w \in \mathcal{H}_\pi$. However, this implies $B^*Bv = 0$ and so $v = 0$ since B^*B is injective. To summarize, we have shown that B^*B is an equivariant, injective, positive, self-adjoint operator with dense image.

We define the self-adjoint operator $A = \sqrt{B^*B}$ by using the functional calculus (see [21, Sec. 12.4]). Also recall from the definition of the continuous functional calculus that A can be obtained as a uniform limit of a sequence of operators $\bigl(p_n(B^*B)\bigr)$, where p_n is a polynomial in one variable for each $n \geqslant 1$. It is easy to see (by the same argument as in (1.10)) that $p_n(B^*B)\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ is equivariant, which implies the same for A by the uniform convergence. We note that $A^2 = B^*B$ implies that A is injective with dense image, since B^*B is injective with dense image.

We now define U, initially as U_0 on the dense subspace $\operatorname{Im} A$ by

$$U_0(Av) = Bv \in \mathcal{H}_\rho,$$

and note that

$$\|U_0(Av)\|^2 = \langle Bv, Bv\rangle = \langle B^*Bv, v\rangle = \left\langle A^2 v, v\right\rangle = \langle Av, Av\rangle = \|Av\|^2$$

for all $v \in \mathcal{H}_\pi$. Hence U_0 is an isometry on the dense subspace $\operatorname{Im} A$ and therefore has a unique extension to an isometry $U\colon \mathcal{H}_\pi \to \mathcal{H}_\rho$. Note that $\operatorname{Im} U_0 = \operatorname{Im} B$ and that $\operatorname{Im} U = \overline{\operatorname{Im} U_0}$ is a closed subspace of $\mathcal{H}_\rho$, since U is an isometry.

Finally, $v \in \mathcal{H}_\pi$ and $g \in G$ implies that

$$U_0\pi_g Av = U_0 A\pi_g v = B\pi_g v = \rho_g Bv = \rho_g U_0 Av,$$

or $U_0\pi_g = \rho_g U_0$. This gives $U\pi_g = \rho_g U$ for all $g \in G$, as claimed. □

1.2.6 The Regular Representations

To give an example of an isomorphism between unitary representations, we wish to return to the left and normalized right-regular representations.

Using the preparations from Section 1.2.4, we define

$$U(f)(g) = \Delta(g)^{-\frac{1}{2}} f(g^{-1})$$

for $g \in G$ and a function $f \in L^2(G)$, and obtain from (1.5)

$$\|U(f)\|_2^2 = \int |f(g^{-1})|^2 \Delta(g)^{-1} \,\mathrm{d}m(g) = \int |f|^2 \,\mathrm{d}m = \|f\|_2^2$$

and

$$U^2(f)(g) = \Delta(g)^{-\frac{1}{2}} U(f)(g^{-1}) = \Delta(g)^{-\frac{1}{2}} \Delta(g^{-1})^{-\frac{1}{2}} f(g) = f(g).$$

Hence $U\colon L^2(G) \to L^2(G)$ is unitary with $U^2 = I$. Moreover, note that

$$U\left(\lambda_{g_0}(f)\right)(g) = \Delta(g)^{-\frac{1}{2}} \lambda_{g_0}(f)(g^{-1}) = \Delta(g)^{-\frac{1}{2}} f(g_0^{-1} g^{-1})$$

and by (1.9) also

$$\rho_{g_0}\left(U(f)\right)(g) = \Delta(g_0)^{\frac{1}{2}} U(f)(g g_0) = \Delta(g_0)^{\frac{1}{2}} \Delta(g g_0)^{-\frac{1}{2}} f(g_0^{-1} g^{-1})$$

for any $g, g_0 \in G$ and $f \in L^2(G)$. Together, these show that U is equivariant between the left-regular and normalized right-regular representations (always using the left Haar measure).

1.2.7 Containment, Invariant Subspaces, and Direct Sums

Let us introduce the following convenient terminology for the conclusion of Lemma 1.18.

Definition 1.19 (Containment). A unitary representation π of G is said to be *contained in* another unitary representation ρ, written $\pi < \rho$, if there exists a ρ-invariant closed subspace $\mathcal{V}$ of $\mathcal{H}_\rho$ with the property that π is isomorphic to the unitary representation $\rho|_{\mathcal{V}}$ obtained by restricting ρ to $\mathcal{V}$.

We conclude the section with a few basic constructions which we will use often without explicit reference, and whose proofs we leave as exercises.

Essential Exercise 1.20 (Invariant subspaces). Let π be a unitary representation of G and suppose $\mathcal{V} < \mathcal{H}_\pi$ is a closed π-invariant subspace. Then the orthogonal projection $P_{\mathcal{V}}\colon \mathcal{H}_\pi \to \mathcal{V} \subseteq \mathcal{H}_\pi$ is equivariant, the orthogonal complement $\mathcal{V}^\perp$ is also π-invariant, and π is isomorphic to the direct sum $\pi_{\mathcal{V}} \oplus \pi_{\mathcal{V}^\perp}$ of the representations obtained by restricting π to the invariant subspace $\mathcal{V}$, respectively $\mathcal{V}^\perp$ (see Exercise 1.21 below).

Essential Exercise 1.21 (Direct sums of unitary representations). Let S be a finite or countable index set, and let π_n be a unitary representation of G on $\mathcal{H}_n$ for every $n \in S$. Show that $\pi_\oplus(g)(v_n)_{n\in S} = (\pi_n(g)v_n)_{n\in S}$ for $g \in G$ and $(v_n)_{n\in S} \in \mathcal{H}_\oplus$ with

$$\mathcal{H}_\oplus = \bigoplus_{n\in S} \mathcal{H}_n = \left\{ (w_n)_{n\in S} \mid w_n \in \mathcal{H}_n, \left\|(w_n)_{n\in S}\right\|_\oplus^2 = \sum_{n\in S} \|w_n\|^2 < \infty \right\}$$

defines a unitary representation $\pi_\oplus = \bigoplus_{n\in S} \pi_n$, which we will refer to as the *direct sum representation.* For $S = \{1, 2\}$ we will simply write $\pi_1 \oplus \pi_2$ for the direct sum representation.

Implicit in Exercise 1.21 is the construction of the *multiple direct sum* of a given Hilbert space, which we will also denote as follows. If $\mathcal{H}$ is a Hilbert space and $n \in \mathbb{N}$, then $\mathcal{H}^n$ is the *direct sum* (or, equivalently, the direct product) of n copies of $\mathcal{H}$ and

$$\mathcal{H}^\infty = \left\{ v = (v_1, v_2, \dots) \mid v_j \in \mathcal{H} \text{ for all } j \in \mathbb{N} \text{ and } \sum_{j\in\mathbb{N}} \|v_j\|^2 < \infty \right\}$$

is the *infinite direct sum* equipped with the inner product

$$\langle (v_j), (w_j) \rangle_{\mathcal{H}^\infty} = \sum_{j\in\mathbb{N}} \langle v_j, w_j \rangle_{\mathcal{H}}.$$

If π is a unitary representation of the group G we will refer to $\pi^n = \bigoplus_{k=1}^n \pi$ as the nth *direct sum unitary representation* on $\mathcal{H}_\pi^n$ for any $n \in \mathbb{N}$ and, if $n = \infty$, the *infinite direct sum unitary representation* on $\mathcal{H}_\pi^\infty$ as in Exercise 1.21.

1.3 Irreducible Representations and Schur's Lemma

Definition 1.22 (Irreducibility). A unitary representation π of G on a non-trivial Hilbert space $\mathcal{H}_\pi$ is *irreducible* if $\{0\}$ and $\mathcal{H}_\pi$ are the only closed π-invariant subspaces of $\mathcal{H}_\pi$.

It is clear that the unitary representation induced by a unitary character is always irreducible, since in that case $\mathcal{H}_\pi = \mathbb{C}$ has no non-trivial proper subspaces. We will, however, see many higher-dimensional irreducible unitary representations, both finite- and infinite-dimensional, in this volume. An irreducible representation is the analogue of an elementary or indivisible particle — it cannot be decomposed into constituent parts. Somewhat surprisingly, we will have many examples of irreducible representations that we will nonetheless analyze using special subspaces — after restricting attention to certain subgroups of G.

One of the goals for a given group might be to find all of its irreducible unitary representations, which makes the following terminology useful. We will achieve this classification goal for many groups, and will also see that it is more or less impossible for some other groups.

Definition 1.23 (Unitary dual). The *unitary dual* $\widehat{G}$ of a group is defined to be the collection of all equivalence classes $[\pi]$ (with respect to isomorphism) of irreducible unitary representations π of G.

As we have no interest in set-theoretic issues (strictly speaking, the collection of all irreducible unitary representations is not a set, but rather a class), we will be happy if we find a complete set of representatives of the equivalence classes of the unitary dual. Moreover, we will frequently write $\pi \in \widehat{G}$ in order to simplify the notation.

Using direct sums it is clear that one can build other, more general, representations from irreducible ones, but it is much less clear whether any unitary representation can be decomposed into irreducible ones.

Essential Exercise 1.24. Let π be a unitary representation on a finite-dimensional Hilbert space $\mathcal{H}_\pi$. Show that π is isomorphic to a finite direct sum of irreducible unitary representations that are contained in π.

For unitary representations on infinite dimensional Hilbert spaces the answer to the question of whether it can be decomposed into irreducible representations depends on the precise formulation of 'decomposed', and also on the type of group whose representations we are studying. We will return to this question in Sections 2.2, 2.7, 5.3 and **??**.

Essential Exercise 1.25. For $n \geqslant 2$ denote the cyclic group with n elements by $\mathrm{C}_n = \mathbb{Z}/n\mathbb{Z}$, and let k_0 be a generator of C_n. Show that every (unitary or otherwise) representation π of C_n can be split into a sum of n invariant subspaces $\mathcal{V}_t$ for $t = 0, \ldots, n-1$ such that $\pi_{k_0}|_{\mathcal{V}_t}$ is multiplication by $\mathrm{e}^{2\pi \mathrm{i} t/n}$.

For now, we content ourselves with studying the properties of irreducible representations.

1.3.1 Twisting (Irreducible) Unitary Representations

The following result can be useful in constructing additional irreducible representations.

Lemma 1.26 (Tensor products with characters). *Let χ be a unitary character of G, and let π be a unitary representation of G. Then*

$$(\chi \otimes \pi)(g) = \chi_g \pi_g$$

for $g \in G$ defines a unitary representation $\chi \otimes \pi$ on $\mathcal{H}_{\chi\otimes\pi} = \mathcal{H}_\pi$. The representation $\chi \otimes \pi$ is irreducible if and only if π is irreducible.

Proof. We note that $(\chi \otimes \pi)(g)$ is unitary for every $g \in G$, and that

$$(\chi\otimes\pi)(g_1 g_2) = \chi_{g_1 g_2}\pi_{g_1 g_2} = \chi_{g_1}\pi_{g_1}\chi_{g_2}\pi_{g_2} = (\chi\otimes\pi)(g_1)(\chi\otimes\pi)(g_2)$$

for every $g_1, g_2 \in G$, since $\chi_{g_2} \in \mathbb{S}^1$ is a scalar. To see continuity of the representation, we let $v \in \mathcal{H}_\pi$ and suppose that the sequence (g_n) in G satisfies $g_n \to e$ as $n \to \infty$. Then we also have

$$\big\|(\chi \otimes \pi)(g_n)v - v\big\| \leqslant \|\chi_{g_n}\pi_{g_n}v - \chi_{g_n}v\| + |\chi_{g_n} - 1|\|v\| \longrightarrow 0$$

as $n \to \infty$. Finally, note that a closed subspace $\mathcal{V} \subseteq \mathcal{H}_\pi$ is invariant under π_g for $g \in G$ if and only if it is invariant under $(\chi \otimes \pi)(g) = \chi_g \pi_g$ for $g \in G$, which implies the last statement in the lemma. □

1.3.2 Schur's Lemma

The most fundamental property of irreducible representations is traditionally called Schur's lemma. Despite the modest title, it and its corollaries in this section will have widespread usage in the theory to come and will be cited more than twenty times in this volume.

Theorem 1.27 (Schur's lemma(1)). *Suppose that π is an irreducible unitary representation and ρ a unitary representation of G.*

(a) *If $B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ is bounded equivariant, then $B = \alpha I$ for some $\alpha \in \mathbb{C}$.*
(b) *If $B\colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ is bounded equivariant, then $B^*B = \alpha I$ for some $\alpha \geqslant 0$. If in addition $B \neq 0$ (equivalently, if $\alpha > 0$), then a scalar multiple of B is an equivariant isometry, and so π is contained in ρ.*

Proof. Suppose first that $B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ is bounded equivariant with $B = B^*$ but with $B \neq \alpha I$ for all $\alpha \in \mathbb{R}$. By the spectral theorem for self-adjoint operators (see [21, Th. 12.55]) this implies that the spectrum $\sigma(B)$ of the operator B consists of more than just one point. This in turn implies that there exist functions $f_j \in C(\sigma(B))$ with $f_1 f_2 = 0$ and with $\|f_j\|_\infty = 1$ for $j = 1, 2$. By the continuous functional calculus (see [21, Sec. 12.4.1]) there exist operators $f_j(B) \in \mathrm{B}(\mathcal{H}_\pi)$ with $\|f_j(B)\| = 1$ for $j = 1, 2$ and with $f_1(B)f_2(B) = 0$. Moreover, each $f_j(B)$ can be obtained as a a uniform limit of polynomials in B, so $f_j(B)$ is equivariant for $j = 1, 2$. It follows that $\ker f_1(B)$ is closed, π-invariant, non-trivial since

$$\{0\} \subsetneq \operatorname{Im} f_2(B) \subseteq \ker f_1(B),$$

and proper since $f_1(B) \neq 0$. As this contradicts our assumption of irreducibility of π we deduce that $B = \alpha I$ for some $\alpha \in \mathbb{R}$.

If $B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ is bounded and equivariant, then B^*, $\frac{B+B^*}{2}$, and $\frac{B-B^*}{2\mathrm{i}}$ are all bounded and equivariant too. Applying the argument above to $\frac{B+B^*}{2}$, and $\frac{B-B^*}{2\mathrm{i}}$ we see that $B = \alpha I$ for some $\alpha \in \mathbb{C}$ as claimed in (a).

To prove (b), again notice that $B^*\colon \mathcal{H}_\rho \to \mathcal{H}_\pi$ is equivariant, and hence $B^*B\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ is also. By (a) this implies that $B^*B = \alpha I$ for some $\alpha \geqslant 0$ since B^*B is a positive operator. If $\alpha > 0$, then we can replace B by the equivariant isometry $U = \alpha^{-\frac{1}{2}} B$, which gives an isomorphism between $\mathcal{H}_\pi$ and a closed subspace of $\mathcal{H}_\rho$. □

Exercise 1.28 (Converse to Schur's lemma). Let π be a unitary representation of G. Suppose that equivariance of $B \in \mathrm{B}(\mathcal{H}_\pi)$ implies that $B = \alpha I$ for some $\alpha \in \mathbb{C}$. Show that π is irreducible.

Exercise 1.29 (Goursat's lemma for unitary representations). Let π and ρ be irreducible unitary representations of G. Describe all closed invariant subspaces of $\mathcal{H}_\pi \oplus \mathcal{H}_\rho$. Show, in particular, that either there are precisely four closed invariant subspaces (including the trivial one), or there are infinitely many.

1.3.3 Irreducible Representations Leading to Characters

Schur's lemma is all we need for the first (rather abstract) classification result for irreducible unitary representations.

Corollary 1.30 (Characters on the centre). *Let π be an irreducible unitary representation of G. Then π_g is equal to $\chi_g I$ for any element g of the centre*

$$C_G = \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

of G, where $\chi = \chi^\pi\colon C_G \to \mathbb{S}^1$ is a unitary character on the closed subgroup C_G. In particular, if G is abelian, then any irreducible unitary representation is one-dimensional and is induced by a unitary character on G.

Proof. For $g_0 \in C_G$ the operator $B = \pi_{g_0}$ is equivariant since

$$B \circ \pi_g = \pi_{g_0 g} = \pi_{g g_0} = \pi_g \circ B$$

for all $g \in G$. Schur's lemma (Theorem 1.27) implies that $\pi_{g_0} = B = \chi_{g_0} I$ for some scalar $\chi_{g_0} \in \mathbb{C}$. Since B is unitary, $\chi_{g_0} \in \mathbb{S}^1$. For $g_0, g_1 \in C_G$ we also have

$$\left(\chi_{g_0} I\right)\left(\chi_{g_1} I\right) = \pi_{g_0}\pi_{g_1} = \pi_{g_0 g_1} = \chi_{g_0 g_1} I,$$

which shows that $\chi\colon C_G \to \mathbb{S}^1$ is a homomorphism. Finally, if $v \in \mathcal{H}_\pi$ is a unit vector, then

$$\chi_{g_0} = \chi_{g_0}\|v\|^2 = \langle \pi_{g_0} v, v\rangle$$

shows that $g_0 \mapsto \chi_{g_0}$ is continuous on C_G.

If in addition G is abelian, then the continuous character χ constructed above is defined on all of G, and $\pi_g = \chi_g I$ for all $g \in G$. This implies that any subspace is π-invariant. By irreducibility of π, we deduce that $\dim \mathcal{H}_\pi = 1$. □

Even if G is not abelian, the corollary above and the following related definition are useful for classifying unitary representations.

Definition 1.31 (Central characters). Let π be a unitary representation of G such that the restriction of π to the centre is given by $\pi_g = \chi_g I$ for a unitary character χ on C_G and all $g \in C_G$. Then χ is called the *central character of* π.

The following exercise highlights the differences in the use of characters in Lemma 1.26 and Corollary 1.30.

Exercise 1.32. Show that $\mathrm{SL}_2(\mathbb{R})$ has no non-trivial characters, but has a non-trivial centre and a non-trivial central character.

Example 1.33. If G is abelian, H satisfies our standing assumptions, and τ is an irreducible representation of $G \times H$, then G belongs to the centre of $G \times H$ and we may apply Corollary 1.30. It follows that $\tau|_G = \chi I$ for a character χ on G. Let $\rho = \tau|_H$ and extend both χ and ρ trivially to the product $G \times H$. Then $\tau = \chi \otimes \rho$ as in Lemma 1.26 for the (necessarily irreducible) unitary representation $\rho = \tau|_H$ of H.

1.3.4 Unbounded Equivariant Operators

Occasionally we will need a generalization of Schur's lemma for densely defined closed equivariant operators (see [21, Ch. 13] for more background on such operators).

Definition 1.34 (Closed operators). Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces with a subspace $D_T \subseteq \mathcal{H}$, and $T\colon D_T \to \mathcal{H}'$ a linear map. We say that T is a *densely defined* operator from $\mathcal{H}$ to $\mathcal{H}'$ if $\overline{D_T} = \mathcal{H}$, and is *closed* if

$$\mathrm{Graph}(T) = \{(v, Tv) \mid v \in D_T\} \subseteq \mathcal{H} \oplus \mathcal{H}'$$

is a closed subspace.

Definition 1.35 (Closed equivariant operators). Let π and ρ be unitary representations of G. A densely defined closed operator T from $\mathcal{H}_\pi$ to $\mathcal{H}_\rho$ is *equivariant* if D_T is invariant under π and $T \circ \pi_g = \rho_g \circ T$ on D_T.

Corollary 1.36 (Schur's lemma for closed operators). *Let π be an irreducible unitary representation of G and ρ another unitary representation of G. If T is a densely defined closed equivariant operator from $\mathcal{H}_\pi$ to $\mathcal{H}_\rho$, then T is bounded and T is either the zero operator or is a multiple of a unitary isomorphism to a closed subspace of $\mathcal{H}_\rho$. Moreover, if $\rho = \pi$ then $T = \alpha I$ for some $\alpha \in \mathbb{C}$.*

Proof. We prove first that T is in fact defined everywhere, which will imply that T is bounded and will allow us to apply Schur's lemma (Theorem 1.27) for bounded operators.

By equivariance of T, we have that $(u, Tu) \in \mathrm{Graph}(T)$ implies

$$(\pi_g u, \rho_g T u) \in \mathrm{Graph}(T)$$

for all $g \in G$. Since we also assumed that $\mathrm{Graph}(T)$ is closed, it follows that $\mathrm{Graph}(T)$ is a closed $\pi \oplus \rho$-invariant subspace of $\mathcal{H}_\pi \oplus \mathcal{H}_\rho$, which implies that its orthogonal projection operator

$$P_{\mathrm{Graph}(T)} \colon \mathcal{H}_\pi \oplus \mathcal{H}_\rho \longrightarrow \mathrm{Graph}(T) \subseteq \mathcal{H}_\pi \oplus \mathcal{H}_\rho$$

is equivariant. Moreover, the embedding operator $\imath_1 \colon \mathcal{H}_\pi \to \mathcal{H}_\pi \oplus \mathcal{H}_\rho$ and the projection $P_1 \colon \mathcal{H}_\pi \oplus \mathcal{H}_\rho \to \mathcal{H}_\pi$ to the first coordinate are also equivariant from π to $\pi \oplus \rho$ respectively from $\pi \oplus \rho$ to π.

Hence we can apply Schur's lemma to the equivariant bounded operator $P_1 P_{\mathrm{Graph}(T)} \imath_1 \colon \mathcal{H}_\pi \to D_T \subseteq \mathcal{H}_\pi$ defined by

$$P_1 P_{\mathrm{Graph}(T)} \imath_1 \colon v \longmapsto P_{\mathrm{Graph}(T)}(v, 0) = (w, Tw) \longmapsto w$$

(see Figure 1.1). It follows that $P_1 P_{\mathrm{Graph}(T)}(v, 0) = \alpha_0 v$ for some $\alpha_0 \in \mathbb{C}$ and

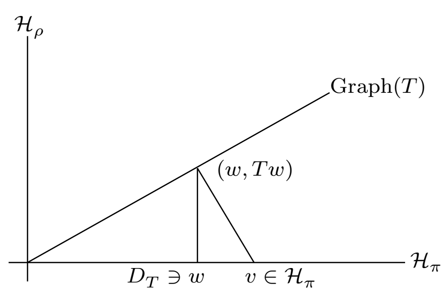

Fig. 1.1: The operator $P_1 P_{\mathrm{Graph}(T)}$.

all $v \in \mathcal{H}$. For a non-zero $v \in D_T$ (which must exist since T is densely defined) we have $(v, 0) \not\perp (v, Tv) \in \text{Graph}(T)$ and so $P_{\text{Graph}(T)}(v, 0) = (w, Tw) \neq 0$, which implies that $w = \alpha_0 v \neq 0$ and so $\alpha_0 \neq 0$. For a general $v \in \mathcal{H}_\pi$ this implies that $P_{\text{Graph}(T)}(v, 0) = (\alpha_0 v, T(\alpha_0 v))$, hence also $\alpha_0 v \in D_T$ and so v lies in D_T. However, this shows that T is a closed operator with domain equal to $D_T = \mathcal{H}_\pi$, and now the closed graph theorem [21, Th. 4.28] implies that T is bounded. Applying Schur's lemma (Theorem 1.27) to this bounded operator gives the corollary. □

The following exercise gives a generalization of Lemma 1.18 to closed equivariant operators.

Exercise 1.37. Suppose π and ρ are unitary representations and T is a densely defined closed equivariant operator from $\mathcal{H}_\pi$ to $\mathcal{H}_\rho$. Show that $\pi|_{(\ker T)^\perp}$ is unitarily isomorphic to $\rho|_{\overline{\operatorname{Im} T}}$, and hence is contained in ρ.

1.3.5 Why is the Hilbert Space Always Complex?

We want to show via a quick example how Schur's lemma in the formulation of Theorem 1.27(a) fails for irreducible representations on *real* Hilbert spaces. For this we define the multiplicative subgroup $G = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$ of the Hamiltonian quaternions

$$\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k},$$

which we will also consider as a real Hilbert space by declaring $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ an orthonormal basis. We define the representation π of G by left-multiplication on $\mathbb{H}$ and obtain that π is unitary and irreducible (since for any non-zero vector $v \in \mathbb{H}$ the vectors $v, \mathbf{i}v, \mathbf{j}v, \mathbf{k}v$ form a real basis of $\mathbb{H}$). Furthermore, for any $b \in \mathbb{H}$ the map $B\colon \mathbb{H} \to \mathbb{H}$ defined by $B\colon v \mapsto vb$ satisfies

$$\pi_g \circ B = B \circ \pi_g$$

for all $g \in G$, and hence Theorem 1.27(a) fails for this irreducible real representation.

In fact Schur's lemma over fields that are not algebraically closed involves in general the study of division algebras over that field. For our purposes and the applications we have in mind, this would only be an algebraic distraction, so we will always work with complex Hilbert spaces.

Exercise 1.38. Show that the representation theory of abelian groups also behaves differently over the reals, by finding a two-dimensional irreducible representation of $\mathrm{C}_3 = \mathbb{Z}/3\mathbb{Z}$ over the reals.

1.3.6 Dihedral Groups*

†As a concrete example of a complete description of the unitary dual of a non-abelian group, we discuss here the dihedral groups.

The dihedral group D_n for $n \geqslant 2$ is defined as the group of all Euclidean symmetries of the regular n-gon in $\mathbb{R}^2$. For convenience we identify $\mathbb{R}^2$ with $\mathbb{C}$, let $M = 0$ be the centre, and let $P_1 = 1$ be a vertex of the regular n-gon. One symmetry is the rotation k_0 through angle $\frac{2\pi}{n}$, defined by $k_0\colon \mathbb{C} \ni z \mapsto \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}} z$, which generates an abelian subgroup $\mathrm{C}_n < \mathrm{D}_n$ isomorphic to $\mathbb{Z}/n\mathbb{Z}$. However, there are also reflections, for example $r_0\colon \mathbb{C} \ni z \mapsto \overline{z}$ belongs to D_n.

More formally, let C_n be the cyclic group of order n generated by $k_0 \in \mathrm{C}_n$ and define the dihedral group D_n as the semi-direct product $\mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{C}_n$ generated by C_n and the involution $r_0 \in \mathbb{Z}/2\mathbb{Z}$ with $r_0 h r_0 = h^{-1}$ for $h \in \mathrm{C}_n$.

We will study irreducible unitary representations of D_n via their restriction to the abelian normal subgroup C_n, leading to two types of representation.

(Old) We say that an irreducible representation π of D_n is an *old representation* with respect to $\mathrm{C}_n \lhd \mathrm{D}_n$ if $\pi|_{\mathrm{C}_n}$ is trivial. It follows that these can also be thought of as unitary representations of $\mathrm{D}_n/\mathrm{C}_n \cong \mathbb{Z}/2\mathbb{Z}$. Thus there are two old unitary representations: the trivial representation $\mathbb{1}$ and the representation defined by the character sign, where

$$\operatorname{sign}(g) = \begin{cases} 1 & \text{if } g \in \mathrm{C}_n, \\ -1 & \text{if } g \in r_0 \mathrm{C}_n. \end{cases}$$

(New) We say that an irreducible representation π of D_n is a *new representation* with respect to $\mathrm{C}_n \lhd \mathrm{D}_n$ if $\pi|_{\mathrm{C}_n}$ is non-trivial. As we will see $\mathcal{H}_\pi$ will be a direct sum of eigenspaces for π_{k_0}. Since π is assumed to be irreducible for D_n, these eigenspaces cannot also be invariant under r_0 unless there is only one.

Construction of New Representations

Let χ be a character of C_n that satisfies $\chi(\mathrm{C}_n) \not\subseteq \mathbb{R}$, so that $\chi \neq \overline{\chi}$. From this (automatically unitary) character, we define a new two-dimensional representation using a very special case of the induced representation construction denoted in this case by $\pi = \operatorname{Ind}_{\mathrm{C}_n}^{\mathrm{D}_n}(\chi)$. We let $\mathcal{H}_\pi = \mathbb{C}^2$ and let C_n act on the first coordinate via the character χ and on the second via $\overline{\chi}$, so

$$\pi_g(\alpha_1, \alpha_2) = (\chi(g)\alpha_1, \overline{\chi}(g)\alpha_2)$$

for all $g \in \mathrm{C}_n$ and $(\alpha_1, \alpha_2) \in \mathbb{C}^2$. For the element r_0 we define

$$\pi_{r_0}(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1)$$

† As explained on page vi, some sections (this being an example) are marked with a * if they are not crucial for the majority of the later chapters.

for all $(\alpha_1, \alpha_2) \in \mathbb{C}^2$. We note that $\pi|_{\mathrm{C}_n}$ and $\pi|_{\langle r_0 \rangle}$ are unitary representations of C_n, resp. $\langle r_0 \rangle \cong \mathbb{Z}/2\mathbb{Z}$. To see that this extends to a representation of D_n, we verify that

$$\begin{aligned} \pi_{r_0} \pi_h \pi_{r_0^{-1}}(\alpha_1, \alpha_2) &= \pi_{r_0} \pi_h (\alpha_2, \alpha_1) \\ &= \pi_{r_0} \big(\chi(h)\alpha_2, \overline{\chi(h)}\alpha_1\big) \\ &= \big(\overline{\chi(h)}\alpha_1, \chi(h)\alpha_2\big) = \pi_{h^{-1}}(\alpha_1, \alpha_2) \end{aligned}$$

for $h \in \mathrm{C}_n$ and $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ and note that this matches the calculation $r_0 h r_0^{-1} = h^{-1}$ for $h \in \mathrm{C}_n$.We now define $\pi(h r_0^j) = \pi(h)\pi(r_0^j)$ for $h \in \mathrm{C}_n$ and $j \in \{0, 1\}$, which implies that

$$\begin{aligned} \pi(h_1 r_0^{j_1})\pi(h_2 r_0^{j_2}) &= \pi(h_1)\pi(r_0)^{j_1}\pi(h_2)\pi(r_0^{j_2}) \\ &= \pi(h_1) \underbrace{\pi(r_0)^{j_1}\pi(h_2)\pi(r_0)^{-j_1}}_{=\pi(r_0^{j_1} h_2 r_0^{-j_1})} \pi(r_0)^{j_1+j_2} \\ &= \pi(h_1 r_0^{j_1} h_2 r_0^{-j_1})\pi(r_0^{j_1+j_2}) = \pi(h_1 r_0^{j_1} h_2 r_0^{j_2}), \end{aligned}$$

for all $h_1, h_2 \in \mathrm{C}_n$ and $j_1, j_2 \in \{0, 1\}$, as required.

To see that this is an irreducible representation, suppose that $\mathcal{V} \subseteq \mathcal{H}_\pi$ is non-trivial and π-invariant. Then it is also invariant under π_{k_0}, which is the diagonal matrix with eigenvalues $\chi(k_0) \neq \overline{\chi(k_0)}$. Since $\mathcal{V}$ is invariant under π_{k_0}, it follows that $\mathcal{V}$ must contain the eigenspace $\mathbb{C} \times \{0\}$ or the eigenspace $\{0\} \times \mathbb{C}$. As π_{r_0} switches these two subspaces, and $\mathcal{V}$ is invariant under π_{r_0}, we must have $\mathcal{V} = \mathbb{C}^2$, and so π is irreducible. We also note that the new unitary representation defined by χ is isomorphic to the unitary representation defined by $\overline{\chi}$.

Suppose now that χ is a character of C_n with $\chi(k_0) = -1$, which is only possible if n is even (as the multiplicative order of $\chi(k_0)$ must be a divisor of n). Then the construction above does not give an irreducible representation (as the diagonal $\{(\alpha, \alpha) \mid \alpha \in \mathbb{C}\}$ and its orthogonal complement are invariant under π). In this case the subgroup $\mathrm{C}_n^2 = \langle k_0^2 \rangle$ is a normal subgroup of D_n with

$$\mathrm{D}_n/\mathrm{C}_n^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and we can extend χ in two ways to a unitary character of D_n. These define two more irreducible unitary representations (corresponding to the diagonal and its orthogonal complement).

Completeness

We now show that the constructions above give the complete list of irreducible unitary representations of D_n. So let π be an irreducible unitary representation of D_n. If $\pi|_{\mathrm{C}_n}$ is trivial, then π is an old representation defined by a representation of $\mathrm{D}_n/\mathrm{C}_n \cong \mathbb{Z}/2\mathbb{Z}$, which is defined by a unitary character

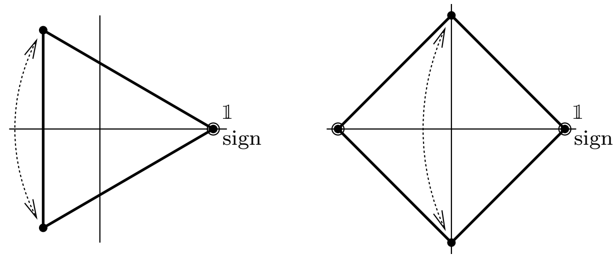

Fig. 1.2: The character of C_n for $n = 3$ and $n = 4$ naturally form an isosceles triangle and square in $\mathbb{C}$ respectively. The real characters give rise to two irreducible unitary representations as indicated by the circles around the dots. The non-real characters arise in pairs, and each pair together gives rise to one two-dimensional unitary representation indicated by the dashed arrows.

by Corollary 1.30. So suppose $\pi|_{C_n}$ is non-trivial. Then (by Exercise 1.25) $\mathcal{H}_\pi$ must have a non-trivial eigenspace $\mathcal{V}_\chi$ for a non-trivial character χ on C_n. For $v \in \mathcal{V}_\chi$ we have $\pi_h v = \chi(h)v$, and so

$$\pi_h \pi_{r_0} v = \pi_{r_0} \pi_{r_0} \pi_h \pi_{r_0} v = \pi_{r_0} \pi_{h^{-1}} v = \overline{\chi(h)} \pi_{r_0} v$$

for all $h \in C_n$. Using the same argument for $\overline{\chi}$ gives that $\pi_{r_0} \mathcal{V}_\chi = \mathcal{V}_{\overline{\chi}}$ is the eigenspace for eigenvalue $\overline{\chi}$.

If now $\chi = \overline{\chi}$, then n is even and $\mathcal{V}_\chi$ is invariant under $\langle C_n, r_0 \rangle = D_n$, and so $\mathcal{V}_\chi = \mathcal{H}_\pi$. Hence π is a representation induced by $D_n / C_n^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and so is defined by a character on this group by Corollary 1.30.

Finally, suppose that $\chi \neq \overline{\chi}$. Then we may choose some unit vector $v \in \mathcal{V}_\chi$ and see that $\mathcal{V} = \mathbb{C}v + \mathbb{C}\pi_{r_0} v$ is invariant under $\pi(C_n)$ and π_{r_0}, and hence must be all of $\mathcal{H}_\pi$. This implies that π is isomorphic to the new two-dimensional unitary representation constructed above.

Generalizations

We will generalize the method used here in an infinite setting in Chapter 3, and prepare for this in Chapter 2, where we will discuss in detail the generalizations of eigenspaces for locally compact abelian groups.

Exercise 1.39 (The affine group over $\mathbb{F}_p$). Let $p \in \mathbb{N}$ be a prime, and define the characteristic p analogue

$$G = \mathbb{F}_p^\times \ltimes \mathbb{F}_p = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\}$$

of the affine group. Show that $\widehat{G}$ consists of $(p-1)$ old representations defined by characters on $\mathbb{F}_p^\times \cong \mathbb{Z}/(p-1)$ and one new irreducible representation on $\mathbb{C}^{p-1}$.

Exercise 1.40 (The symmetric group S_3). (a) Use the above to describe the unitary dual of the symmetric group $S_3 \cong D_3$ explicitly. Note that the alternating subgroup $A_3 \lhd S_3$ corresponds to the normal subgroup $C_3 \lhd D_3$ considered above.

(b) Let V be a complex vector space carrying a linear representation ρ of S_3. Define the *symmetric subspace* $V_{\mathrm{s}} = V^{\mathrm{S}_3}$ of invariant vectors, the *skew-symmetric subspace*

$$V_{\mathrm{ss}} = \{v \in V \mid \rho(\tau)v = \mathrm{sign}(\tau)v \text{ for all } \tau \in \mathrm{S}_3\},$$

and the *cycle-symmetric subspace*

$$V_{\mathrm{cs}} = \{v \in V \mid v + \rho(\sigma)v + \rho(\sigma^2)v = 0\},$$

where $\sigma = (231)$ is the cyclic permutation and generator of A_3. Show that these subspaces are ρ-invariant, and that

$$V = V_{\mathrm{s}} \oplus V_{\mathrm{ss}} \oplus V_{\mathrm{cs}}.$$

Find equivariant projection maps from V to V_{s}, to V_{ss}, and to V_{cs}. Furthermore, show that for any non-zero $v \in V_{\mathrm{s}} \cup V_{\mathrm{ss}}$ the restriction of ρ to $\langle \rho(\mathrm{S}_3)v\rangle$ is isomorphic to one of the old irreducible unitary representations of S_3 described in (a). What happens for $v \in V_{\mathrm{cs}}$?
(c) Apply part (b) to the vector space $V = \{f : \mathbb{R}^3 \to \mathbb{C}\}$ equipped with the representation induced by permutation of the coordinates in $\mathbb{R}^3$. Show that in this case each of the subspaces in part (b) are non-trivial.

For the group S_4 the above procedure can be repeated to a large extent. However, for $n \geqslant 5$ the normal subgroup $\mathrm{A}_n \lhd \mathrm{S}_n$ is simple, and so the procedure discussed here does not apply. We will not discuss the representations of S_n in detail, referring to Fulton and Harris [26, Ch. 4] or Kowalski [41] for the details — but note that many general results about the representation theory of finite groups will follow as a special case of the discussions in Chapter 5 concerning compact groups.

1.4 Convolution Operators

In this section we introduce two Banach algebras related to the group G which will allow us to create additional linear structures associated to unitary representations.

1.4.1 The Banach Algebra $L^1(G)$

The multiplication of $f_1, f_2 \in L^1(G)$ in the Banach algebra $L^1(G)$ is defined by the convolution

$$\begin{aligned} f_1 * f_2(g) &= \int f_1(h) f_2(h^{-1}g)\,\mathrm{d}m(h) \\ &= \int f_1(gk) f_2(k^{-1})\,\mathrm{d}m(k) \qquad (1.11) \\ &= \int f_1(g\ell^{-1}) f_2(\ell) \Delta(\ell)^{-1}\,\mathrm{d}m(\ell) \qquad (1.12) \end{aligned}$$

for any $g \in G$, where we used the measure-preserving substitution $h = gk$ and in (1.12) the possibly non-measure-preserving substitution $\ell = \imath(k) = k^{-1}$ from Lemma 1.13. The defining Banach algebra inequality follows by using Fubini's theorem since

$$\begin{aligned}\|f_1 * f_2\|_1 &= \int |f_1 * f_2(g)| \,\mathrm{d}m(g)\\ &\leqslant \iint |f_1(h)||f_2(h^{-1}g)| \,\mathrm{d}m(h)\,\mathrm{d}m(g)\\ &= \int |f_1(h)| \int |f_2(g)| \,\mathrm{d}m(g)\,\mathrm{d}m(h) = \|f_1\|_1 \|f_2\|_1\end{aligned}$$

(which also proves that the integral defining $f_1 * f_2(g)$ exists for m-almost every $g \in G$). As G is assumed to be locally compact σ-compact metric, the Banach algebra $L^1(G)$ is separable (see [21, Prop. 3.91]).

On occasion we will look at *convolution powers*, that is, powers of functions $f \in L^1(G)$ with respect to convolution. We will use the notation f^{*n} defined inductively by $f^{*1} = f$ and $f^{*(n+1)} = f * f^{*n}$ for $n \in \mathbb{N}$.

Exercise 1.41. Show that $L^1(G)$ has a unit if and only if G is discrete.

In the absence of a convolution identity, an often adequate replacement is provided by the next lemma.

Proposition 1.42 (Approximate convolution identity). *Let (B_n) be a decreasing sequence of compact neighbourhoods of $e \in G$ that form a basis of the topology at e, and let (ψ_n) be a sequence in $L^1(G)$ with the property that* $\operatorname{supp}\psi_n \subseteq B_n$, $\psi_n \geqslant 0$, *and* $\int \psi_n \,\mathrm{d}m = 1$ *for all* $n \geqslant 1$. *Then we have*

$$\lim_{n\to\infty} \psi_n * f = \lim_{n\to\infty} f * \psi_n = f \tag{1.13}$$

for all $f \in L^1(G)$, where the convergence is in $L^1(G)$.

Proof. By the density of $C_c(G) \subseteq L^1(G)$ (see [21, Prop. 2.41]) and the Banach algebra inequality, it is enough to consider $f \in C_c(G)$. Then, by continuity of f,

$$\left|\psi_n * f(g) - f(g)\right| \leqslant \int_{B_n} \psi_n(h) \left|f(h^{-1}g) - f(g)\right| \,\mathrm{d}m(h) \longrightarrow 0$$

as $n \to \infty$ for every $g \in G$. Clearly

$$|\psi_n * f(g)| \leqslant \|f\|_\infty$$

for all $n \geqslant 1$ and $\psi_n * f(g) = 0$ for $g \notin B_1 \operatorname{supp} f$ which gives $\psi_n * f \to f$ in $L^1(G)$ as $n \to \infty$ by dominated convergence.

For convolution on the right we use the third formula (1.12) in the definition of convolution in the form

$$f * \psi_n(g) = \int f(g\ell^{-1})\Delta(\ell)^{-1}\psi_n(\ell)\,\mathrm{d}m(\ell)$$

and argue as before using continuity of $f \in C_c(G)$ and of Δ. In fact we again have

$$|f * \psi_n(g) - f(g)| \leqslant \int_{B_n} \big|f(g\ell^{-1})\Delta(\ell)^{-1} - f(g)\big|\psi_n(\ell)\,\mathrm{d}m(\ell) \longrightarrow 0$$

as $n \to \infty$ for every $g \in G$, $\|f * \psi_n\|_\infty \leqslant \|f\|_\infty \|\Delta^{-1}|_{B_1}\|_\infty$ and

$$\operatorname{supp}(f * \psi_n) \subseteq (\operatorname{supp} f)B_1$$

for all $n \geqslant 1$. Together with dominated convergence, this gives the lemma. □

The Banach algebra $L^1(G)$ is also a Banach $*$-algebra, where the involution or star operator $f \mapsto f^*$ is defined by

$$f^*(g) = \overline{f(g^{-1})}\Delta(g)^{-1}.$$

Indeed, it is clear that f^* is a semi-linear function of f,

$$\|f^*\|_1 = \int |f(g^{-1})|\Delta(g)^{-1}\,\mathrm{d}m(g) = \|f\|_1$$

by Lemma 1.13, and

$$\left(f^*\right)^*(g) = \overline{f^*(g^{-1})}\Delta(g)^{-1} = f(g)\Delta(g^{-1})^{-1}\Delta(g)^{-1} = f(g)$$

for all $g \in G$. Finally, for $f_1, f_2 \in L^1(G)$ we have

$$\begin{aligned}
f_1^* * f_2^*(g) &= \int f_1^*(h) f_2^*(h^{-1}g)\,\mathrm{d}m(h)\\
&= \int \overline{f_1(h^{-1})}\Delta(h)^{-1}\overline{f_2(g^{-1}h)}\Delta(h^{-1}g)^{-1}\,\mathrm{d}m(h)\\
&= \overline{\int f_2(g^{-1}h) f_1(h^{-1})\,\mathrm{d}m(h)}\Delta(g)^{-1}\\
&= \overline{f_2 * f_1(g^{-1})}\Delta(g)^{-1} = \left(f_2 * f_1\right)^*(g)
\end{aligned}$$

by definition and the second formula (1.11) in the definition of convolution.

We conclude our discussion of $L^1(G)$ by noting that if G is abelian, then $L^1(G)$ is also commutative. Indeed, for $f_1, f_2 \in L^1(G)$ and $g \in G$

$$f_1 * f_2(g) = \int f_1(h) f_2(g-h)\,\mathrm{d}m(h) = \int f_2(k) f_1(g-k)\,\mathrm{d}m(k) = f_2 * f_1(g)$$

by the measure preserving substitution $k = g - h$ (formally by $\Delta \equiv 1$ and Lemma 1.13).

Exercise 1.43. Generalize Proposition 1.42 by relaxing the condition on ψ_n and assuming that $\psi_n \geqslant 0$, $\int \psi_n \, \mathrm{d}m = 1$, and $\int_{B_n} \psi_n \, \mathrm{d}m \to 1$ as $n \to \infty$ instead of $\operatorname{supp} \psi_n \subseteq B_n$ for all $n \in \mathbb{N}$.

Essential Exercise 1.44. Show, for $f_1 \in L^1(G)$ and $f_2 \in L^\infty(G)$ (and also for $f_1 \in C_c(G)$ and $f_2 \in C(G)$), that the convolution product $f_1 * f_2$ exists everywhere, and defines a continuous function on G. Show that if, in addition, f_1 and f_2 have compact support, then $f_1 * f_2 \in C_c(G)$.

1.4.2 The Banach Algebra $M(G)$

The Banach $*$-algebra $L^1(G)$ is actually a closed sub-algebra of the much larger Banach $*$-algebra $M(G)$, called the *measure algebra*, which we introduce now.

For this, recall that a complex-valued Borel measure ν on G can be defined† by $\mathrm{d}\nu = f_\nu \, \mathrm{d}\mu$, where μ is a σ-finite measure on G and $f_\nu \in L^1_\mu(G)$ (see [21, App. B.5]). We define $M(G)$ to be the space of all complex-valued measures $\mathrm{d}\nu = f_\nu \, \mathrm{d}\mu$ on G with the norm $\|\nu\| = \int |f_\nu| \, \mathrm{d}\mu$, which by the Riesz representation theorem (see [21, Thm. 7.54]) can be identified as a Banach space with the dual‡ $C_0(G)'$. The convolution on $M(G)$ is defined by

$$\int F \, \mathrm{d}(\nu_1 * \nu_2) = \iint F(g_1 g_2) \, \mathrm{d}\nu_1(g_1) \, \mathrm{d}\nu_2(g_2)$$

for $F \in C_0(G)$ or $F = \mathbb{1}_B$ for a Borel measurable $B \subseteq G$. The convolution is associative since for $\nu_1, \nu_2, \nu_3 \in M(G)$ and $F \in C_0(G)$ we have

$$\begin{aligned}
\int F \, \mathrm{d}(\nu_1 * (\nu_2 * \nu_3)) &= \int F(g_1 h) \, \mathrm{d}\nu_1(g_1) \, \mathrm{d}(\nu_2 * \nu_3)(h) \\
&= \int F(g_1 g_2 g_3) \, \mathrm{d}\nu_1(g_1) \, \mathrm{d}\nu_2(g_2) \, \mathrm{d}\nu_3(g_3) \\
&= \int F \, \mathrm{d}((\nu_1 * \nu_2) * \nu_3).
\end{aligned}$$

Also note that the definition immediately implies that the convolution of measures is commutative if the group G is abelian.

The Banach algebra inequality $\|\nu_1 * \nu_2\| \leqslant \|\nu_1\| \|\nu_2\|$ follows from

† In effect, this is describing a measure by how it integrates functions. For convenience, we will talk about both ν and $\mathrm{d}\nu$ as a measure.

‡ The dual of a Banach space V is often denoted V^*, but we have chosen to write V' to avoid confusion with the $*$-operator in $L^1(G)$, the adjoint, and several convolution operators.

$$\begin{aligned}\left|\int F \,\mathrm{d}(\nu_1 * \nu_2)\right| &\leqslant \iint |F(g_1g_2)||f_{\nu_1}(g_1)||f_{\nu_2}(g_2)| \,\mathrm{d}\mu_1(g_1)\,\mathrm{d}\mu_2(g_2)\\ &\leqslant \|F\|_\infty \int |f_{\nu_1}(g_1)| \,\mathrm{d}\mu_1(g_1) \int |f_{\nu_2}(g_2)| \,\mathrm{d}\mu_2(g_2)\\ &= \|F\|_\infty \|\nu_1\|\|\nu_2\|,\end{aligned}$$

where $\mathrm{d}\nu_1 = f_{\nu_1}\,\mathrm{d}\mu_1$ and $\mathrm{d}\nu_2 = f_{\nu_2}\,\mathrm{d}\mu_2$ as above, and $F \in C_0(G)$. Notice that $M(G)$ is always a unital algebra, since the Dirac measure δ_e is a multiplicative unit in $M(G)$ (recall that the Dirac measure δ_g for a given $g \in G$ is defined by $\delta_g(B) = 1$ if $g \in B$ and $\delta_g(B) = 0$ otherwise).

We define the involution $\nu \mapsto \nu^*$ on $M(G)$ by

$$\int F \,\mathrm{d}\nu^* = \int F(g^{-1}) \,\mathrm{d}\overline{\nu}(g) = \int F(g^{-1})\overline{f_\nu}(g) \,\mathrm{d}\mu(g)$$

and applying this with $F = \mathbb{1}_B$ this becomes the definition $\nu^*(B) = \overline{\nu(B^{-1})}$, which implies that $(\nu^*)^* = \nu$. The involution is again semi-linear, satisfies $\|\nu^*\| = \|\nu\|$ since

$$\left|\int F \,\mathrm{d}\nu^*\right| \leqslant \int |F(g^{-1})||f_\nu(g)| \,\mathrm{d}\mu(g) \leqslant \|F\|_\infty \|\nu\|$$

for $F \in C_0(G)$, and has $(\nu_1 * \nu_2)^* = \nu_2^* * \nu_1^*$ since

$$\begin{aligned}\int F \,\mathrm{d}(\nu_1 * \nu_2)^* &= \int F(g^{-1}) \,\mathrm{d}\overline{\nu_1 * \nu_2}(g)\\ &= \iint F((g_1g_2)^{-1}) \,\mathrm{d}\overline{\nu_1}(g_1)\overline{\nu_2}(g_2)\\ &= \iint F(g_2^{-1}g_1^{-1}) \,\mathrm{d}\overline{\nu_1}(g_1) \,\mathrm{d}\overline{\nu_2}(g_2) = \int F \,\mathrm{d}(\nu_2^* * \nu_1^*).\end{aligned}$$

We may identify $f \in L^1(G)$ with the finite complex-valued measure ν_f defined by $\mathrm{d}\nu_f = f\,\mathrm{d}m$, which shows that $L^1(G)$ is a closed subspace of the Banach space $M(G)$. It is also a sub-algebra, meaning that the convolutions of functions in $L^1(G)$ and of measures in $M(G)$ are compatible. Indeed, for any $F \in C_0(G)$ we have

$$\begin{aligned}\int F \,\mathrm{d}\nu_{f_1 * f_2} &= \int F(g) f_1 * f_2(g) \,\mathrm{d}m\\ &= \iint F(g) f_1(h) f_2(h^{-1}g) \,\mathrm{d}m(h) \,\mathrm{d}m(g)\\ &= \iint F(hk) f_1(h) f_2(k) \,\mathrm{d}m(h) \,\mathrm{d}m(k) = \int F \,\mathrm{d}(\nu_{f_1} * \nu_{f_2})\end{aligned}$$

by the substitution $k = h^{-1}g$ for a fixed h. In particular — something we failed to verify above — it follows that convolution in $L^1(G)$ is associative.

Moreover, we claim that the involution on $M(G)$, when restricted to $L^1(G)$, also gives the previously defined involution on $L^1(G)$. To see this, assume that $\mathrm{d}\nu_f = f\,\mathrm{d}m$ for some $f \in L^1(G)$. Then

$$\int F\,\mathrm{d}\nu_f^* = \int F(g^{-1})\overline{f(g)}\,\mathrm{d}m = \int F(g)\overline{f(g^{-1})}\Delta(g)^{-1}\,\mathrm{d}m = \int F\,\mathrm{d}\nu_{f^*}$$

for any $F \in C_0(G)$ by Lemma 1.13.

Finally, notice that the convolution of the Dirac measure δ_g for $g \in G$ and the measure ν_f corresponding to some $f \in L^1(G)$ is given by

$$\delta_g * \nu_f = \nu_{\lambda_g f} \tag{1.14}$$

since

$$\begin{aligned}\int F\,\mathrm{d}(\delta_g * \nu_f) &= \iint F(hk)\,\mathrm{d}\delta_g(h)\,\mathrm{d}\nu_f(k) = \int F(gk)f(k)\,\mathrm{d}m(k)\\ &= \int F(h')f(g^{-1}h')\,\mathrm{d}m(h') = \int F\,\mathrm{d}\nu_{\lambda_g f}\end{aligned}$$

for any $F \in C_0(G)$.

In many ways $M(G)$ is too big for its own good. For example, it is not separable (unless G is discrete, in which case $M(G) = L^1(G)$), and the dual space of $M(G)$ is (depending on mathematical taste) monstrous. For this and other reasons we will mostly restrict ourselves to $L^1(G)$, whose dual space is simply $L^\infty(G)$.

Exercise 1.45. Show that $L^1(G) \subseteq M(G)$ is a two-sided ideal.

Exercise 1.46. Prove the analogue $\nu_f * \delta_g = \nu_{\rho_{g^{-1}} f}$ to (1.14) for all $f \in L^1(G)$ and $g \in G$, where $\rho_g f(h) = \Delta(g)f(hg)$ for $g, h \in G$ and $f \in L^1(G)$.

1.4.3 Convolution and Unitary Representations

Now let π be a unitary representation of G on the Hilbert space $\mathcal{H}_\pi$. As we now show, $\mathcal{H}_\pi$ is automatically a module over the Banach algebras $L^1(G)$ and $M(G)$. Indeed, for any $\nu \in M(G)$ and $u \in \mathcal{H}_\pi$ we can define the operator $\pi_*(\nu)$ by the weak integral

$$\pi_*(\nu)u = \int \pi_g u\,\mathrm{d}\nu(g),$$

which means we define $\pi_*(\nu)u$ by the Fréchet–Riesz representation theorem (see [21, Cor. 3.19] for example) and the formula

$$\langle \pi_*(\nu)u, w\rangle = \int \langle \pi_g u, w\rangle \,\mathrm{d}\nu(g) \tag{1.15}$$

for all $w \in \mathcal{H}_\pi$. Indeed, the right-hand side of (1.15) depends linearly on u, semi-linearly on w, and satisfies

$$\begin{aligned} \left|\int \langle \pi_g u, w\rangle \,\mathrm{d}\nu\right| &= \left|\int \langle \pi_g u, w\rangle f(g)\,\mathrm{d}\mu(g)\right| \\ &\leqslant \int |\langle \pi_g u, w\rangle||f(g)|\,\mathrm{d}\mu(g) \leqslant \|u\|\|w\|\|\nu\| \end{aligned} \tag{1.16}$$

where $\mathrm{d}\nu = f\,\mathrm{d}\mu$ for $f \in L^1_\mu(G)$, and μ is a σ-finite measure. This allows us to apply the Fréchet–Riesz representation theorem to define the vector $\pi_*(\nu)u$ as the unique vector satisfying (1.15). By (1.16), the so defined operator $\pi_*(\nu)$ also satisfies

$$\|\pi_*(\nu)\| \leqslant \|\nu\|,$$

which we will use frequently without explicit reference. We refer to [21, Sec. 3.5.4] for more details,[†] to Exercise 1.52 for a definition using a strong (or Riemann) integral, and to Exercise 1.53 for another equivalent definition in the case of a Koopman representation. We will call $\pi_*(\nu)$ the *convolution operator* associated to $\nu \in M(G)$.

Next we verify that $\pi_*(\nu_1^*) = \pi_*(\nu_1)^*$ and $\pi_*(\nu_1 * \nu_2) = \pi_*(\nu_1)\pi_*(\nu_2)$ for any $\nu_1, \nu_2 \in M(G)$. To see this, let $u, w \in \mathcal{H}_\pi$. Then

$$\begin{aligned} \big\langle \pi_*(\nu_1^*)u, w\big\rangle &= \int \langle \pi_g u, w\rangle \,\mathrm{d}\nu_1^*(g) = \int \langle \pi_{g^{-1}} u, w\rangle \,\mathrm{d}\overline{\nu_1}(g) \\ &= \int \langle u, \pi_g w\rangle \,\mathrm{d}\overline{\nu_1}(g) = \int \overline{\langle \pi_g w, u\rangle}\,\mathrm{d}\overline{\nu_1}(g) \\ &= \overline{\langle \pi_*(\nu_1)w, u\rangle} = \langle u, \pi_*(\nu_1)w\rangle \end{aligned}$$

shows the first claim. For the second, we have

$$\begin{aligned} \langle \pi_*(\nu_1 * \nu_2)u, w\rangle &= \int \langle \pi_g u, w\rangle \,\mathrm{d}\nu_1 * \nu_2(g) \\ &= \iint \langle \pi_{g_1}\pi_{g_2} u, w\rangle \,\mathrm{d}\nu_1(g_1)\,\mathrm{d}\nu_2(g_2) \\ &= \int \underbrace{\int \langle \pi_{g_2} u, \pi_{g_1}^* w\rangle \,\mathrm{d}\nu_2(g_2)}_{=\langle \pi_*(\nu_2)u, \pi(g_1)^* w\rangle} \,\mathrm{d}\nu_1(g_1) \\ &= \int \langle \pi_{g_1}\pi_*(\nu_2)u, w\rangle \,\mathrm{d}\nu_1(g_1) = \langle \pi_*(\nu_1)\pi_*(\nu_2)u, w\rangle, \end{aligned}$$

† In [21] we denoted $\pi_*(\nu)u$ by $\nu \underset{\pi}{*} u$ to emphasise the similarity between the operator $\pi_*(\nu)$ and convolution. We have chosen here the notation $\pi_*(\nu)$ as it is closer to the standard notation $\pi(\nu)$ but still contains a reminder of the similarities to convolution.

as required.

As mentioned above, we often identify $f \in L^1(G)$ with the associated finite complex-valued measure $\mathrm{d}\nu_f = f\,\mathrm{d}m$, which by the above defines a bounded operator $\pi_*(f) = \pi_*(\nu_f)$ on $\mathcal{H}_\pi$ with $\|\pi_*(f)\| \leqslant \|f\|_1$.

Notice that the unitary representation π is automatically contained in the module structure of $\mathcal{H}_\pi$ with respect to $M(G)$ since for any Dirac measure δ_g we have†

$$\pi_*(\delta_g) = \int \pi_h \,\mathrm{d}\delta_g(h) = \pi_g.$$

The same is not true when we restrict the module structure to $L^1(G)$ (as we often will), but it nearly is.

Proposition 1.47 (Operators for an approximate identity). *Let (ψ_n) be an approximate identity as in Proposition 1.42 and let $g \in G$. Then*

$$\pi_*\left(\lambda_g \psi_n\right) = \pi_g \pi_*(\psi_n) \longrightarrow \pi_g$$

in the strong operator topology as $n \to \infty$.

PROOF. By (1.14) we have $\lambda_g \psi_n = \delta_g * \psi_n$ if we identify any $f \in L^1(G)$ with its associated complex-valued measure $\mathrm{d}\nu_f = f\,\mathrm{d}m$. Therefore

$$\pi_*(\lambda_g \psi_n) = \pi_*(\delta_g * \psi_n) = \pi_*(\delta_g)\pi_*(\psi_n) = \pi_g \pi_*(\psi_n)$$

by the discussion above. Since π_g is continuous, it suffices to consider the case $g = e$.

Let $u \in \mathcal{H}_\pi$ be non-trivial. By continuity of the representation, for every $\varepsilon > 0$ there exists some n_0 with the property that $g \in B_{n_0}$ implies $\|\pi_g u - u\| < \varepsilon \|u\|$. Thus for $n \geqslant n_0$ we have

$$\begin{aligned}\left|\langle \pi_*(\psi_n)u, u\rangle - \|u\|^2\right| &= \left|\int_{B_n} \left(\left\langle \pi_g u, u\right\rangle - \|u\|^2\right) \psi_n(g)\,\mathrm{d}m(g)\right| \\ &\leqslant \int_{B_n} \left|\left\langle \pi_g u - u, u\right\rangle\right| \psi_n(g)\,\mathrm{d}m(g) \leqslant \varepsilon \|u\|^2\end{aligned}$$

by Cauchy–Schwarz and since $\|\psi_n\|_1 = 1$, and so

$$\Re \langle \pi_*(\psi_n)u, u\rangle \geqslant \|u\|^2 - \varepsilon \|u\|^2.$$

Also note that $\|\pi_*(\psi_n)u\| \leqslant \|u\|$. For the distance of $\pi_*(\psi_n)u$ to u this gives

$$\begin{aligned}\|\pi_*(\psi_n)u - u\|^2 &= \langle \pi_*(\psi_n)u, \pi_*(\psi_n)u\rangle - 2\Re \langle \pi_*(\psi_n)u, u\rangle + \langle u, u\rangle \\ &\leqslant 2\|u\|^2 - 2\|u\|^2 + 2\varepsilon \|u\|^2 = 2\varepsilon \|u\|^2.\end{aligned}$$

† Strictly speaking, we should verify this identify by applying the operators to u in $\mathcal{H}_\pi$ and taking the inner product with w in $\mathcal{H}_\pi$ as in the definition of $\pi_*(\nu)$, but when these vectors are just distracting decorations we sometimes do not write them down.

Since $\varepsilon > 0$ was arbitrary, we see that $\pi_*(\psi_n)u \to u$ as $n \to \infty$. □

Corollary 1.48 (Invariance). *For a closed subspace $\mathcal{V} \subseteq \mathcal{H}_\pi$, the following are equivalent:*

(a) $\mathcal{V}$ *is invariant under* π,
(b) $\mathcal{V}$ *is invariant under* $\pi_*(L^1(G))$, *and*
(c) $\mathcal{V}$ *is invariant under* $\pi_*(M(G))$.

PROOF. Suppose that $\mathcal{V}$ is invariant under $\pi(G)$ as in (a). Then for $\mu \in M(G)$ and $v \in \mathcal{V}$ we have

$$\langle \pi_*(\mu)v, w\rangle = \int \langle \pi_g v, w\rangle \,\mathrm{d}\mu(g) = 0$$

for any $w \in \mathcal{V}^\perp$, so $\pi_*(\mu)v \in \mathcal{V}$ as required for (c). Clearly (c) implies (b), and finally (b) implies (a) by Proposition 1.47. □

Essential Exercise 1.49. Let π and ρ be unitary representations of G, and let $B\colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ be an equivariant bounded operator. Show that

$$B \circ \pi_*(f) = \rho_*(f) \circ B$$

for all $f \in L^1(G)$, and $B \circ \pi_*(\nu) = \rho_*(\nu) \circ B$ for all $\nu \in M(G)$.

Exercise 1.50. Generalize Proposition 1.47 by allowing the more general approximate identities from Exercise 1.43.

Exercise 1.51. Suppose G is unimodular, and let $f \in L^1(G)$ have the property that for every $g_0 \in G$ we have $f(g_0 g) = f(g g_0)$ for almost every $g \in G$. Show that

$$\pi_{g_0}\pi_*(f) = \pi_*(f)\pi_{g_0}$$

for any unitary representation π of G.

The following two exercises may help the reader to familiarize herself with the notion of convolution operators.

Exercise 1.52. Let π be a unitary representation of G on $\mathcal{H}_\pi$.
(a) Let ν be a compactly supported complex-valued Borel measure on G and $K = \operatorname{supp}\nu$. For $v \in \mathcal{H}_\pi$ and a finite partition $\mathcal{P} = \{B_1, \dots, B_n\}$ of K and sample points $g_j \in B_j$ for $j = 1, \dots, n$, define the Riemann sum by

$$R_\nu(\mathcal{P}, (g_1, \dots, g_n)) = \sum_{j=1}^{n} \nu(B_j)\pi_{g_j} v.$$

Use uniform continuity of the map $K \ni g \mapsto \pi_g v \in \mathcal{H}_\pi$ to show that the Riemann sums converge in $\mathcal{H}_\pi$ as the diameter

$$\operatorname{diam}(\mathcal{P}) = \max_{j=1,\dots,n} \sup_{g_1,g_2 \in B_j} \mathsf{d}(g_1, g_2)$$

of the partition goes to zero. Show that the limit ${}^{R}\!\!\int_K \pi_g v \,\mathrm{d}\nu$ of the above Riemann sums satisfies

$$\left\| {}^{R}\!\!\int_K \pi_g v \,\mathrm{d}\nu \right\| \leqslant \|\nu\|\|v\|.$$

(b) Let $\nu \in M(G)$ and define $\nu_m = \nu|_{K_m}$ for $m \in \mathbb{N}$, where $K_1 \subseteq K_2 \subseteq \cdots$ is an increasing sequence of compact subsets of G with $G = \bigcup_{m=1}^{\infty} K_m$. Extend the above notion of a Riemann integral to an improper Riemann integral by showing that the limit

$${}^{R}\!\!\int_G \pi_g v \,\mathrm{d}\nu = \lim_{m\to\infty} {}^{R}\!\!\int_{K_m} \pi_g v \,\mathrm{d}\nu_m$$

exists and is independent of the choice of the sequence (K_m).
(c) Show that ${}^{R}\!\!\int_G \pi_g v \,\mathrm{d}\nu = \pi_*(\nu)(v)$ for any $v \in \mathcal{H}_\pi$ and any $\nu \in M(G)$.

Exercise 1.53. Suppose that G acts continuously on X preserving a locally finite measure μ on X, giving rise to the Koopman representation π as in Proposition 1.3. Show that for any $f \in L^2_\mu(X)$ and $\nu \in M(G)$ the integral

$$\int_G f(g^{-1} \bullet x) \,\mathrm{d}\nu(g)$$

exists for μ-almost every $x \in X$ and equals $\pi_*(\nu)f$ almost surely.

Exercise 1.54. Let π be a unitary representation of G on $\mathcal{H}_\pi$, let ν be a probability measure on G, $\varepsilon > 0$, $v, w \in \mathcal{H}_\pi$ with $\|\pi_g v - w\| < \varepsilon$ for all $g \in \operatorname{supp}\nu$. Show that

$$\|\pi_*(\nu)v - w\| < \varepsilon.$$

1.5 Cyclic Representations

In this section we introduce several fundamental notions for the study of unitary representations, all of which relate to the question of how to decompose a given unitary representation into simpler constituents.

1.5.1 Decomposition into Cyclic Subspace

It is natural to study the smallest invariant subspaces containing a particular element.

Definition 1.55 (Cyclic subspaces). Let π be a unitary representation of G and $v \in \mathcal{H}_\pi$. The smallest closed π-invariant subspace of $\mathcal{H}_\pi$ containing v is called the *cyclic subspace generated by* v and denoted by $\langle v \rangle_\pi$. We say that π is *cyclic* if there exists a vector $v \in \mathcal{H}_\pi$, called a *generator*, with $\langle v \rangle_\pi = \mathcal{H}_\pi$.

We note that due to Corollary 1.48 we may equivalently use invariance under the unitary representation or invariance under the associated module structures to define the cyclic subspace generated by a vector. More formally we have for a unitary representation π of G and a vector $v \in \mathcal{H}_\pi$ the identity

$$\langle v\rangle_\pi = \overline{\langle\pi(G)v\rangle_{\mathbb{C}}} = \overline{\pi_*(L^1(G))v} = \overline{\pi_*(M(G))v},$$

where $\langle\cdot\rangle_{\mathbb{C}}$ again denotes the linear hull over $\mathbb{C}$.

Lemma 1.56 (Decomposition into cyclic subspaces). *Let π be a unitary representation of G. Then there exists a finite or countable*[†] *collection of orthogonal cyclic subspaces $\langle v_n\rangle_\pi \subseteq \mathcal{H}_\pi$ such that $\mathcal{H}_\pi = \bigoplus_{n\geqslant 1}\langle v_n\rangle_\pi$.*

PROOF. Let $w_1, w_2, \ldots \in \mathcal{H}$ be an orthonormal basis. Define $v_1 = w_1$ and $\mathcal{V}_1 = \langle v_1\rangle_\pi$. Let v_2 be the orthogonal projection of w_2 onto $\mathcal{V}_1^\perp$. By invariance of $\mathcal{V}_1$ we have $\pi_g(v_2) \perp \mathcal{V}_1$ for all $g \in G$, which implies that $\langle v_2\rangle_\pi \perp \mathcal{V}_1$. In particular we see that $\mathcal{V}_2 = \mathcal{V}_1 \oplus \langle v_2\rangle_\pi$ is an orthogonal direct sum of two cyclic subspaces that contains both w_1 and w_2.

Next define v_3 as the orthogonal projection of w_3 onto $\mathcal{V}_2^\perp$ and obtain

$$w_3 \in \mathcal{V}_3 = \mathcal{V}_2 \oplus \langle v_3\rangle_\pi$$

as before. Repeating the argument inductively defines the cyclic subspaces. Since all basis vectors belong to the direct sum of these cyclic subspaces, the direct sum equals $\mathcal{H}_\pi$ and the lemma follows. □

Exercise 1.57. Let G be a finite group. Show that every unitary representation of G can be written as a direct sum of irreducible representations.

Exercise 1.58. Give an example of a non-cyclic representation.

1.5.2 Matrix Coefficients

The following definition is both fundamental for the theory to come, as well as of great interest in applications.

Definition 1.59 (Matrix coefficients). Let π be a unitary representation of G and $v, w \in \mathcal{H}_\pi$. The function $\varphi_v^\pi \in C_b(G) = C(G) \cap L^\infty(G)$ defined for $g \in G$ by

$$\varphi_v^\pi(g) = \langle \pi_g v, v\rangle$$

is a *diagonal (or principal) matrix coefficient for v*. More generally, the function $\varphi_{v,w} \in C_b(G)$ defined by

$$\varphi_{v,w}^\pi(g) = \langle \pi_g v, w\rangle$$

[†] Recall the standing assumption that $\mathcal{H}_\pi$ is separable.

is a *(non-diagonal) matrix coefficient*[†] *for* v, w. If the unitary representation is clear from the context we will also simply write $\varphi_v = \varphi_v^\pi$ and $\varphi_{v,w} = \varphi_{v,w}^\pi$ for the matrix coefficients of $v, w \in \mathcal{H}_\pi$.

We note that for the diagonal matrix coefficient of a vector $v \in \mathcal{H}_\pi$ we have $\|\varphi_v\|_\infty = \varphi_v(e) = \|v\|^2$ and, more generally, we have $\|\varphi_{v,w}\|_\infty \leqslant \|v\| \|w\|$ for $v, w \in \mathcal{H}_\pi$. The following observation, together with Lemma 1.56, shows that the study of matrix coefficients is in a sense equivalent to the study of unitary representations.

Proposition 1.60 (Equal matrix coefficients). *Let π and ρ be two unitary representations of G. Suppose that the principal matrix coefficients defined by vectors $v \in \mathcal{H}_\pi$ and $w \in \mathcal{H}_\rho$ agree, meaning that $\langle \pi_g v, v \rangle = \langle \rho_g w, w \rangle$ for all $g \in G$. Then π restricted to the cyclic subspace $\langle v \rangle_\pi$ is unitarily isomorphic to ρ restricted to the cyclic subspace $\langle w \rangle_\rho$ via a unitary isomorphism that sends v to w.*

PROOF. Fix $g_1, \ldots, g_n \in G$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. Then the norm of the element $\sum_k \alpha_k \pi_{g_k}(v)$ can be expressed in terms of matrix coefficients, since

$$\begin{aligned}\Big\| \sum_k \alpha_k \pi_{g_k}(v) \Big\|^2 &= \Big\langle \sum_k \alpha_k \pi_{g_k}(v), \sum_\ell \alpha_\ell \pi_{g_\ell}(v) \Big\rangle \\ &= \sum_{k,\ell} \alpha_k \overline{\alpha_\ell} \left\langle \pi_{g_\ell^{-1} g_k} v, v \right\rangle = \sum_{k,\ell} \alpha_k \overline{\alpha_\ell} \varphi_v^\pi (g_\ell^{-1} g_k). \qquad (1.17)\end{aligned}$$

In particular, equality of the matrix coefficients $\varphi_v^\pi = \varphi_w^\rho$ implies that

$$\Big\| \sum_k \alpha_k \pi_{g_k}(v) \Big\| = \Big\| \sum_k \alpha_k \rho_{g_k}(w) \Big\|, \qquad (1.18)$$

and so $\sum_k \alpha_k \pi_{g_k}(v) = 0$ if and only if $\sum_k \alpha_k \rho_{g_k}(w) = 0$. Thus the map U defined by

$$U \colon \sum_k \alpha_k \pi_{g_k}(v) \longmapsto \sum_k \alpha_k \rho_{g_k}(w) \qquad (1.19)$$

is well-defined, injective, and is also an isometry by (1.18). Thus U extends to an isometry between $\langle v \rangle_\pi$ and $\langle w \rangle_\rho$. It is clear from the definition that $U(\pi_g(u)) = \rho_g(U(u))$ for all $g \in G$ whenever u is a finite sum as in (1.19). By continuity of π_g and ρ_g, this implies that U is equivariant and the proposition follows. □

We note that the matrix coefficients already featured implicitly in Section 1.4.3 in the definition of the convolution operators $\pi_*(\nu)$ and $\pi_*(f)$ for $\nu \in M(G)$ and $f \in L^1(G)$ since

[†] If v, w belong to an orthonormal basis, then the function indeed represents an entry of a possibly infinite matrix describing the operator π_g.

$$\int \varphi_{u,w}\,\mathrm{d}\nu = \int \left\langle \pi_g u, w\right\rangle\,\mathrm{d}\nu(g) = \langle \pi_*(\nu)u, w\rangle$$
$$\int \varphi_{u,w} f\,\mathrm{d}m = \int \left\langle \pi_g u, w\right\rangle f(g)\,\mathrm{d}m(g) = \langle \pi_*(f)u, w\rangle$$

for all $u, w \in \mathcal{H}_\pi$. We will frequently use these formulas, often without explicit reference.

Finally, the map that sends $u, w \in \mathcal{H}_\pi$ to their matrix coefficient $\varphi_{u,w}$ has interesting equivariance properties for any unitary representation π of G, which will be important later. Indeed, for $u, w \in \mathcal{H}_\pi$ and $g, h \in G$ we have

$$\varphi_{\pi_g u,w}(h) = \left\langle \pi_h \pi_g u, w\right\rangle = \varphi_{u,w}(hg)$$

and

$$\varphi_{u,\pi_g w}(h) = \left\langle \pi_h u, \pi_g w\right\rangle = \left\langle \pi_g^{-1}\pi_h u, w\right\rangle = \varphi_{u,w}(g^{-1}h).$$

Exercise 1.61. Let

$$\pi_\oplus = \bigoplus_{n=1}^\infty \pi_n$$

be as in Exercise 1.21 and let $v = \sum_{n=1}^\infty v_n$ and $w = \sum_{n=1}^\infty w_n$ be elements of $\mathcal{H}_{\pi_\oplus}$ with $v_n, w_n \in \mathcal{H}_n = \mathcal{H}_{\pi_n}$ for $n \geqslant 1$. Show that

$$\varphi^{\pi_\oplus}_{v,w} = \sum_{n=1}^\infty \varphi^{\pi_n}_{v_n,w_n},$$

and that the series converges uniformly on G.

Exercise 1.62. Let π be a cyclic unitary representation of G with generator $w \in \mathcal{H}_\pi$. Show that the map $\mathcal{H}_\pi \ni v \mapsto \varphi_{v,w} \in C(G)$ is linear and injective.

Exercise 1.63. Let π and ρ be irreducible unitary representations of G. Let $v, w \in \mathcal{H}_\pi$ and $a, b \in \mathcal{H}_\rho$ be non-zero with $\|v\| = \|a\|$, $\|w\| = \|b\|$, and $\varphi^\pi_{v,w} = \varphi^\rho_{a,b}$. Show that π and ρ are isomorphic, and that there exists a unitary isomorphism $U\colon \mathcal{H}_\pi \to \mathcal{H}_\rho$ between π and ρ with $U(v) = a$ and $U(w) = b$.

Exercise 1.64. Let G be a compact group, and let π be an irreducible unitary representation of G. Show that $\pi < \lambda$.

1.5.3 Positive-Definite Functions and Continuous GNS

The following notion identifies a crucial property of matrix coefficients.

Definition 1.65 (Continuous positive-definite function). A continuous function ϕ on G is called *positive-definite* if

$$\sum_{k,\ell} \alpha_k \overline{\alpha_\ell} \phi(g_\ell^{-1} g_k) \geqslant 0$$

for any finite list $g_1, \ldots, g_n \in G$ and scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$.

Lemma 1.66 (Diagonal matrix coefficient). *A diagonal matrix coefficient of a unitary representation is a positive-definite function.*

Proof. This follows at once from (1.17). □

We will adopt where convenient the notational convention that ϕ is used for an abstract positive-definite function, while φ_v is used for concrete matrix coefficients.

The converse of Lemma 1.66 is given by (our introductory version of) the Gelfand–Naimark–Segal (GNS) construction.[(2)]

Proposition 1.67 (Gelfand–Naimark–Segal: continuous version). *Let the function ϕ on G be continuous and positive-definite. Then there exists a cyclic representation π^ϕ and a generator $v_\phi \in \mathcal{H}_\phi = \mathcal{H}_{\pi^\phi}$ such that ϕ is the diagonal matrix coefficient of v_ϕ for π^ϕ. In particular, $\|\phi\|_\infty = \phi(e) = \|v_\phi\|^2$.*

For the proof, the following abstract principle will be helpful. Suppose $\mathcal{V}$ is a complex vector space and $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is sesqui-linear and positive semi-definite. Then $\langle \cdot, \cdot \rangle$ satisfies the *polarization identity*

$$\langle v, w \rangle = \frac{1}{4} \sum_{n=0}^{3} \mathrm{i}^n \langle v + \mathrm{i}^n w, v + \mathrm{i}^n w \rangle$$

for all $v, w \in \mathcal{V}$ by sesqui-linearity (and expanding the right hand side). Using the positive semi-definiteness and sesqui-linearity again, we also obtain the conjugate-symmetry

$$\begin{aligned} \overline{\langle v, w \rangle} &= \frac{1}{4} \sum_{n=0}^{3} \underbrace{\mathrm{i}^{-n} \mathrm{i}^{-n} \mathrm{i}^{n}}_{=\mathrm{i}^{-n}} \langle v + \mathrm{i}^n w, v + \mathrm{i}^n w \rangle \\ &= \frac{1}{4} \sum_{n=0}^{3} \mathrm{i}^{-n} \left\langle \mathrm{i}^{-n} v + w, \mathrm{i}^{-n} v + w \right\rangle = \langle w, v \rangle \end{aligned} \tag{1.20}$$

for all $v, w \in \mathcal{V}$. In particular, sesqui-linearity and positive semi-definiteness of $\langle \cdot, \cdot \rangle$ together imply that $\langle \cdot, \cdot \rangle$ is a semi-inner product on $\mathcal{V}$.

Proof. For the first step of the conjuring trick that defines $\mathcal{H}_\phi$ and π^ϕ, we define the group algebra

$$\mathbb{C}[G] = \left\{ \sum_{k=1}^{n} \alpha_k \delta_{g_k} \mid n \in \mathbb{N}, g_1, \ldots, g_n \in G, \alpha_1, \ldots, \alpha_n \in \mathbb{C} \right\} \subseteq M(G)$$

to be the space of finitely supported complex-valued measures on G. For elements $\sum_k \alpha_k \delta_{g_k}$ and $\sum_\ell \beta_\ell \delta_{h_\ell}$ in $\mathbb{C}[G]$ we define the ϕ-product

$$\left\langle \sum_k \alpha_k \delta_{g_k}, \sum_\ell \beta_\ell \delta_{h_\ell} \right\rangle_\phi = \sum_{k,\ell} \alpha_k \overline{\beta_\ell} \phi(h_\ell^{-1} g_k).$$

It is clear that $\langle\cdot,\cdot\rangle_\phi$ depends linearly on the first argument and semi-linearly on the second. Moreover, the definition of positive-definiteness shows that

$$\left\langle \sum_k \alpha_k \delta_{g_k}, \sum_k \alpha_k \delta_{g_k} \right\rangle_\phi \geqslant 0$$

for any $\sum_k \alpha_k \delta_{g_k} \in \mathbb{C}[G]$.

Using the argument before the proof, we see that $\langle\cdot,\cdot\rangle_\phi$ is also conjugate-symmetric, and hence defines a semi-inner product on $\mathbb{C}[G]$. We define the associated semi-norm

$$\|\nu\|_\phi = \sqrt{\langle \nu, \nu\rangle_\phi}$$

for $\nu \in \mathbb{C}[G]$, its kernel $W_0 = \{\nu \in \mathbb{C}[G] \mid \|\nu\|_\phi = 0\}$, and define $\mathcal{H}_\phi$ to be the completion of $\mathbb{C}[G]/W_0$ with respect to the norm $\|\cdot\|_\phi$ induced on it. Note that $\mathcal{H}_\phi$ is a Hilbert space with respect to the inner product, again denoted by $\langle\cdot,\cdot\rangle_\phi$, induced by the ϕ-product.

For $g \in G$ we define the linear map $\pi_g\colon \mathbb{C}[G] \to \mathbb{C}[G]$ by

$$\pi_g\colon \sum_k \alpha_k \delta_{g_k} \longmapsto \sum_k \alpha_k \delta_{g g_k},$$

which satisfies for $\nu = \sum_k \alpha_k \delta_{g_k} \in \mathbb{C}[G]$

$$\|\pi_g(\nu)\|_\phi^2 = \left\langle \sum_k \alpha_k \delta_{g g_k}, \sum_k \alpha_k \delta_{g g_k} \right\rangle_\phi = \sum_k \alpha_k \overline{\alpha_\ell} \phi(g_\ell^{-1} g_k) = \|\nu\|_\phi^2$$

by definition of the ϕ-product. This shows that $\pi_g(W_0) = W_0$ and that π_g defines an isometry, again denoted π_g, from $\mathcal{H}_\phi$ to $\mathcal{H}_\phi$. Since $\pi_e = I$ and since we have

$$\pi_g(\pi_h(\nu)) = \pi_g\left(\sum_k \alpha_k \delta_{h g_k}\right) = \sum_k \alpha_k \delta_{g h g_k} = \pi_{gh}(\nu)$$

for $\nu = \sum_k \alpha_k \delta_{g_k} \in \mathbb{C}[G]$ and all $g, h \in G$, we see that $g \mapsto \pi_g$ is a homomorphism from G into the group of unitary operators on $\mathcal{H}_\phi$.

It remains to show the continuity requirement, that is to show that

$$G \ni g \longmapsto \pi_g v \in \mathcal{H}_\phi$$

is continuous for every $v \in \mathcal{H}_\phi$. For this, consider $v_\phi = \delta_e + W_0$ and suppose that $g_n \to g$ as $n \to \infty$. Then

$$\begin{aligned}\|\pi_{g_n}(v_\phi) - \pi_g(v_\phi)\|_\phi^2 &= \left\langle \delta_{g_n} - \delta_g, \delta_{g_n} - \delta_g \right\rangle_\phi \\ &= \phi(e) - 2\Re\phi(g_n^{-1} g) + \phi(e) \longrightarrow 0\end{aligned}$$

as $n \to \infty$ by continuity of ϕ. Using linear combinations and density this extends to continuity of the representation π — more formally set $D = \{v_\phi\}$, note that

$$\langle \pi_G(D) \rangle_{\mathbb{C}} = \langle \pi_G(v_\phi) \rangle_{\mathbb{C}} = \langle \pi_G(\delta_e + W_0) \rangle_{\mathbb{C}} = \mathbb{C}[G] + W_0$$

is dense in $\mathcal{H}_\pi$ and apply Lemma 1.9.(1).

By our standing assumption G is separable, which implies the same for $\pi_G(v_\phi)$ and hence also for the cyclic representation $\left\langle v_\phi \right\rangle_\pi = \mathcal{H}_\phi$ with generator $v_\phi = \delta_e + W_0$. Finally, the principal matrix coefficient of v_ϕ is given by

$$\varphi_{v_\phi}(g) = \left\langle \pi_g(\delta_e + W_0), \delta_e + W_0 \right\rangle_\phi = \left\langle \delta_g, \delta_e \right\rangle_\phi = \phi(g)$$

for all $g \in G$. □

Proposition 1.60, Lemma 1.66, and Proposition 1.67 together give a one-to-one correspondence between positive-definite functions ϕ and their *associated cyclic representations* π^ϕ. Knowing that every unitary representation can be decomposed into cyclic representations (by Lemma 1.56), this suggest that we should study the set of positive-definite functions more closely. Since the latter is contained in the linear space $C_b(G)$, this gives a common setting in which to discuss the relationship between different cyclic unitary representations (on otherwise unrelated Hilbert spaces).

1.5.4 Irreducibility and Extremality

Notice first that an irreducible representation is always cyclic since any non-zero vector can be used as a generator. The property of irreducibility of a cyclic representation has a very satisfying characterization in terms of the associated matrix coefficients.

Proposition 1.68 (Extremality). *Let*

$$\mathcal{P}(G) = \{\phi \in C_b(G) \mid \phi \textit{ is positive-definite}\}$$

and

$$\mathcal{P}^1(G) = \{\phi \in \mathcal{P}(G) \mid \phi(e) = 1\}.$$

Then $\mathcal{P}(G) = [0,\infty)\mathcal{P}^1(G)$ is a convex cone and $\mathcal{P}^1(G)$ is a convex subset of $C_b(G)$. An element ϕ of $\mathcal{P}^1(G)$ is an extremal point of $\mathcal{P}^1(G)$ if and only if the associated cyclic representation π^ϕ is irreducible.

We recall that a point p of a convex set P is called extremal if the relation $\alpha_1 p_1 + \alpha_2 p_2 = p$ for $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1 + \alpha_2 = 1$ and $p_1, p_2 \in P$ forces $p_1 = p_2 = p$.

Proof of Proposition 1.68. Clearly if $\alpha_1, \alpha_2 \geqslant 0$ and $\phi_1, \phi_2 \in \mathcal{P}(G)$ then the linear combination $\alpha_1\phi_1 + \alpha_2\phi_2$ also belongs to $\mathcal{P}(G)$, and so $\mathcal{P}(G)$ is a convex cone. If in addition $\alpha_1 + \alpha_2 = 1$ and $\phi_1, \phi_2 \in \mathcal{P}^1(G)$, then

$$\alpha_1\phi_1 + \alpha_2\phi_2 \in \mathcal{P}^1(G),$$

and hence $\mathcal{P}^1(G)$ is convex. Finally, if $\phi \in \mathcal{P}(G)\smallsetminus\{0\}$, then $\phi(e) = \|\phi\|_\infty > 0$ by the correspondence between positive-definite functions and matrix coefficients (Proposition 1.67). Hence $\phi_1 = \phi(e)^{-1}\phi \in \mathcal{P}^1(G)$ and as $\phi \in \mathcal{P}(G)\smallsetminus\{0\}$ was arbitrary it follows that $\mathcal{P}(G) = [0,\infty)\mathcal{P}^1(G)$.

It remains to prove the equivalence in the last part of the proposition.

Extremality implies irreducibility. Suppose first that $\phi \in \mathcal{P}^1(G)$ is an extremal point and suppose for the purpose of a contradiction that $\mathcal{H}_\phi$ is a sum $\mathcal{V}_1 \oplus \mathcal{V}_2$ for two non-trivial closed π^ϕ-invariant subspaces $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{H}_\phi$. Let us write $\pi = \pi^\phi$ for brevity, let v_ϕ denote the generator of $\mathcal{H}_\phi$, and split v_ϕ into its components by writing $v_\phi = v_1 + v_2$ with $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$. Since $\left\langle \pi_G v_\phi \right\rangle_{\mathbb{C}}$ is dense in $\mathcal{H}_\phi$, it follows that $\left\langle \pi_G v_j \right\rangle_{\mathbb{C}}$ is dense in $\mathcal{V}_j$, so $\mathcal{V}_j$ is cyclic with generator v_j, for $j = 1, 2$. Since $\mathcal{V}_j$ is assumed to be non-trivial, this implies that $\|v_j\|^2 > 0$ for $j = 1, 2$. Thus

$$\begin{aligned}\phi(g) = \left\langle \pi_g(v_\phi), v_\phi \right\rangle &= \left\langle \pi_g v_1, v_1 \right\rangle + \left\langle \pi_g v_2, v_2 \right\rangle \\ &= \|v_1\|^2 \left\langle \pi_g \widetilde{v_1}, \widetilde{v_1} \right\rangle + \|v_2\|^2 \left\langle \pi_g \widetilde{v_2}, \widetilde{v_2} \right\rangle\end{aligned}$$

for $g \in G$, where we set $\widetilde{v_j} = \|v_j\|^{-1} v_j$ for $j = 1, 2$. Notice that

$$\phi_j(g) = \left\langle \pi_g \widetilde{v_j}, \widetilde{v_j} \right\rangle$$

for $j = 1, 2$ belong to $\mathcal{P}^1(G)$. Thus we have shown that

$$\phi = \|v_1\|^2 \phi_1 + \|v_2\|^2 \phi_2$$

(with $\|v_1\|^2 + \|v_2\|^2 = \|v_\phi\|^2 = 1$ and $\|v_1\|^2, \|v_2\|^2 > 0$) is expressing ϕ as a convex combination in terms of $\phi_1, \phi_2 \in \mathcal{P}^1(G)$. The assumed extremality therefore implies that $\phi = \phi_1 = \phi_2$. However, by Proposition 1.60, this implies that the cyclic representation $\mathcal{V}_1$ is unitarily isomorphic to $\mathcal{V}_2$, where the unitary isomorphism $U\colon \mathcal{V}_1 \to \mathcal{V}_2$ sends the generator $\widetilde{v_1} \in \mathcal{V}_1$ to the generator $\widetilde{v_2} \in \mathcal{V}_2$. Therefore the subspace

$$\mathcal{W} = \left\{ w + \tfrac{\|v_2\|}{\|v_1\|} U w \mid w \in \mathcal{V}_1 \right\} \subseteq \mathcal{V}_1 \oplus \mathcal{V}_2 = \mathcal{H}_\phi$$

(the graph of $\tfrac{\|v_2\|}{\|v_1\|} U$) is closed and π-invariant. However,

$$v_\phi = v_1 + v_2 = v_1 + \frac{\|v_2\|}{\|v_1\|} U v_1 \in \mathcal{W}$$

now shows that the generator v_ϕ of $\mathcal{H}_\phi$ belongs to the π-invariant closed subspace $\mathcal{W}$. Since $\mathcal{W}$ is a proper subspace this is a contradiction. It follows that π^ϕ must be irreducible (see also the left-hand part of Figure 1.3).

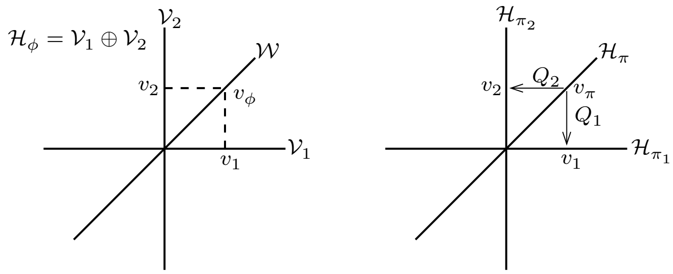

Fig. 1.3: In both directions of the proof, we relate the given representation with a direct sum representation. However, the logic of the argument is quite different in the two parts.

Irreducibility implies extremality. Suppose now that π is an irreducible unitary representation and $\phi = \varphi_{v_\pi} \in \mathcal{P}^1(G)$ for some $v_\pi \in \mathcal{H}_\pi$. We need to show that ϕ is an extremal point in $\mathcal{P}^1(G)$.

Suppose therefore that $\phi = \beta_1\phi_1 + \beta_2\phi_2$ with $\beta_1, \beta_2 \in (0,1)$, $\beta_1 + \beta_2 = 1$, and $\phi_1, \phi_2 \in \mathcal{P}^1(G)$. We wish to show that $\phi_1 = \phi_2$. Applying Proposition 1.67 to ϕ_j for $j = 1,2$ gives cyclic representations π_j with generators $v_j \in \mathcal{H}_{\pi_j}$ of norm $\|v_j\| = \phi_j(e) = 1$ and principal matrix coefficients $\phi_j(g) = \varphi_{v_j}(g) = \langle \pi_j(g)(v_j), v_j \rangle$ for $g \in G$. This shows that

$$\beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2 \in \mathcal{H}_{\pi_1} \oplus \mathcal{H}_{\pi_2}$$

has the principal matrix coefficient

$$\begin{aligned}\Big\langle (\pi_1 \oplus \pi_2)(g)\Big(\beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2\Big), \beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2 \Big\rangle& \\ &= \beta_1 \langle \pi_1(g)v_1, v_1\rangle + \beta_2 \langle \pi_2(g)v_2, v_2\rangle \\ &= (\beta_1\phi_1 + \beta_2\phi_2)(g) = \phi(g)\end{aligned}$$

for $g \in G$. Using Proposition 1.60 we see that $\mathcal{H}_\pi$, its representation π, and its generator v_π can be identified with the closed subspace

$$\mathcal{H}_\pi = \Big\langle \beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2 \Big\rangle_{\pi_1 \oplus \pi_2} \subseteq \mathcal{H}_{\pi_1} \oplus \mathcal{H}_{\pi_2},$$

the restriction of $\pi_1 \oplus \pi_2$ to this subspace, and the generator $v_\pi = \beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2$.

To obtain the desired conclusion we consider the equivariant projection operators

$$P_j\colon \mathcal{H}_{\pi_1} \oplus \mathcal{H}_{\pi_2} \to \mathcal{H}_{\pi_j}$$

and their restrictions $Q_j = P_j|_{\mathcal{H}_\pi}$ to the subspace $\mathcal{H}_\pi = \langle v_\pi \rangle_{\pi_1 \oplus \pi_2}$ (see also the right-hand part of Figure 1.3). By Schur's lemma (Theorem 1.27) we have $Q_j^* Q_j = \alpha_j I$ for some $\alpha_j \geqslant 0$. Applying this to the vector

$$v_\pi = \beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2$$

of norm one, we obtain

$$\alpha_j = \langle Q_j^* Q_j(v_0), v_0 \rangle = \left\langle P_j \left(\beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2 \right), P_j \left(\beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2 \right) \right\rangle = \beta_j > 0$$

for $j = 1, 2$. It follows that $\alpha_j^{-\frac{1}{2}} Q_j\colon \mathcal{H}_\pi \to \mathcal{H}_{\pi_j}$ is an equivariant isometry from $\mathcal{H}_\pi$ into $\mathcal{H}_{\pi_j}$ that sends $v_\pi = \beta_1^{\frac{1}{2}} v_1 \oplus \beta_2^{\frac{1}{2}} v_2$ to the generator $v_j \in \mathcal{H}_{\pi_j}$. This shows that the matrix coefficient ϕ of $v_\pi \in \mathcal{H}_\pi$ coincides with the matrix coefficient ϕ_j of $v_j \in \mathcal{H}_{\pi_j}$ for $j = 1, 2$. To summarize: we have shown that

$$\phi = \beta_1 \phi_1 + \beta_2 \phi_2$$

with $\phi_1, \phi_2 \in \mathcal{P}^1(G)$, $\beta_1, \beta_2 \in (0, 1)$ and $\beta_1 + \beta_2 = 1$ implies $\phi = \phi_1 = \phi_2$, so $\phi = \varphi_{v_\pi}$ is an extremal point of $\mathcal{P}^1(G)$. □

1.6 GNS Construction and Corollaries

The Krein–Milman and Choquet theorems (see [21, Th. 8.80 & 8.90]) both explain ways in which elements of a convex set can be determined by (or even expressed in terms of) extreme points of the convex set. However, both of the theorems require the assumption that the convex set under consideration is compact with respect to some locally convex topology. As compactness is frequently obtained from the Banach–Alaoglu theorem concerning the closed unit ball of a dual Banach space (see [21, Th. 8.10]), we wish to view $\mathcal{P}^1(G)$ as a subset of a dual space. In order to achieve this objective, we have to (seemingly) relax the definition of positive-definite functions.

As we will see, the results of this section form the functional analytic foundations for many of the following chapters.

1.6.1 Measurable Positive-Definite Functions

Recall that $L^\infty(G) = L^1(G)'$, where $L^\infty(G)$ is equipped with the *essential supremum norm*

$$\|f\|_\infty = \inf\left\{\lambda > 0 \mid m\left(\{g \in G \mid |f(g)| > \lambda\}\right) = 0\right\}.$$

Definition 1.69 (Measurable positive-definite function). A function ϕ in $L^\infty(G)$ is called *measurable positive-definite* if

$$\iint f(g_1)\overline{f(g_2)}\phi(g_2^{-1}g_1)\,\mathrm{d}m(g_1)\,\mathrm{d}m(g_2) \geqslant 0$$

for any $f \in L^1(G)$.

Essential Exercise 1.70. Show that every continuous positive-definite function is also a measurable positive-definite function.

The converse to the exercise above is more surprising, and is provided by the full version of the GNS-construction.

Theorem 1.71 (Measurable Gelfand–Naimark–Segal construction). *Every measurable positive-definite function $\phi \in L^\infty(G)$ has a continuous representative, which in turn is a matrix coefficient of a generator $v_\phi \in \mathcal{H}_\pi$ for some cyclic unitary representation $\pi = \pi^\phi$.*

For the proof the following lemma will be useful.

Lemma 1.72 (The ϕ-product). *For $\phi \in L^\infty(G)$ we define the ϕ-*product *of $f_1, f_2 \in L^1(G)$ by*

$$\langle f_1, f_2\rangle_\phi = \iint f_1(g_1)\overline{f_2(g_2)}\phi(g_2^{-1}g_1)\,\mathrm{d}m(g_1)\,\mathrm{d}m(g_2).$$

Then the ϕ-product exists for all $f_1, f_2 \in L^1(G)$ and satisfies the continuity bound

$$\left|\langle f_1, f_2\rangle_\phi\right| \leqslant \|\phi\|_\infty \|f_1\|_1 \|f_2\|_1. \tag{1.21}$$

Moreover, it can also be expressed by the convolution formula

$$\langle f_1, f_2\rangle_\phi = \int f_2^* * f_1 \phi\,\mathrm{d}m \tag{1.22}$$

for any $f_1, f_2 \in L^1(G)$. In particular, ϕ is positive-definite if and only if

$$\langle f, f\rangle_\phi = \int f^* * f \phi\,\mathrm{d}m \geqslant 0 \tag{1.23}$$

for all $f \in L^1(G)$.

Proof. Assume first $\phi \in \mathscr{L}^\infty(G)$. Let $f_1, f_2 \in L^1(G)$. Using the measure-preserving substitution replacing g_1 by $h = g_2^{-1}g_1$ and the potentially non-measure preserving substitution $k = g_2^{-1}$ from Lemma 1.13 we obtain

$$\begin{aligned}\langle f_1, f_2\rangle_\phi &= \iint f_1(g_1)\overline{f_2(g_2)}\phi(g_2^{-1}g_1)\,\mathrm{d}m(g_1)\,\mathrm{d}m(g_2)\\ &= \iint f_1(g_2h)\overline{f_2(g_2)}\phi(h)\,\mathrm{d}m(h)\,\mathrm{d}m(g_2)\\ &= \iint \overline{f_2(k^{-1})}\Delta(k)^{-1}f_1(k^{-1}h)\phi(h)\,\mathrm{d}m(k)\,\mathrm{d}m(h)\\ &= \int f_2^* * f_1(h)\phi(h)\,\mathrm{d}m(h),\end{aligned}$$

where as before $f_2^*(k) = \overline{f_2(k^{-1})}\Delta(k)^{-1}$ for $k \in G$. This implies the convolution formula (1.22) but also the continuity bound (1.21) by bounding $\phi(h)$ by its essential supremum and using the properties of the involution and convolution in $L^1(G)$. In particular, it follows that the ϕ-inner product only depends on the equivalence class $\phi \in L^\infty(G)$.

The last claim in the lemma follows by noting that the expression appearing in Definition 1.69 is $\langle f, f\rangle_\phi$ and combining this with (1.22). □

Proof of Theorem 1.71. The proof is another conjuring trick and, in particular, is similar to the proof of the continuous version in Proposition 1.67. However, since ϕ is not defined pointwise we have to start the construction with $L^1(G)$ instead of $\mathbb{C}[G]$, which requires additional arguments. In fact, it may be helpful for the reader to compare the various steps below to the easier but to a large extent analogous steps in the proof of Proposition 1.67.

Defining the Hilbert space $\mathcal{H}_\phi$. We begin by studying the ϕ-product in Lemma 1.72. Clearly $\langle\cdot,\cdot\rangle_\phi$ depends linearly on the first, and semi-linearly on the second, argument. By assumption ϕ is measurable positive-definite and so we also have $\langle f, f\rangle_\phi \geqslant 0$ for all $f \in L^1(G)$. By the argument just after Proposition 1.67, specifically (1.20), it follows that the ϕ-product is a semi-inner product on $L^1(G)$. We also define the associated semi-norm

$$\|f\|_\phi = \sqrt{\langle f, f\rangle_\phi}$$

for $f \in L^1(G)$ and the kernel $W_0 = \{f \in L^1(G) \mid \|f\|_\phi = 0\}$. The Hilbert space $\mathcal{H}_\phi$ is defined as the completion of $L^1(G)/W_0$ with respect to the norm induced on $L^1(G)/W_0$, which we again denote by $\|\cdot\|_\phi$. From (1.21) we also obtain another form of the continuity bound

$$\|f + W_0\|_\phi \leqslant \|\phi\|_\infty^{\frac{1}{2}}\|f\|_1 \tag{1.24}$$

for all $f \in L^1(G)$. In particular, the map from $L^1(G)$ to $\mathcal{H}_\phi$ is continuous. Moreover, $\mathcal{H}_\phi$ is separable since $L^1(G)$ is separable and its image $L^1(G)/W_0$ is dense in $\mathcal{H}_\phi$.

Defining the representation π^ϕ. To define the unitary representation, notice that

$$\begin{aligned}\|\lambda_g f\|_\phi^2 &= \iint f(g^{-1}g_1)\overline{f(g^{-1}g_2)}\phi(g_2^{-1}g_1)\,\mathrm{d}m(g_1)\,\mathrm{d}m(g_2)\\ &= \iint f(h_1)\overline{f(h_2)}\phi(h_2^{-1}h_1)\,\mathrm{d}m(h_1)\,\mathrm{d}m(h_2) = \|f\|_\phi^2\end{aligned}$$

for $f \in L^1(G)$ and $g \in G$, by definition of the ϕ-product and the measure preserving substitution $h_j = g^{-1}g_j$ for $j = 1, 2$. We define the unitary operator π_g as the unique extension of λ_g (which remains well-defined modulo W_0) to the completion $\mathcal{H}_\phi$.

Continuity of this unitary representation follows by applying Lemma 1.9 with $D = C_c(G)/W_0$. Indeed, for $f \in C_c(G)$, continuity of

$$G \ni g \longmapsto \lambda_g f \in L^1(G)$$

follows by the same argument as for the left-regular representation on $L^2(G)$ (see also [21, Lem. 3.74]). Finally, we apply the fact that the map from $L^1(G)$ to $L^1(G)/W_0 \subseteq \mathcal{H}_\phi$ is continuous by (1.24) to conclude that π is indeed a unitary representation.

The inner product with an approximate identity. In the following, (ψ_n) will be an approximate identity as in Proposition 1.42 (and will be needed as a substitute for δ_e in the proof of Proposition 1.67). Since

$$L^1(G) \ni f \longmapsto f^* \in L^1(G)$$

is an isometric involution, we see that $\left(\psi_n^*\right)$ is also an approximate identity and so the convolution formula of the ϕ-product in (1.22) and Proposition 1.42 imply the limit formula

$$\lim_{n\to\infty}\langle f, \psi_n\rangle_\phi = \lim_{n\to\infty}\int(\psi_n^* * f)\phi\,\mathrm{d}m = \int f\phi\,\mathrm{d}m \tag{1.25}$$

for any $f \in L^1(G)$.

Defining the generator v_ϕ. In the proof of Proposition 1.67 we introduced the vector $v_\phi = \delta_e + W_0$ which finished the proof very quickly by realizing the continuous function as the principal matrix coefficient of v_ϕ. The analogue of this is the following key step. We claim that for the approximate identity (ψ_n) the limit

$$v_\phi = \lim_{n\to\infty}(\psi_n + W_0) \in \mathcal{H}_\phi \tag{1.26}$$

exists with respect to the norm $\|\cdot\|_\phi$. As we will see the proof of this claim is similar to the continuity proof for the representation in Proposition 1.67.

In fact, we will again use the inner product and the geometry of Hilbert spaces. We start by investigating the norms $\|\psi_m\|_\phi$ for $m \geqslant 1$ and claim that $\|\psi_m\|_\phi \to \|\phi\|_\infty^{\frac{1}{2}}$ as $m \to \infty$. By the continuity bound in (1.24) we have

$$\|\psi_m\|_\phi \leqslant \|\phi\|_\infty^{\frac{1}{2}} \|\psi_m\|_1 = \|\phi\|_\infty^{\frac{1}{2}} \tag{1.27}$$

for every $m \geqslant 1$. By Cauchy–Schwarz applied to the inner product $\langle \cdot, \cdot \rangle_\phi$ we also have

$$\left| \langle f, \psi_m \rangle_\phi \right| \leqslant \|f\|_\phi \|\psi_m\|_\phi$$

for all $f \in L^1(G)$, which together with the limit formula (1.25) and the continuity bound (1.24) gives

$$\begin{aligned}\left| \int f\phi \,\mathrm{d}m \right| &= \lim_{m\to\infty} \left| \langle f, \psi_m \rangle_\phi \right| \\ &\leqslant \liminf_{m\to\infty} \|f\|_\phi \|\psi_m\|_\phi \leqslant \|f\|_1 \|\phi\|_\infty^{\frac{1}{2}} \liminf_{m\to\infty} \|\psi_m\|_\phi\end{aligned}$$

for all $f \in L^1(G)$. By the isometric identification of $\phi \in L^\infty(G)$ with the linear functional $L^1(G) \ni f \mapsto \int f\phi \,\mathrm{d}m$ it follows that

$$\|\phi\|_\infty \leqslant \|\phi\|_\infty^{\frac{1}{2}} \liminf_{m\to\infty} \|\psi_m\|_\phi,$$

which together with (1.27) gives the earlier claim

$$\lim_{m\to\infty} \|\psi_m\|_\phi = \|\phi\|_\infty^{\frac{1}{2}}.$$

We now upgrade this to a proof of the existence of the limit in (1.26), by showing that $(\psi_m + W_0)$ is a Cauchy sequence in $\mathcal{H}_\phi$. So let $\varepsilon \in (0, \|\phi\|_\infty)$ and find some $k \geqslant 1$ with $\|\psi_k\|_\phi > \sqrt{\|\phi\|_\infty - \varepsilon}$. Then

$$\|\phi\|_\infty - \varepsilon < \|\psi_k\|_\phi^2 = \langle \psi_k, \psi_k \rangle_\phi = \int (\psi_k^* * \psi_k)\phi \,\mathrm{d}m \tag{1.28}$$

by the convolution formula (1.22) for the ϕ-product. We note that, in a sense, this shows that most values $\phi(g)$ for g close to e are close to $\|\phi\|_\infty$ and so hints at the continuity of ϕ at e that we ultimately want to prove. We now define $f_\varepsilon = \psi_k^* * \psi_k$ and apply the limit formula (1.25) to see that there exists some N such that

$$\Re \langle f_\varepsilon, \psi_n \rangle_\phi > \int \psi_k^* * \psi_k \phi \,\mathrm{d}m - \varepsilon > \|\phi\|_\infty - 2\varepsilon$$

for all $n \geqslant N$. Recall (1.27) and note that $\|f_\varepsilon\|_\phi^2 \leqslant \|\phi\|_\infty \|f_\varepsilon\|_1^2 \leqslant \|\phi\|_\infty$ also by the continuity bound (1.24). We now simply use the geometry of Hilbert spaces to calculate

$$\begin{aligned}\|\psi_n - f_\varepsilon\|_\phi^2 &= \|\psi_n\|_\phi^2 + \|f_\varepsilon\|_\phi^2 - 2\Re \langle f_\varepsilon, \psi_n \rangle_\phi \\ &\leqslant \|\phi\|_\infty + \|\phi\|_\infty - 2\|\phi\|_\infty + 4\varepsilon = 4\varepsilon\end{aligned}$$

for all $n \geqslant N$. Therefore for every $\varepsilon > 0$ we have found some N so that

$$\|\psi_m - \psi_n\|_\phi \leqslant 4\varepsilon^{\frac{1}{2}}$$

for all $m, n \geqslant N$, and the claim follows.

THE MATRIX COEFFICIENT OF v_ϕ. Having obtained the existence of the limit in (1.26) we write

$$\varphi_{v_\phi}(g) = \left\langle \pi_g(v_\phi), v_\phi \right\rangle_\phi$$

for the matrix coefficient of v_ϕ. To conclude the proof of the theorem we now prove that $\varphi_{v_\phi} = \phi$ almost surely by showing $\pi_*(f)v_\phi = f + W_0$ for $f \in L^1(G)$ and calculating $\left\langle \pi_*(f)v_\phi, v_\phi \right\rangle_\phi$ in two different ways.

For $f_1, f_2 \in L^1(G)$, we may use the definition of $\pi_*(f)$ and Fubini's theorem to calculate

$$\begin{aligned}
\left\langle \pi_*(f)(f_1 + W_0), f_2 + W_0 \right\rangle_\phi &= \int \left\langle \pi_g(f_1 + W_0), f_2 + W_0 \right\rangle_\phi f(g)\,\mathrm{d}m(g) \\
&= \iiint f(g) f_1(g^{-1}g_1)\,\mathrm{d}m(g)\overline{f_2(g_2)}\phi(g_2^{-1}g_1)\,\mathrm{d}m(g_1)\,\mathrm{d}m(g_2) \\
&= \langle f * f_1 + W_0, f_2 + W_0 \rangle_\phi\,,
\end{aligned}$$

which shows that $\pi_*(f)(f_1 + W_0) = f * f_1 + W_0$ for all $f_1 \in L^1(G)$. For $f_1 = \psi_n$ we can combine this with continuity of $\pi_*(f)$ to obtain

$$\pi_*(f)v_\phi = \lim_{n\to\infty} (f * \psi_n + W_0) = f + W_0 \tag{1.29}$$

since (ψ_n) is an approximate identity and since the map from $L^1(G)$ to $\mathcal{H}_\pi$ is continuous by the continuity bound (1.24).

We now wish to prove that $\varphi_{v_\phi} = \phi$ almost everywhere. Using first the definition of $\pi_*(f)$ we see that

$$\left\langle \pi_*(f)v_\phi, v_\phi \right\rangle_\phi = \int \left\langle \pi_g v_\phi, v_\phi \right\rangle_\phi f(g)\,\mathrm{d}m(g) = \int f \varphi_{v_\phi}\,\mathrm{d}m.$$

Combining (1.29) with the definition of v_ϕ and the limit formula (1.25), we also have

$$\left\langle \pi_*(f)v_\phi, v_\phi \right\rangle_\phi = \left\langle f + W_0, v_\phi \right\rangle_\phi = \lim_{n\to\infty} \langle f, \psi_n \rangle_\phi = \int f\phi\,\mathrm{d}m$$

for every $f \in L^1(G)$. Since $\phi \in L^\infty(G)$ is uniquely determined by its associated functional on $L^1(G)$, we see that $\varphi_{v_\phi} \in C_b(G)$ is a continuous representative of the equivalence class $\phi \in L^\infty(G)$. Note that (1.29) also shows that

$$\pi_*(L^1(G))v_\phi = L^1(G) + W_0$$

is dense in $\mathcal{H}_\phi$, and so $\mathcal{H}_\phi$ is cyclic and generated by v_ϕ. □

1.6.2 Approximation by Irreducible Representations

By combining the GNS construction with the Krein–Milman theorem, we can now show that all cyclic representations can be approximated by convex combinations of irreducible representations. In the following we will refer to the topology on $C(G)$ corresponding to uniform convergence within compact subsets of G as the compact-open topology.

Corollary 1.73 (Approximation by convex combination). *The set*

$$\mathcal{P}^{\leqslant 1}(G) = \big\{\phi \in \mathcal{P}(G) \mid \phi(e) \in [0,1]\big\}$$

is convex and weak compact when considered as a subset of $L^\infty(G)$. Every element $\phi \in \mathcal{P}^1(G)$ can be approximated by a finite convex combination*

$$\sum_{j=1}^n \alpha_j \phi_j$$

of extremal elements $\phi_1, \dots, \phi_n \in \mathcal{P}^1(G)$ (corresponding to unit vectors in irreducible representations) with $\alpha_1, \dots, \alpha_n \in [0,1]$ and $\alpha_1 + \cdots + \alpha_n = 1$.

In fact, ϕ can even be approximated in the compact-open topology. That is, for every compact subset $K \subseteq G$ and every $\varepsilon > 0$ we can find a finite convex combination as above such that

$$\Big\|\phi - \sum_{j=1}^n \alpha_j \phi_j\Big\|_{K,\infty} = \sup_{g \in K}\Big|\phi(g) - \sum_{j=1}^n \alpha_j \phi_j(g)\Big| < \varepsilon.$$

Proof. By the GNS construction (Theorem 1.71) and Exercise 1.70 there is no need to distinguish between a continuous or a measurable positive-definite function if we identify a measurable positive-definite function with its continuous representative. Using the reformulation in (1.23) of the measurable definition of positive-definiteness (Definition 1.69) it is clear that the subset $\mathcal{P}(G) \subseteq L^\infty(G)$ is weak* closed. Since $\|\phi\|_\infty = \phi(e)$ (see Proposition 1.67), we obtain from the Banach–Alaoglu theorem (see [21, Th. 8.10]) that

$$\mathcal{P}^{\leqslant 1}(G) = \mathcal{P}(G) \cap \overline{B_1^{L^\infty(G)}(0)}$$

is weak* compact.

By the Krein–Milman theorem (see [21, Th. 8.80]) the set $\mathcal{P}^{\leqslant 1}(G)$ is equal to the weak* closure of the convex hull of its extremal points. Since $\phi(e) \geqslant 0$ for every $\phi \in \mathcal{P}(G)$ and $\phi(e) = \|\phi\|_\infty$, it is clear that $0 \in \mathcal{P}^{\leqslant 1}(G)$ is an extremal point of $\mathcal{P}^{\leqslant 1}(G)$. Also, any $\phi \in \mathcal{P}^{\leqslant 1}(G)$ with $\phi(e) \in (0,1)$ cannot be extremal since $\phi = (1-\phi(e))0+\phi(e)\left(\phi(e)^{-1}\phi\right)$ with $0 \neq \phi(e)^{-1}\phi \in \mathcal{P}^{\leqslant 1}(G)$. Therefore, the extremal points of $\mathcal{P}^{\leqslant 1}(G)$ are 0 and the extremal points of

$$\mathcal{P}^1(G) = \{\phi \in \mathcal{P}(G) \mid \phi(e) = 1\}.$$

By Proposition 1.68 each extremal point among the latter positive-definite functions corresponds to a unit vector in an irreducible unitary representation.

So let $\phi \in \mathcal{P}^1(G)$, let $\mathcal{N}_0$ be an open neighbourhood of 0 in the weak* topology, and let $\varepsilon \in (0,1)$. Since $\overline{B_{1-\varepsilon}^{L^\infty(G)}(0)}$ is weak* closed,

$$\mathcal{N} = \left(\phi + \tfrac{1}{2}\mathcal{N}_0\right) \smallsetminus \overline{B_{1-\varepsilon}^{L^\infty(G)}(0)}$$

is an open neighbourhood of ϕ with respect to the weak* topology. Thus, by the Krein–Milman theorem, there exists a finite convex combination

$$\sum_{j=1}^{n} \alpha_j \phi_j$$

of pairwise distinct extremal points that belongs to $\mathcal{N}$. If the extreme point $0 \in \mathcal{P}^{\leqslant 1}(G)$ does not appear in the sum, we have obtained the desired approximation of ϕ by a convex combination of extremal points of $\mathcal{P}^1(G)$.

So suppose now that 0, say as $\phi_n = 0$, appears in the convex combination. Then the corresponding coefficient satisfies $\alpha_n < \varepsilon$, as otherwise we would have $\sum_{j=1}^n \alpha_j \phi_j \in \overline{B_{1-\varepsilon}(0)}$. This implies that the convex combination

$$\phi' = (1-\alpha_n)^{-1} \sum_{j=1}^{n-1} \alpha_j \phi_j$$

of extremal points of $\mathcal{P}^1(G)$ also approximates ϕ. To see this formally, we note that $\| \sum_{j=1}^{n-1} \alpha_j \phi_j \| \leqslant \sum_{j=1}^{n-1} \alpha_j = 1 - \alpha_n$ and obtain

$$\begin{aligned}\phi - \phi' &= \phi - \sum_{j=1}^{n} \alpha_j \phi_j + \left(1 - \tfrac{1}{1-\alpha_n}\right) \sum_{j=1}^{n-1} \alpha_j \phi_j \\ &\in \tfrac{1}{2}\mathcal{N}_0 + \left(\tfrac{\alpha_n}{1-\alpha_n}\right) \overline{B_{1-\alpha_n}^{L^\infty(G)}(0)} \subseteq \tfrac{1}{2}\mathcal{N}_0 + \varepsilon \overline{B_1^{L^\infty(G)}(0)} \subseteq \mathcal{N}_0\end{aligned}$$

if we choose ε sufficiently small. This proves the first half of the corollary. The fact that the approximation is possible even with respect to the compact-open topology follows from the result in the next subsection. □

1.6.3 Upgrading to Uniform Convergence on Compact Subsets

Proposition 1.74 (Weak* and compact-open topology). *The compact-open topology on $L^\infty(G)$ is stronger than the weak* topology, and on the subset $\mathcal{P}^1(G) \subseteq C_b(G)$ the two topologies agree.*

Proof. Let us fix a unitary representation π and a unit vector $v \in \mathcal{H}_\pi$. We will compare the neighbourhoods of $\varphi_v = \varphi_v^\pi$ with respect to the two topologies.

Compact-open is stronger. For the easier half of the proposition let ϕ_0 be in $L^\infty(G)$ (for example, $\phi_0 = \varphi_v$), let f_0 be a function in $L^1(G)$, and let $\varepsilon > 0$, which together define the weak* neighbourhood

$$\mathcal{N}_{f_0;\varepsilon} = \left\{\phi \in \mathcal{P}^1(G) \mid \left|\int (\phi - \phi_0) f_0 \,\mathrm{d}m\right| < \varepsilon\right\}$$

of ϕ_0. By the density of $C_c(G)$ in $L^1(G)$ there exists $f \in C_c(G)$ with

$$\|f - f_0\|_1 \leqslant \tfrac{1}{4}\varepsilon,$$

which implies that

$$\left\{\phi \in \mathcal{P}^1(G) \mid \|\phi - \phi_0\|_{\operatorname{supp} f,\infty} < \tfrac{\varepsilon}{2(1+\|f\|_1)}\right\} \subseteq \mathcal{N}_{f;\frac{\varepsilon}{2}} \subseteq \mathcal{N}_{f_0;\varepsilon}.$$

For the latter inclusion, note that

$$\left|\int(\phi-\phi_0) f_0 \,\mathrm{d}m\right| \leqslant \left|\int \phi(f_0 - f)\,\mathrm{d}m\right| + \left|\int(\phi-\phi_0) f\,\mathrm{d}m\right| + \left|\int \phi_0(f-f_0)\,\mathrm{d}m\right| < \varepsilon$$

for any $\phi \in \mathcal{N}_{f;\frac{\varepsilon}{2}}$. Since finite intersections of neighbourhoods of the form $\mathcal{N}_{f_0;\varepsilon}$ form a neighbourhood basis for the weak* topology, this proves that the compact-open topology is stronger than the weak* topology.

Weak* is stronger. For the (more surprising) converse, we let $\varepsilon \in (0, 1]$ and let $K \subseteq G$ be a compact subset. Using the continuity property of the representation π we find a compact neighbourhood $U = U^{-1}$ of the identity $e \in G$ with

$$\|\pi_h v - v\| < \varepsilon \tag{1.30}$$

for all $h \in U$. We define $f_0 = \frac{1}{m(U)} \mathbb{1}_U$. By continuity of the left-regular representation on $L^1(G)$ there exists an open neighbourhood $V = V^{-1} \subseteq U$ of $e \in G$ such that

$$\left\|\lambda_h f_0 - f_0\right\|_1 < \varepsilon \tag{1.31}$$

for all $h \in V$. Applying compactness of K, we can choose a finite cover

$$\bigcup_{\ell=1}^{n} g_\ell V \supseteq K. \tag{1.32}$$

Finally, we define $f_\ell = \frac{1}{m(U)} \mathbb{1}_{g_\ell U}$ for $\ell = 1, \ldots, n$, and use these to define a weak* neighbourhood $\mathcal{N}_{f_0, f_1, \ldots, f_n; \varepsilon} = \mathcal{N}_{f_0; \varepsilon} \cap \ldots \cap \mathcal{N}_{f_n; \varepsilon}$ of φ_v. We will show that any $\phi \in \mathcal{N}_{f_0, f_1, \ldots, f_n; \varepsilon}$ is within $O(\varepsilon^{\frac{1}{2}})$ of φ_v on K.

So suppose that $\phi \in \mathcal{N}_{f_0, \ldots, f_n; \varepsilon}$ and use the continuous GNS construction (Proposition 1.67) to find a unitary representation ρ and some $a \in \mathcal{H}_\rho$ with $\|a\| = 1$ and $\phi = \varphi_a^\rho$ so that

$$\left| \int_G (\varphi_a^\rho - \varphi_v) \, f_\ell \, \mathrm{d}m \right| < \varepsilon \tag{1.33}$$

for $\ell = 0, \ldots, n$. We define $a' = \rho_*(f_0) a \in \mathcal{H}_\rho$, so that $\|a'\| \leqslant \|f_0\|_1 \|a\| = 1$, and notice that

$$\begin{aligned} \left\langle a', a \right\rangle &= \int f_0(h) \left\langle \rho_h a, a \right\rangle \, \mathrm{d}m(h) = \int f_0 \varphi_a^\rho \, \mathrm{d}m \\ &= \int f_0 \varphi_v \, \mathrm{d}m + O(\varepsilon) && \text{(by (1.33))} \\ &= \frac{1}{m(U)} \int_U \left\langle \pi_h v, v \right\rangle \, \mathrm{d}m(h) + O(\varepsilon) = 1 + O(\varepsilon) \end{aligned}$$

by (1.30) and Cauchy–Schwarz applied to $\langle \pi_h v - v, v \rangle$ for $h \in U$. Hence we obtain $\Re \left\langle a', a \right\rangle > 1 - O(\varepsilon)$ and

$$\|a' - a\|^2 = \|a'\|^2 - 2\Re \left\langle a', a \right\rangle + \|a\|^2 \ll \varepsilon,$$

which gives

$$\|a' - a\| \ll \varepsilon^{\frac{1}{2}}. \tag{1.34}$$

For $h \in V$ we also obtain

$$\|\rho_h a' - a'\| = \|(\rho_h \rho_*(f_0) - \rho_*(f_0)) a\| = \|\rho_*(\lambda_h f_0 - f_0) a\| = O(\varepsilon) \tag{1.35}$$

by (1.31).

We now use these uniform continuity estimates together with the cover in (1.32). In fact for every $g \in K$ there exists some $\ell \in \{1, \ldots, n\}$ with $g = g_\ell h$ for some $h \in V$. Recall that

$$\rho_{g_\ell} a' = \rho_{g_\ell} \rho_*(f_0) a = \rho_*(\lambda_{g_\ell} f_0) a = \rho_*(f_\ell) a. \tag{1.36}$$

Together with Cauchy–Schwarz the above bounds yield

$$\begin{aligned}
\varphi_a^\rho(g) &= \langle \rho_{g_\ell}\rho_h a', a\rangle + O(\varepsilon^{\frac{1}{2}}) && \text{(by (1.34))}\\
&= \langle \rho_{g_\ell} a', a\rangle + O(\varepsilon^{\frac{1}{2}}) && \text{(by (1.35))}\\
&= \langle \rho_*(f_\ell)a, a\rangle + O(\varepsilon^{\frac{1}{2}}) && \text{(by (1.36))}\\
&= \int f_\ell \varphi_a^\rho \,\mathrm{d}m + O(\varepsilon^{\frac{1}{2}}).
\end{aligned}$$

By assumption, we have

$$\int f_\ell \varphi_a^\rho \,\mathrm{d}m = \int f_\ell \varphi_v \,\mathrm{d}m + O(\varepsilon).$$

Combining this with the above, and using the fact that the above applies to any positive-definite function $\phi = \varphi_a^\rho$ in the neighbourhood $\mathcal{N}_{f_0,\dots,f_n;\varepsilon}$ of φ_v and so in particular to itself, we obtain

$$\varphi_a^\rho(g) = \varphi_v(g) + O(\varepsilon^{\frac{1}{2}})$$

for all $g \in K$. Since $K \subseteq G$ was an abitrary compact subset and $\varepsilon > 0$ was arbitrary, this concludes the proof that the weak* topology is stronger. □

1.6.4 Separation of Points by Irreducible Representations

Corollary 1.75 (Gelfand–Raikov). *Let g_1 and g_2 be distinct elements of the group G. Then there exists an irreducible unitary representation π of G with $\pi_{g_1} \neq \pi_{g_2}$.*

Proof. As $g_0 = g_1^{-1} g_2 \in G\smallsetminus\{e\}$, there exists a neighbourhood U of e with $g_0 \notin U$. By continuity of the map $G \ni g \mapsto g^{-1}$ and by continuity of multiplication there exists a compact neighbourhood V of $e \in G$ with $V \cdot V^{-1} \subseteq U$. We define

$$f = m(V)^{-\frac{1}{2}} \mathbb{1}_V,$$

which satisfies $\varphi_f(e) = \|f\|_2^2 = 1$ and

$$\varphi_f(g_0) = \frac{1}{m(V)} \int \mathbb{1}_V\big(g_0^{-1} g\big)\mathbb{1}_V(g)\,\mathrm{d}m(g) = 0$$

since $g_0^{-1}g \in V$ and $g \in V$ would imply that $g_0 = g\big(g_0^{-1}g\big)^{-1} \in VV^{-1} \subseteq U$.

In other words, the above defines a principal matrix coefficient $\varphi_f \in \mathcal{P}^1(G)$ that separates g_0 from the identity $e \in G$. By Corollary 1.73, there exist irreducible representations π_j, unit vectors $v_j \in \mathcal{H}_{\pi_j}$, and scalars $\alpha_j \in [0,1]$ for $j = 1, \dots, n$ with $\sum_{j=1}^n \alpha_j = 1$ and with

$$\Big\|\varphi_f - \sum_{j=1}^{n} \alpha_j \varphi_{v_j}^{\pi_j}\Big\|_{\{g_0\},\infty} < \frac{1}{3}.$$

Thus

$$\Big|\sum_{j=1}^{n} \alpha_j \varphi_{v_j}^{\pi_j}(g_0)\Big| < \frac{1}{3}.$$

Therefore, there exists some j with

$$\varphi_{v_j}^{\pi_j}(g_0) = \big\langle (\pi_j)_{g_0} v_j, v_j \big\rangle \neq 1 = \|v_j\|^2.$$

This implies for $\pi = \pi_j$ that $\pi_{g_1}^{-1}\pi_{g_2} = \pi_{g_0} \neq I$ and the corollary follows. □

Exercise 1.76. Suppose that every irreducible unitary representation of G is one-dimensional (that is, defined by a unitary character of G). Show that this implies that G is abelian.

1.7 First Semi-Simple Phenomena*

To start to see the stark contrast between abelian groups and semi-simple groups (with compact and solvable groups in between in many ways), we prove two fundamental results for $\mathrm{SL}_2(\mathbb{R})$. We restrict to this specific group solely in the interests of brevity, since the results hold more generally for any non-compact simple Lie group (and with some complications also for non-compact semi-simple Lie groups).

Theorem 1.77 (Mautner phenomenon for $\mathrm{SL}_2(\mathbb{R})$). *If π is a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ and $v \in \mathcal{H}_\pi$ is invariant (meaning that $\pi_g v = v$) under g either for*

$$g = a_t = \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix}$$

for some $t \neq 0$, or for

$$g = u_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$$

for some $s \neq 0$, then v is invariant under all of $\mathrm{SL}_2(\mathbb{R})$.

Proof. Suppose that $v \in \mathcal{H}_\pi$ is invariant under a_t for $t \neq 0$. Replacing a_t by a_t^{-1} if necessary, we may assume that $t > 0$. Then

$$a_t^{-n} u_s a_t^n = \begin{pmatrix} \mathrm{e}^{-nt} & \\ & \mathrm{e}^{nt} \end{pmatrix} \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{nt} & \\ & \mathrm{e}^{-nt} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{e}^{-2nt} s \\ & 1 \end{pmatrix}$$

so that

$$\|\pi_{u_s} v - v\| = \|\pi_{a_t^{-n}}(\pi_{u_s}\pi_{a_t^n} v - \pi_{a_t^n} v)\| = \|\pi_{a_t^{-n} u_s a_t^n} v - v\| \longrightarrow 0$$

as $n \to \infty$ by continuity of the representation. For $\begin{pmatrix} 1 & \\ s & 1 \end{pmatrix}$ we can use the same argument but taking $n \to -\infty$ instead. Together we deduce that v is invariant under

$$H = \left\langle \left\{ \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ s & 1 \end{pmatrix} \Big| s \in \mathbb{R} \right\} \right\rangle.$$

However, it is easy to see (cf. Section A.1) that $H = \mathrm{SL}_2(\mathbb{R})$, which proves the first case of the theorem.

Now suppose that $v \in \mathcal{H}_\pi$ is invariant under $u_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$ for $s \neq 0$. Let us define the shorthand $g_\varepsilon = \begin{pmatrix} 1 & \\ \varepsilon & 1 \end{pmatrix}$ for $\varepsilon \in \mathbb{R}$ for convenience. Then

$$u_s^m g_\varepsilon u_s^n = \begin{pmatrix} 1 & ms \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & ns \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 + \varepsilon ms & ms(1 + \varepsilon ns) + ns \\ \varepsilon & 1 + \varepsilon ns \end{pmatrix}.$$

Choosing $\varepsilon = \varepsilon(n) = -\frac{1}{2ns}$ and $m = -2n$, this simplifes to give

$$u_s^{-2n} g_{\varepsilon(n)} u_s^n = \begin{pmatrix} 2 & 0 \\ \varepsilon(n) & \frac{1}{2} \end{pmatrix}.$$

Using continuity of the representation again, we see that

$$\|\pi_{a_{\log 2}} v - v\| = \lim_{n\to\infty} \|\pi_{u_s^{-2n} g_{\varepsilon(n)} u_s^n} v - v\| = \lim_{n\to\infty} \|\pi_{g_{\varepsilon(n)}} v - v\| = 0.$$

It follows that v is invariant under $a_{\log 2}$, and the first argument now applies to show that v is invariant under $\mathrm{SL}_2(\mathbb{R})$. □

Essential Exercise 1.78. Extend Theorem 1.77 by proving that if π is a unitary representation of $\mathrm{SL}_3(\mathbb{R})$ and $v \in \mathcal{H}_\pi$ is invariant under

$$\left\{ \begin{pmatrix} 1 & & x_1 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} \Big| x_1, x_2 \in \mathbb{R} \right\} \cong \mathbb{R}^2$$

then v is invariant under $\mathrm{SL}_3(\mathbb{R})$.

Exercise 1.79 (No non-trivial finite-dimensional representations). Use the method used to prove Theorem 1.77 to show that if π is a finite-dimensional unitary representation of $\mathrm{SL}_2(\mathbb{R})$, then $\pi_g = I$ for all $g \in \mathrm{SL}_2(\mathbb{R})$.

Our second initial result about $\mathrm{SL}_2(\mathbb{R})$ gives a significant strengthening of the Mautner phenomenon. We say that π *has invariant vectors* if there exists a non-zero vector $v \in \mathcal{H}_\pi$ that is invariant.

Theorem 1.80 (Howe–Moore for $\mathrm{SL}_2(\mathbb{R})$). *Let π be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ that does not have invariant vectors. Then every matrix coefficient $\varphi_{v,w}(g) = \langle \pi_g v, w \rangle$ converges to zero as $g \to \infty$ in $\mathrm{SL}_2(\mathbb{R})$.*

We recall that for a sequence $g_n \in G$ we write $g_n \to \infty$ as $n \to \infty$ if for every compact set $K \subseteq G$ there is some $N(K)$ with the property that $n > N(K)$ implies that $g_n \notin K$. The formulation in the theorem is a shorthand for $\varphi_{v,w}(g_n) \to 0$ for any such sequence.

Proof of Theorem 1.80. Let (g_n) be a sequence in $\mathrm{SL}_2(\mathbb{R})$ with $g_n \to \infty$ as $n \to \infty$.

Diagonal case. Assume first that $g_n = \begin{pmatrix} \mathrm{e}^{t_n} & \\ & \mathrm{e}^{-t_n} \end{pmatrix}$ for some sequence (t_n) with $t_n \to \infty$ as $n \to \infty$. Notice that $\pi_{g_n} v \in \overline{B_{\|v\|}^{\mathcal{H}_\pi}(0)}$ belongs to a fixed weak* compact subset of $\mathcal{H}_\pi$ so that there exists a weak* convergent subsequence (see [21, Prop. 8.11]), which for simplicity we again denote $(\pi_{g_n} v)$. Thus

$$\widetilde{v} = \lim_{n\to\infty} \pi_{g_n} v$$

exists in the weak* topology. We claim that $\widetilde{v} = 0$, which will follow from Theorem 1.77 and our assumption on π if we show that $\widetilde{v}$ is invariant under $u = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$ for all $s \in \mathbb{R}$. To prove the latter, let $w \in \mathcal{H}_\pi$ and calculate

$$\langle \pi_u \widetilde{v}, w \rangle = \left\langle \widetilde{v}, \pi_u^* w \right\rangle = \lim_{n\to\infty} \left\langle \pi_{g_n} v, \pi_u^* w \right\rangle = \lim_{n\to\infty} \left\langle \pi_{g_n} \pi_{g_n^{-1} u g_n} v, w \right\rangle.$$

Now notice that $\lim_{n\to\infty} \|\pi_{g_n^{-1} u g_n} v - v\| = 0$ by continuity of the representation since $g_n^{-1} u g_n = \begin{pmatrix} 1 & \mathrm{e}^{-2t_n} s \\ & 1 \end{pmatrix} \to I$ as $n \to \infty$. Hence

$$\begin{aligned} |\langle \pi_u \widetilde{v}, w \rangle - \langle \widetilde{v}, w \rangle| &= \lim_{n\to\infty} \left| \left\langle \pi_{g_n} \pi_{g_n^{-1} u g_n} v, w \right\rangle - \left\langle \pi_{g_n} v, w \right\rangle \right| \\ &\leqslant \lim_{n\to\infty} \|\pi_{g_n^{-1} u g_n} v - v\| \|w\| = 0, \end{aligned}$$

which shows that $\pi_u \widetilde{v} = \widetilde{v}$ since w was arbitrary. As discussed above, this implies that $\widetilde{v} = 0$. By weak* compactness of $\overline{B_{\|v\|}^{\mathcal{H}_\pi}(0)}$ this implies that $\lim_{n\to\infty} \pi_{g_n} v = 0$ without any need to pass to a subsequence. Indeed, if this were not the case then we would be able to find a weak* neighbourhood N of 0 and a subsequence that is entirely outside N. Choosing a weak* convergent subsequence and applying the argument above then gives a contradiction.

Extending to the general case. If $g_n \in \mathrm{SL}_2(\mathbb{R})$ for all $n \geqslant 1$ and we have that $g_n \to \infty$ as $n \to \infty$, then we can apply the Cartan decomposition (see Section A.2 in Appendix A) to write $g_n = k_n a_n k_n'$ with $k_n, k_n' \in K = \mathrm{SO}_2(\mathbb{R})$ and

$$a_n = \begin{pmatrix} \mathrm{e}^{t_n} & \\ & \mathrm{e}^{-t_n} \end{pmatrix}$$

with $t_n \to \infty$ as $n \to \infty$. Using compactness of K in the same way as above, we may assume that $k_n \to k$ and $k_n' \to k'$ as $n \to \infty$. By continuity of the representation and the case for (a_n) proved above, we may then find for $\varepsilon > 0$ some n_0 such that $n \geqslant n_0$ implies the three estimates indicated in the braces underneath the expression

$$\left\langle \pi_{a_n} \underbrace{\pi_{k_n'} v - \pi_{k'} v}_{\|\cdot\|<\varepsilon}, \pi_{k_n^{-1}} w \right\rangle + \left\langle \pi_{a_n} \pi_{k'} v, \underbrace{\pi_{k_n^{-1}} w - \pi_{k^{-1}} w}_{\|\cdot\|<\varepsilon} \right\rangle + \underbrace{\left\langle \pi_{a_n} \pi_{k'} v, \pi_{k^{-1}} w \right\rangle}_{\|\cdot\|<\varepsilon}.$$

However, recalling that $g_n = k_n a_n k_n'$, this sum is actually equal to $\langle \pi_{g_n} v, w \rangle$. Hence the Cauchy–Schwarz inequality gives

$$\left| \left\langle \pi_{g_n} v, w \right\rangle \right| \leqslant (\|w\| + \|v\| + 1)\,\varepsilon$$

for all $n \geqslant n_0$, which proves the theorem. □

Exercise 1.81 (No non-trivial finite-dimensional representations). Prove the same statement as in Exercise 1.79, but this time as a consequence of Theorem 1.80.

One of the motivations for the material in this volume is to render Theorem 1.80 effective. That is, to obtain a concrete estimate on the rate of decay of the matrix coefficients. As we will see in Chapter 7, this is relatively straightforward for some groups (with $\mathrm{SL}_3(\mathbb{R})$ being the prime example), but is more delicate for $\mathrm{SL}_2(\mathbb{R})$, see Chapters 7 and 8.

Exercise 1.82 (Frobenius). Let $p > 2$ be a prime and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field with p elements. Show that any non-trivial unitary representation of the finite group $\mathrm{SL}_2(\mathbb{F}_p)$ has dimension at least $\frac{p-1}{2}$.

1.8 Summary and Outlook

The results of this chapter form the basis of all the discussions that follow. Most notably, we will need:

- Schur's lemma and its corollaries,
- familiarity with convolution in $L^1(G)$ and the convolution operators for unitary representations,
- the notions of matrix coefficients, positive-definite functions and their equivalences, and
- the corollaries to the Gelfand–Naimark–Seagal construction.

We did not yet see many examples of unitary duals, and this should be corrected as soon as possible. There are two general classes of groups that are reasonable candidates to discuss next. In fact the unitary representation theory of abelian groups and of compact groups are the easiest to understand, and we could continue with either of these.

We decided to continue with abelian groups in the next chapter for several reasons. Indeed, achieving a complete understanding of the structure of unitary representations of abelian groups allows us to achieve a classification of irreducible unitary representations for certain semi-direct products in Chapter 3. In that setting we can give many concrete examples of infinite-dimensional irreducible unitary representations.

A second reason for delaying the quite basic theory of unitary representations of compact groups to Chapter 5 is that we also would like to accompany the abstract results with some discussions of unitary duals of concrete compact non-abelian groups. However, this immediately brings simple groups like $\mathrm{SU}_2(\mathbb{R})$ into the picture, and leads us naturally into the discussions of unitary representations of Lie groups in Chapters 6 to 9.

The reader may continue with Chapter 2 or move straight to Chapter 5, returning to Chapters 2 to 4 only when needed later.