A Pólya–Carlson dichotomy for algebraic dynamics?

(joint work with Jason Bell, Robert Royals and Richard Miles)

Tom Ward (Durham) October 2014, UEA

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These series are also called lacunary as the early examples had missing powers.

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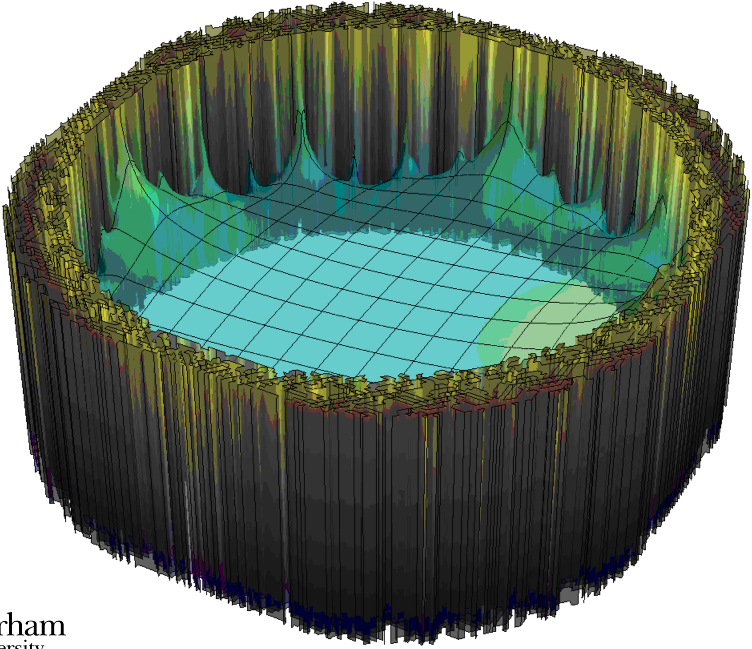

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and generating function

$$F(z) = \sum_{n \geqslant 1} Fix_\theta(n) z^n$$

are invariants of topological conjugacy.

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Context

Hasse–Weil introduced the function

$$\zeta_{HW}(z) = \exp \sum_{n \geqslant 1} \frac{z^n}{n} Fix(f^n)$$

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$$\exp \sum_{n\geqslant 1} \frac{z^n}{n} \mathrm{trace}(A^n) = \frac{1}{\det(I - zA)}$$

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for any operator (or matrix) where this makes sense.In particular: if periodic points are counted by traces of some operator, then we expect rationality and a link to the spectrum of that operator.

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Another reason to view the $\exp \sum$ formalism as natural is an Euler product formula:

$$\zeta_f(z) = \prod_{\tau} (1 - z^{|\tau|})^{-1},$$

where the product is taken over all closed periodic orbits τ with $|\tau|$ the length of τ.

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- More generally, toral endomorphisms / automorphisms have rational zeta functions.
- Many natural dynamical systems have rational zeta functions (shifts of finite type, Axiom A maps,...).

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- The map $x \mapsto 1 - \mu x^2$ for $\mu = 1.401155\dots$ (the 'Feigenbaum constant') has exactly one closed orbit of length 2^n for each $n \geqslant 1$, so
$$\zeta(z) = \prod_{n=0}^{\infty}(1 - z^{2^n})^{-1} = \prod_{n=0}^{\infty}(1 + z^{2^n})^{n+1}.$$
Notice that $\zeta(z^2) = (1-z)\zeta(z)$, so this has a natural boundary for 'lacunary' reasons.

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Cautionary example: The function $f(z) = \frac{1}{(1-z)(1-z^5)}$ is the dynamical zeta function of the permutation $\tau = (1)(23456)$ on the set $\{1, 2, 3, 4, 5, 6\}$. The sequence $(Fix_\tau(n))$ is a linear recurrent divisibility sequence greater than or equal to 1, but f is not the zeta function of any group automorphism.

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The 'next simplest' automorphism, the dual of $x \mapsto 2x$ on $\mathbb{Z}[1/6]$ has

$$Fix(n) = (2^n - 1)|2^n - 1|_3,$$

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We claim that $|z| = 1$ is a natural boundary for R, and hence $|z| = \frac{1}{2}$ is one for F (and hence for ζ).

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Write

$$R(z) = \frac{1}{3} \sum_{2|n} |n|_3 z^n + \sum_{2\nmid n} z^n,$$

so $R(z) = \frac{1}{3} G(z^2) + H_2(z)$, where $G(z) = \sum_{n \geqslant 1} |n|_3 z^n$.

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Writing $n = 3^e k$, where $e \geqslant 0$ and $3 \nmid k$, gives

$$G(z) = \sum_{e\geqslant 0} \frac{1}{3^e} \sum_{3\nmid k} z^{3^e k} = \sum_{e\geqslant 0} \frac{1}{3^e} H_3(z^{3^e})$$

$$= H_3(z) + \frac{1}{3}\sum_{e\geqslant 0} \frac{1}{3^e} H_3(z^{3^{e+1}}).$$

It follows that

$$G(z) = H_3(z) + \frac{1}{3}G(z^3).$$

Using this functional equation inductively, we deduce that there are dense singularities of G on the unit circle, occurring at 3^e-th roots of unity, $e \in \mathbb{N}$.

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Remark: This is not a reasonable proof – its only method is luck.

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Question: Do zeta functions for compact group automorphisms enjoy the same dichotomy?

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Hadamard: Let $\mathbb{K}$ be a field of characteristic zero, and suppose that $\sum_{n \geqslant 0} b_n z^n$ and $\sum_{n \geqslant 1} c_n z^n$ in $\mathbb{K}[[z]]$ are expansions of rational functions. If there is a finitely-generated ring R over $\mathbb{Z}$ with $a_n = \frac{b_n}{c_n} \in R$ for all $n \geqslant 1$, then $\sum_{n \geqslant 0} a_n z^n$ is also the expansion of a rational function.

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Fabry: If $0 < p_1 < p_2 < \cdots$ are integers with $\frac{p_n}{n} \to \infty$ as $n \to \infty$ and (a_n) is a sequence of complex numbers for which $\sum_{n\geqslant 1} a_n z^{p_n}$ has radius of convergence 1, then the series admits $|z| = 1$ as a natural boundary.

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Warning: The radius of convergence of the zeta function of a group automorphism is rarely 1, and is usually unknown.

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The simplest case is to assume that X is a one-dimensional solenoid, so (roughly) the automorphism is dual to the map $x \mapsto rx$ on the ring $R = \mathbb{Z}[\frac{1}{p} : p \in S]$ for some subset S of the primes.

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Write $f_S(n) = |r^n - 1| \cdot |r^n - 1|_S$ and $F_S(z) = \sum_{n \geqslant 1} f_S(n) z^n$, where $|x|_S = \prod_{p \in S} |x|_p$.

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To see how Hadamard arises, we claim that F_S is rational if and only if $|r|_p \neq 1$ for all $p \in S$ ('hyperbolicity').

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Arithmetic arguments can then be used to show that f takes on infinitely many values infinitely often, which is impossible.

For S co-finite it is easy to show that the Pólya–Carlson dichotomy holds because the theorem itself applies.

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4) If the dichotomy is really there, to explain this rigidity.

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