Chapter 6
Lifting Entropy

In this chapter we will extend the theory of topological entropy to uniformly continuous maps on metric spaces, and use this method (introduced by Bowen [25]) to compute the topological entropy of automorphisms of solenoids (including the torus) and other examples of homogeneous dynamical systems. This will lead to many generalizations of Theorem 1.32.

Most of these generalizations could (individually) be proven much faster than the almost axiomatic approach taken here, but the discussion here gives a common framework from which the various instances of calculating entropy for automorphisms on a torus, a solenoid or for flows on homogeneous spaces (real or p-adic) all follow quite quickly.

6.1 Entropy for Uniformly Continuous Maps

In this section we suppose that (X, d) is a locally compact σ-compact metric space, and that

$$T\colon X \longrightarrow X$$

is a uniformly continuous map (that is, for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\mathsf{d}(x,y) < \delta$ implies $\mathsf{d}(f(x), f(y)) < \varepsilon$ for all $x, y \in X$).

Definition 5.16 extends to this setting as follows. For a compact set $K \subseteq X$, we say that a subset $F_{\text{span}} \subseteq K$ (n, ε)-spans K if, for every $x \in K$ there is a point $y \in F_{\text{span}}$ with

$$\mathsf{d}_n(x,y) = \max_{0 \leqslant i \leqslant n-1} \mathsf{d}(T^i x, T^i y) \leqslant \varepsilon.$$

A subset $F_{\text{sep}} \subseteq K$ is (n, ε)-separated if for any two distinct points $x, y \in F_{\text{sep}}$,

$$\mathsf{d}_n(x,y) = \max_{0 \leqslant i \leqslant n-1} \mathsf{d}(T^i x, T^i y) > \varepsilon.$$

We let $s_{\mathrm{span}}(n,\varepsilon,K,\mathsf{d})$ be the smallest cardinality of any set F_{span} which (n,ε)-spans K with respect to T, and let $s_{\mathrm{sep}}(n,\varepsilon,K,\mathsf{d})$ denote the largest cardinality of any (n,ε)-separated set with respect to T contained in K.

As before, the compactness of K ensures that the numbers $s_{\mathrm{sep}}(n,\varepsilon,K,\mathsf{d})$ and $s_{\mathrm{span}}(n,\varepsilon,K,\mathsf{d})$ are finite for any $n \geqslant 1$ and $\varepsilon > 0$. Notice that the number s_{sep} is a monotone functions of K in the sense that if $K \subseteq K'$ then

$$s_{\mathrm{sep}}(n,\varepsilon,K,\mathsf{d}) \leqslant s_{\mathrm{sep}}(n,\varepsilon,K',\mathsf{d}). \tag{6.1}$$

Just as in Lemma 5.17, we have that

$$s_{\mathrm{span}}(n,\varepsilon,K,\mathsf{d}) \leqslant s_{\mathrm{sep}}(n,\varepsilon,K,\mathsf{d}) \leqslant s_{\mathrm{span}}(n,\varepsilon/2,K,\mathsf{d}) < \infty;$$

it is also clear that if $\varepsilon < \varepsilon'$ then

$$\limsup_{n\to\infty} \frac{1}{n} \log s_{\mathrm{span}}(n,\varepsilon,K,\mathsf{d}) \geqslant \limsup_{n\to\infty} \frac{1}{n} \log s_{\mathrm{span}}(n,\varepsilon',K,\mathsf{d})$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log s_{\mathrm{sep}}(n,\varepsilon,K,\mathsf{d}) \geqslant \limsup_{n\to\infty} \frac{1}{n} \log s_{\mathrm{sep}}(n,\varepsilon',K,\mathsf{d}).$$

An immediate consequence of the above equations is that the following definition makes sense.

Definition 6.1. The Bowen entropy of T with respect to K is

$$\begin{aligned} h_{\mathsf{d}}(T,K) &= \lim_{\varepsilon\searrow 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{\mathrm{span}}(n,\varepsilon,K,\mathsf{d}) \\ &= \lim_{\varepsilon\searrow 0} \limsup_{n\to\infty} \frac{1}{n} \log s_{\mathrm{sep}}(n,\varepsilon,K,\mathsf{d}), \end{aligned}$$

and the Bowen entropy of T is

$$h_{\mathsf{d}}(T) = \sup_{K\subseteq X \text{ compact}} h_{\mathsf{d}}(T,K).$$

From Theorem 5.19 it is clear that $h_{\mathsf{d}}(T) = h_{\mathrm{top}}(T)$ whenever T is a continuous map on a compact metric space (X,d).

The notation reflects the fact that the quantities all depend(29) on the choice of the metric d (see also Exercise 6.1.1).

Lemma 6.2 (Entropy of unions). *Let T be a uniformly continuous map on a locally compact σ-compact metric space (X,d). For compact sets K_1, K_2 in X and $K \subseteq K_1 \cup K_2$,*

$$h_{\mathsf{d}}(T,K) \leqslant h_{\mathsf{d}}(T,K_1\cup K_2) = \max\bigl(h_{\mathsf{d}}(T,K_1), h_{\mathsf{d}}(T,K_2)\bigr).$$

Proof. By (6.1) we have $h_{\mathsf{d}}(T,K) \leqslant h_{\mathsf{d}}(T,K_1\cup K_2)$ and in particular

$$\max\bigl(h_{\mathsf{d}}(T,K_1),h_{\mathsf{d}}(T,K_2)\bigr)\leqslant h_{\mathsf{d}}(T,K_1\cup K_2).$$

For the converse inequality we note that

$$\begin{aligned}s_{\text{span}}(n,\varepsilon,K_1\cup K_2,\mathsf{d})&\leqslant s_{\text{span}}(n,\varepsilon,K_1,\mathsf{d})+s_{\text{span}}(n,\varepsilon,K_2,\mathsf{d})\\&\leqslant 2\max\bigl(s_{\text{span}}(n,\varepsilon,K_1,\mathsf{d}),s_{\text{span}}(n,\varepsilon,K_2,\mathsf{d})\bigr)\end{aligned}$$

for any n and $\varepsilon>0$. Therefore,

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log s_{\text{sep}}(n,\varepsilon,K_1\cup K_2,\mathsf{d})\\&\qquad\leqslant\limsup_{n\to\infty}\frac{1}{n}\log\bigl(2\max\left(s_{\text{sep}}(n,\varepsilon,K_1,\mathsf{d}),s_{\text{sep}}(n,\varepsilon,K_2,\mathsf{d})\right)\bigr)\\&\qquad=\max\Bigl(\limsup_{n\to\infty}\frac{1}{n}\log s_{\text{sep}}(n,\varepsilon,K_1,\mathsf{d}),\limsup_{n\to\infty}\frac{1}{n}s_{\text{sep}}(n,\varepsilon,K_2,\mathsf{d})\Bigr)\\&\qquad=\max\left(h_{\mathsf{d}}(T,K_1),h_{\mathsf{d}}(T,K_2)\right).\end{aligned}$$

□

Proposition 6.3 (Small compact sets). *For any $\delta>0$, the supremum of $h_{\mathsf{d}}(T,K)$ over all compact sets K of diameter no more than δ coincides with $h_{\mathsf{d}}(T)$.*

Proof. Any compact set $K'\subseteq X$ has a finite cover

$$K'\subseteq B_{\delta/2}(x_1)\cup\cdots\cup B_{\delta/2}(x_k)$$

by metric open balls of radius $\delta/2$. By Lemma 6.2 and induction,

$$h_{\mathsf{d}}(T,K')\leqslant\max_{1\leqslant j\leqslant k}\{h_{\mathsf{d}}(T,K'\cap\overline{B_{\delta/2}(x_j)})\},$$

completing the proof. □

As discussed in Lemmas 5.20 and 5.22, topological entropy has functorial properties.(30) This is also true for uniformly continuous maps discussed here (see Exercise 6.1.2) but a small complication does arise in the non-compact setting, which will be explained in Lemma 6.4. We note that Lemma 6.4 in particular gives a proof of Lemma 5.21. Below we will find spanning and separating sets for various maps, so we add a superscript to s_{span} and s_{sep} to denote this when needed.

Lemma 6.4 (Entropy of products). *If $T_i\colon X_i\to X_i$ are uniformly continuous maps of metric spaces (X_i,d_i) for $i=1,2$, then*

$$h_{\mathsf{d}}(T_1\times T_2)\leqslant h_{\mathsf{d}_1}(T_1)+h_{\mathsf{d}_2}(T_2),$$

where the metric on $X_1\times X_2$ is

$$\mathsf{d}\big((x_1,x_2),(y_1,y_2)\big)=\max\big(\mathsf{d}_1(x_1,y_1),\mathsf{d}_2(x_2,y_2)\big).$$

If we assume† *that*

$$h_{\mathsf{d}_1}(T_1,K_1)=\lim_{\varepsilon\searrow 0}\liminf_{n\to\infty}\frac{1}{n}\log s_{\mathrm{sep}}(n,\varepsilon,K_1,\mathsf{d}_1) \tag{6.2}$$

for all $K_1\subseteq X_1$*, then*

$$h_{\mathsf{d}}(T_1\times T_2)=h_{\mathsf{d}_1}(T_1)+h_{\mathsf{d}_2}(T_2).$$

Proof. Let K_i be compact in X_i, and let $F_{\mathrm{span},i}$ be an (n,ε)-spanning set for K_i. Then $F_{\mathrm{span},1}\times F_{\mathrm{span},2}$ is (n,ε)-spanning for $K_1\times K_2$ with respect to $T_1\times T_2$. It follows that

$$s_{\mathrm{span}}^{T_1\times T_2}(n,\varepsilon,K_1\times K_2,\mathsf{d})\leqslant s_{\mathrm{span}}^{T_1}(n,\varepsilon,K_1,\mathsf{d}_1)s_{\mathrm{span}}^{T_2}(n,\varepsilon,K_2,\mathsf{d}_2),$$

so

$$h_{\mathsf{d}}(T_1\times T_2,K_1\times K_2)\leqslant h_{\mathsf{d}_1}(T_1,K_1)+h_{\mathsf{d}_2}(T_2,K_2).$$

Write $\pi_i\colon X_1\times X_2\to X_i$ for the projection map for $i=1,2$. If $K\subseteq X_1\times X_2$ is compact, then $K\subseteq\pi_1(K)\times\pi_2(K)$ and

$$h_{\mathsf{d}}(T,K)\leqslant h_{\mathsf{d}}(T_1\times T_2,\pi_1(K)\times\pi_2(K))\leqslant h_{\mathsf{d}_1}(T_1)+h_{\mathsf{d}_2}(T_2)$$

This proves the first half of the lemma.

For the reverse inequality, we assume (6.2). If $F_{\mathrm{sep},i}$ is (n,ε)-separated for the compact set $K_i\subseteq X_i$ under T_i and for $i=1,2$, then $F_{\mathrm{sep},1}\times F_{\mathrm{sep},2}$ is (n,ε)-separated for $K_1\times K_2$ under $T_1\times T_2$, so

$$s_{\mathrm{sep}}^{T_1\times T_2}(n,\varepsilon,K_1\times K_2,\mathsf{d})\geqslant s_{\mathrm{sep}}^{T_1}(n,\varepsilon,K_1,\mathsf{d}_1)s_{\mathrm{sep}}^{T_2}(n,\varepsilon,K_2,\mathsf{d}_2).$$

Therefore

$$\begin{aligned}\limsup\nolimits_{n\to\infty}&\frac{1}{n}\log s_{\mathrm{sep}}^{T_1\times T_2}(n,\varepsilon,K_1\times K_2,\mathsf{d})\\ &\geqslant\limsup_{n\to\infty}\frac{1}{n}\left(\log s_{\mathrm{sep}}^{T_1}(n,\varepsilon,K_1,\mathsf{d}_1)+\log s_{\mathrm{sep}}^{T_2}(n,\varepsilon,K_2,\mathsf{d}_2)\right)\\ &\geqslant\liminf_{n\to\infty}\frac{1}{n}\log s_{\mathrm{sep}}^{T_1}(n,\varepsilon,K_1,\mathsf{d}_1)+\limsup_{n\to\infty}\frac{1}{n}\log s_{\mathrm{sep}}^{T_2}(n,\varepsilon,K_2,\mathsf{d}_2).\end{aligned}$$

Thus $h_{\mathsf{d}}(T_1\times T_2,K_1\times K_2)\geqslant h_{\mathsf{d}_1}(T_1,K_1)+h_{\mathsf{d}_2}(T_2,K_2)$ by (6.2), completing the proof. □

One of the ways in which topological entropy for maps on non-compact spaces will be useful is to linearize certain entropy calculations, initially on compact spaces. A simple example (see Proposition 6.10) involves lifting a

† Note that this holds for any compact X_1 by Theorem 5.19.

toral automorphism: The r-torus $\mathbb{T}^r$ is defined as the quotient $\mathbb{R}^r/\mathbb{Z}^r$, and the usual Euclidean metric d on $\mathbb{R}^r$ induces a metric d' on $\mathbb{T}^r$ by setting

$$\mathsf{d}'(x+\mathbb{Z}^r, y+\mathbb{Z}^r) = \min_{n\in\mathbb{Z}^r} \mathsf{d}(x, y+n).$$

Notice that this makes the quotient map $\pi\colon \mathbb{R}^r \to \mathbb{T}^r$ into a local isometry: every point in $\mathbb{R}^r$ has a neighbourhood that is mapped isometrically onto an open set in $\mathbb{T}^r$. The next result gives general conditions under which entropy is preserved in this kind of lift.

Also recall that the definition of (n,ε)-separated or (n,ε)-spanning points for a uniformly continuous map T on a metric space (X,d) can conveniently be phrased in terms of the n-th Bowen metric defined by

$$\mathsf{d}_n(x,y) = \max_{0\leqslant j\leqslant n-1} \mathsf{d}(T^j x, T^j y)$$

for all $x,y \in X$. Below we will have a second metric space (X',d') and uniformly continuous T' and will write d'_n for its Bowen metric.

Theorem 6.5 (Lifting entropy). *Let $\pi\colon (X,\mathsf{d}) \to (X',\mathsf{d}')$ be a continuous surjective map with the property that for some $\delta > 0$ the map π restricted to $B_{\delta,\mathsf{d}}(x)$ is an isometric surjection onto $B_{\delta,\mathsf{d}'}(\pi(x))$ for every $x \in X$. If*

$$\begin{array}{ccc} (X,\mathsf{d}) & \xrightarrow{\;T\;} & (X,\mathsf{d}) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ (X',\mathsf{d}') & \xrightarrow[\;T'\;]{} & (X',\mathsf{d}') \end{array}$$

is a commutative diagram and both T and T' are uniformly continuous, then $h_{\mathsf{d}}(T) = h_{\mathsf{d}'}(T')$.

Moreover, there exists some $\delta_T > 0$ such that $x,y \in X$, $\mathsf{d}(x,y) < \delta$, and $n \geqslant 1$ implies that $\mathsf{d}_n(x,y) < \delta_T$ if and only if $\mathsf{d}'_n(\pi(x),\pi(y)) < \delta_T$; and if that holds we also have $\mathsf{d}_n(x,y) = \mathsf{d}'_n(\pi(x),\pi(y))$.

Proof. We start by proving the last part of the theorem and suppose first that $\mathsf{d}_n(x,y) < \delta$ for some $x,y \in X$ and $n \geqslant 1$. In particular we then have $\mathsf{d}(T^j x, T^j y) < \delta$ and so

$$\mathsf{d}'(T'^j\pi(x), T'^j\pi(y)) = \mathsf{d}(T^j x, T^j y)$$

for $j = 0,\dots,n-1$. This shows that $\mathsf{d}_n(x,y) < \delta$ implies that

$$\mathsf{d}'_n(\pi(x),\pi(y)) = \mathsf{d}_n(x,y).$$

For the converse we need to choose (by uniform continuity) some $\delta_T \in (0,\delta)$ so that $\mathsf{d}(x,y) \leqslant \delta_T$ implies $\mathsf{d}(Tx,Ty) < \delta$ for all $x,y \in X$. Assume that x,y

are points in X, $\mathsf{d}(x,y)<\delta$, and $\mathsf{d}'_n(\pi(x),\pi(y))<\delta_T$ for some $n\geqslant 1$. Using this assumption we will show by induction that $\mathsf{d}_n(x,y)=\mathsf{d}'_n(\pi(x),\pi(y))$.

First note that $\mathsf{d}(x,y)<\delta$ implies $\mathsf{d}(x,y)=\mathsf{d}'(\pi(x),\pi(y))$ or equivalently $\mathsf{d}_1(x,y)=\mathsf{d}'_1(\pi(x),\pi(y))$. For the induction suppose we have already shown $\mathsf{d}_k(x,y)=\mathsf{d}'_k(\pi(x),\pi(y))$ for some $k<n$. This gives

$$\mathsf{d}(T^{k-1}x,T^{k-1}y)\leqslant \mathsf{d}_k(x,y)=\mathsf{d}'_k(\pi(x),\pi(y))\leqslant \mathsf{d}'_n(\pi(x),\pi(y))<\delta_T,$$

which implies $\mathsf{d}(T^kx,T^ky)<\delta$ by choice of δ_T. The property of δ now again implies

$$\mathsf{d}(T^kx,T^ky)=\mathsf{d}'(T'^k\pi(x),T'^k\pi(y))$$

and also

$$\mathsf{d}_{k+1}(x,y)=\mathsf{d}'_{k+1}(\pi(x),\pi(y)).$$

The induction stops once we have reached $k=n$, in which case we have shown the last part of the theorem.

If $K\subseteq X$ is compact with $\operatorname{diam}(K)<\delta$ then by the local isometry property the image $\pi(K)$ is compact with $\operatorname{diam}(K)<\delta$; moreover any compact set $K'\subseteq X'$ with $\operatorname{diam}(K')<\delta$ is an image of such a set. Fix some such $K\subseteq X$. The above property of d_n resp. d'_n shows that $s^T_{\text{sep}}(n,\varepsilon,K,\mathsf{d})=s^{T'}_{\text{sep}}(n,\varepsilon,\pi(K),\mathsf{d}')$ for all $n\geqslant 1$ and $\varepsilon\in(0,\delta_T)$. In particular, $h_{\mathsf{d}}(T,K)=h_{\mathsf{d}'}(T',\pi(K))$. Using Proposition 6.3 the theorem follows. □

Exercises for Section 6.1

Exercise 6.1.1. Two metrics d and d' on a space X are called *uniformly equivalent* if for any $\varepsilon>0$ there is a constant $\delta>0$ for which

$$\mathsf{d}(x,y)<\delta \implies \mathsf{d}'(x,y)<\varepsilon,$$

and

$$\mathsf{d}'(x,y)<\delta \implies \mathsf{d}(x,y)<\varepsilon$$

for all $x,y\in X$. Show that if d is uniformly equivalent to d', then

$$h_{\mathsf{d}}(T)=h_{\mathsf{d}'}(T).$$

Exercise 6.1.2. Let $T\colon X\to X$ be a uniformly continuous map on a metric space. Show that $h_{\mathsf{d}}(T^k)=kh_{\mathsf{d}}(T)$ for any $k\geqslant 1$.

Exercise 6.1.3. Let T be a uniformly continuous map on a metric space (X,d). Consider $T\times T$ on $X\times X$ (with the product metric d' as defined in Lemma 6.4) and show that $h_{\mathsf{d}'}(T\times T)=2h_{\mathsf{d}}(T)$.

Exercise 6.1.4. For the map $x\mapsto 2x \pmod 1$ compute $h_{\text{top}}(T)$ directly from the definition using separating sets in $\mathbb{T}$, and then compute it using Theorem 6.5.

Exercise 6.1.5. Assume that the Theorem 6.5 is weakened slightly so that it says that for each $x \in X$ there is a $\delta = \delta(x) > 0$ with the property that the map π restricted to $B_{\delta,\mathsf{d}}(x)$ is an isometric surjection onto $B_{\delta,\mathsf{d}'}(\pi(x))$. Show that in this case we can deduce that $h_{\mathsf{d}}(T) \geqslant h_{\mathsf{d}'}(T')$.

Exercise 6.1.6. Show that the canonical projection map

$$\pi\colon \mathrm{SL}_2(\mathbb{R}) \longrightarrow X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$$

satisfies the weaker hypothesis in Exercise 6.1.5 but does not satisfy the hypothesis of Theorem 6.5. In this and similar cases, equality of topological entropy does nonetheless hold for certain algebraic maps.
(a) Show that the horocycle time-one map, defined on X by

$$\mathrm{SL}_2(\mathbb{Z})g \longmapsto \mathrm{SL}_2(\mathbb{Z})g \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has zero Bowen entropy.
(b) Compute the Bowen entropy of the time-one geodesic flow, defined on X by

$$\mathrm{SL}_2(\mathbb{Z})g \longmapsto \mathrm{SL}_2(\mathbb{Z})g \begin{pmatrix} \mathrm{e}^{1/2} & 0 \\ 0 & \mathrm{e}^{-1/2} \end{pmatrix}.$$

Exercise 6.1.7. Let $T\colon X \to X$ be a uniformly continuous homeomorphism of a locally compact metric space (X, d), and let $X^* = X \cup \{\infty\}$ be the one-point compactification of X. Define $T^*\colon X^* \to X^*$ by setting $T^*(x) = T(x)$ for $x \in X \subseteq X^*$ and $T^*(\infty) = \infty$. Show that

$$h_{\mathsf{d}}(T) \geqslant h_{\mathrm{top}}(T^*).$$

Find an example with $h_{\mathsf{d}}(T) > 0$ and $h_{\mathrm{top}}(T^*) = 0$.

Exercise 6.1.8. Strengthen the inequality for entropy of topological factors from Exercise 5.2.3 as follows. If $T_k\colon (X_k, \mathsf{d}_k) \to (X_k, \mathsf{d}_k)$ for $k = 1, 2$ are continuous maps of compact metric spaces, and $\pi\colon X_1 \to X_2$ is a continuous surjective factor map, then

$$h_{\mathsf{d}_1}(T_1) \leqslant h_{\mathsf{d}_2}(T_2) + \sup_{x \in X_2} h_{\mathsf{d}_1}(T_1, \pi^{-1}(x)).$$

6.2 Homogeneous Measures

As before we let $T\colon (X, \mathsf{d}) \to (X, \mathsf{d})$ be a uniformly continuous map on a locally compact metric space.

Definition 6.6. A *Bowen (* or *Bowen–Dinaburg) ball about* x is a set of the form

$$D(x, n, \varepsilon) = D_T(x, n, \varepsilon) = \bigcap_{k=0}^{n-1} T^{-k}\left(B_\varepsilon(T^k x)\right) = \left\{y \in X \mid \mathsf{d}_n(y, x) < \varepsilon\right\},$$

where $B_\varepsilon(y) = \{z \in X \mid \mathsf{d}(y, z) < \varepsilon\}$ denotes the metric open ball around y of radius ε and d_n denotes the Bowen metric.

A Borel measure μ on X is called *T-homogeneous* if

(1) $\mu(K) < \infty$ for all compact sets $K \subseteq X$,
(2) $\mu(K) > 0$ for some compact set $K \subseteq X$,
(3) for each $\varepsilon > 0$ there is a $\delta > 0$ and a $c > 0$ with the property that†

$$c^{-1}\mu\left(D(x,n,\varepsilon)\right) \leqslant \mu\left(D(y,n,\delta)\right) \leqslant c\mu\left(D(x,n,\varepsilon)\right)$$

for all $n \geqslant 1$ and $x, y \in X$.

We shall see later that T-homogeneous measures exist in many useful situations; on the other hand they are so special that the rate of the decay of the measure (with respect to any T-homogeneous measure) of the Bowen balls computes the topological entropy. We define the *volume decay entropy* for the homogeneous measure by

$$\mathrm{vol}_{\mu}(T) = \lim_{\varepsilon \searrow 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(D(x,n,\varepsilon)\right); \tag{6.3}$$

notice that property (3) of Definition 6.6 implies that the definition in (6.3) is independent of $x \in X$.

Theorem 6.7 (Bowen's volume decay entropy). *Let $T\colon (X,\mathsf{d}) \to (X,\mathsf{d})$ be a uniformly continuous map on a locally compact metric space and let μ be a T-homogeneous Borel measure on X. Then*

$$h_{\mathsf{d}}(T) = \mathrm{vol}_{\mu}(T).$$

If X is compact, $\mu(X) = 1$, and μ is T-invariant and T-homogeneous, then

$$h_{\mu}(T) = \mathrm{vol}_{\mu}(T).$$

Proof. Let $K \subseteq X$ be compact. By property (2) of Definition 6.6 and the local compactness of X, there is an open set $U \supseteq K$ with $\mu(U) < \infty$; choose $\varepsilon > 0$ small enough to ensure that

$$B_{\varepsilon}(K) = \bigcup_{x\in K} B_{\varepsilon}(x) \subseteq U.$$

Let $F_{\mathrm{sep}} \subseteq K$ be an (n,ε)-separated set of maximal cardinality. For distinct points $x_1, x_2 \in F_{\mathrm{sep}}$ the sets $D(x_1,n,\varepsilon/2)$ and $D(x_2,n,\varepsilon/2)$ are disjoint and

$$\bigsqcup_{x\in F_{\mathrm{sep}}} D(x,n,\varepsilon/2) \subseteq U$$

is a disjoint union. By property (3) of Definition 6.6 there are positive constants δ, c with

$$\mu\left(D(y,n,\delta)\right) \leqslant c\mu\left(D(x,n,\varepsilon/2)\right)$$

† It is sufficient to require the inequality on the right and the one on the left then follows.

for all x, y. Thus for a fixed y we have

$$\mu\left(D(y,n,\delta)\right) s_{\mathrm{sep}}(n,\varepsilon,K) \leqslant \sum_{x\in F_{\mathrm{sep}}} c\mu\left(D(x,n,\varepsilon/2)\right)$$
$$= c\mu\Bigg(\bigsqcup_{x\in F_{\mathrm{sep}}} D(x,n,\varepsilon/2)\Bigg) \leqslant c\mu(U),$$

so

$$\limsup_{n\to\infty}\frac{1}{n}\log s_{\mathrm{sep}}(n,\varepsilon,K) \leqslant \limsup_{n\to\infty} -\frac{1}{n}\log\mu\left(D(y,n,\delta)\right).$$

Taking $\varepsilon \to 0$ gives $h_{\mathsf{d}}(T,K) \leqslant \mathrm{vol}_\mu(T)$, so $h_{\mathsf{d}}(T) \leqslant \mathrm{vol}_\mu(T)$.

The reverse inequality is similar: let K be a given compact set and let $\varepsilon > 0$ be given. We may assume that $\mu(K) > 0$. Choose $\delta > 0$ and $c > 0$ as in property (3) of Definition 6.6 for the given $\varepsilon > 0$, and let F_{span} be an (n,δ)-spanning set for K. Then, by definition of a spanning set,

$$\bigcup_{x\in F_{\mathrm{span}}} D(x,n,\delta) \supseteq K,$$

which together with property (3) of Definition 6.6 implies that

$$c\mu\left(D(y,n,\varepsilon)\right) s_{\mathrm{span}}(n,\delta,K) \geqslant \mu(K) > 0.$$

Therefore

$$\limsup_{n\to\infty}\frac{1}{n}\log s_{\mathrm{span}}(n,\delta,K) \geqslant \limsup -\frac{1}{n}\log\mu\left(D(y,n,\varepsilon)\right)$$

Taking $\varepsilon \to 0$ gives $h_{\mathsf{d}}(T,K) \geqslant \mathrm{vol}_\mu(T)$, so $h_{\mathsf{d}}(T) = \mathrm{vol}_\mu(T)$.

Now assume that X is compact and μ is a T-invariant, T-homogeneous probability measure. By the variational principle (Theorem 5.24),

$$h_\mu(T) \leqslant h_{\mathrm{top}}(T) = h_{\mathsf{d}}(T) = \mathrm{vol}_\mu(T).$$

For the reverse inequality, fix $\varepsilon > 0$ and choose $\delta > 0$, $c > 0$ with the property that

$$\mu\left(D(x,n,\delta)\right) \leqslant c\mu\left(D(y,n,\varepsilon)\right)$$

for all $x, y \in X$ and all $n \geqslant 1$, and let $\xi = \{A_1,\dots,A_r\}$ be a measurable partition of X into sets of diameter no more than δ. Then for x in an element A of ξ_0^{n-1}, we have $A \subseteq D(x,n,\delta)$, and so $\mu(A) \leqslant c\mu\left(D(y,n,\varepsilon)\right)$. It follows that

$$\begin{aligned} H\left(\xi_0^{n-1}\right) &= -\sum_{A\in\xi_0^{n-1}} \mu(A)\log\mu(A) \\ &\geqslant -\sum_{A\in\xi_0^{n-1}} \mu(A)\log c\mu\left(D(y,n,\varepsilon)\right) \\ &= -\log c - \log\mu\left(D(y,n,\varepsilon)\right). \end{aligned}$$

Thus

$$h_\mu(T) \geqslant h_\mu(T,\xi) \geqslant \limsup_{n\to\infty} -\frac{1}{n}\log\mu\left(D(y,n,\varepsilon)\right);$$

taking $\varepsilon \to 0$ shows that $h_\mu(T) \geqslant \mathrm{vol}_\mu(T)$. □

We have seen that topological entropy for continuous maps on compact metric spaces adds over products in Lemma 5.22. In general the topological entropy for uniformly continuous maps on a non-compact space does not add over products without an additional assumption (Lemma 6.4 gives one such assumption that ensures additivity). Here we record two such situations in which we recover additivity, the first in the context of homogeneous measures.

Corollary 6.8 (Entropy of products). *Let $T_i\colon (X_i,\mathsf{d}_i) \to (X_i,\mathsf{d}_i)$ be uniformly continuous maps on metric spaces for $i = 1,2$, and define a metric on the product space by $\mathsf{d} = \max\{\mathsf{d}_1,\mathsf{d}_2\}$. Suppose that μ_i is a T_i-homogeneous measure on X_i such that $-\frac{1}{n}\log\mu_i\left(D_{T_i}(x_i,n,\varepsilon)\right)$ converges for any $x_i \in X_i$, then*

$$h_{\mathsf{d}}(T_1\times T_2) = h_{\mathsf{d}_1}(T_1) + h_{\mathsf{d}_2}(T_2).$$

PROOF. If μ_i is T_i-homogeneous, then $\mu_1\times\mu_2$ is $T_1\times T_2$-homogeneous with respect to the maximum metric d on $X_1\times X_2$. Thus the lemma follows from Theorem 6.7. □

Exercises for Section 6.2

Exercise 6.2.1. (a) For the time-one map T of the geodesic flow on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ defined as in Exercise 6.1.6 and lifted to $\mathrm{SL}_2(\mathbb{R})$, compute $\mathrm{vol}_\mu(T)$ and $h_{\mathsf{d}}(T)$.
(b) Compute $\mathrm{vol}_\mu(T)$ for the time-one map of the horocycle flow, and deduce that the horocycle flow has zero entropy.

6.3 Calculating Topological Entropy on the torus

For continuous maps that are highly homogeneous (that is, their action on each part of the space looks the same) Theorem 6.5 and Corollary 6.8 together make it easy to compute[(31)] the topological entropy locally.

To demonstrate this we compute in this section the entropy of a toral automorphism.

Theorem 6.9 (Entropy of toral automorphisms). *The entropy of the automorphism T_A of the r-torus $\mathbb{T}^r$ associated to a matrix $A \in \mathrm{GL}_r(\mathbb{Z})$ is given by*

$$h_{\mathrm{top}}(T_A) = h_m(T_A) = \sum_{\lambda} \log^+ |\lambda|$$

where m denotes Lebesgue measure, the sum is taken over all the eigenvalues of A (with repetitions according to the algebraic multiplicity of each eigenvalue), and $\log^+(x) = \max\{\log x, 0\}$. If A is diagonalizable then there exists a constant $c \geqslant 1$ (depending only on A and d) such that

$$c^{-1}\varepsilon^r e^{-h_{\mathrm{top}}(T_A)n} \leqslant m(D_{T_A}(x,n,\varepsilon)) \leqslant c\varepsilon^r e^{-h_{\mathrm{top}}(T_A)n}$$

for all $n \geqslant 1$ and $\varepsilon \in (0, \delta_{T_A})$.

In particular this shows that for toral automorphism the Lebesgue measure is a measure of maximal entropy. In fact, it is the unique measure of maximal entropy in many cases, but the methods developed in this section do not show that, see Chapter 8.

6.3.1 Entropy of linear maps

For the proof of the entropy formula in Theorem 6.9 we are going to use the linear case.

Proposition 6.10 (Entropy of linear maps). *Let $A\colon \mathbb{R}^r \to \mathbb{R}^r$ be the linear automorphism defined by the matrix $A \in \mathrm{GL}_r(\mathbb{R})$. Then, if d is the usual Euclidean metric on $\mathbb{R}^r$,*

$$h_{\mathsf{d}}(A) = \sum_{\lambda} \log^+ |\lambda| \tag{6.4}$$

where the sum is taken over all the eigenvalues of A (repeated according to the algebraic multiplicity of each eigenvalue), and $\log^+(x) = \max\{\log x, 0\}$.

Moreover, the Lebesgue measure m on $\mathbb{R}^r$ is A-homogeneous and for any $\varepsilon > 0$ and any metric d induced from a norm on $\mathbb{R}^r$ we have

$$\lim_{n\to\infty} \frac{-\log m(D_A(0,n,\varepsilon))}{n} = h_{\mathsf{d}}(A).$$

If A is diagonalizable then there exists a constant $c \geqslant 1$ (depending only on A and d) such that

$$c^{-1}\varepsilon^r e^{-h_{\mathrm{top}}(A)n} \leqslant m(D_A(x,n,\varepsilon)) \leqslant c\varepsilon^r e^{-h_{\mathrm{top}}(A)n}$$

for all $n \geqslant 1$ *and* $\varepsilon > 0$.

PROOF OF THEOREM 6.9. By Theorem 5.19, $h_{\text{top}}(T_A) = h_{\mathsf{d}}(T_A)$ where d is the metric induced on $\mathbb{T}^r$ by the usual metric on $\mathbb{R}^r$. By Theorem 6.5, it is enough to compute the topological entropy of the map lifted to $\mathbb{R}^r$, and Proposition 6.10 gives the formula. Finally Theorem 6.7 shows that the measure-theoretic entropy with respect to Lebesgue measure has the same value, since Lebesgue measure is T_A-homogeneous (as both the Lebesgue measure and the metric are translation invariant).

For the final claim of the theorem we apply the final claim of Theorem 6.5 to see that sufficiently small Bowen balls for T_A on $\mathbb{T}^r$ are images of the corresponding Bowen balls for A on $\mathbb{R}^r$, and apply the final claim of Proposition 6.10. □

Before proving Proposition 6.10, we note a particularly simple instance. If A is a real diagonal matrix,

$$A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_r \end{pmatrix},$$

then we may replace d with the uniformly equivalent metric

$$\mathsf{d}'(x, y) = \max_{1 \leqslant i \leqslant r} \{|x_i - y_i|\},$$

where $x = (x_1, \dots, x_r)^t$, so that $B_{\mathsf{d}',\varepsilon}(0)$ is an r-dimensional cube with side 2ε centered at 0. Since A is diagonal, the map $x \mapsto A^{-1}x$ dilates the ith axis by the factor λ_i^{-1}; this is illustrated in Figure 6.1 for the situation

$$r = 3,\ 0 < \lambda_1 < 1,\ \lambda_2 > 1,\ \lambda_3 > 1.$$

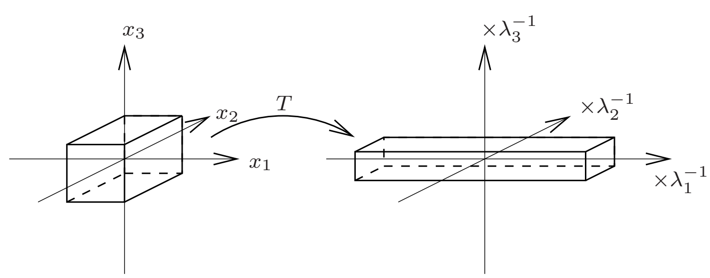

Fig. 6.1: Action of A^{-1} with $0 < \lambda_1 < 1$, $\lambda_2 > 1$, $\lambda_3 > 1$.

Thus the Bowen ball $D_A(0,n,\varepsilon)$ is an r-dimensional rectangular parallelepiped (using the metric d' to simplify matters); the ith side has length 2ε if $|\lambda_i| \leqslant 1$ and length $2\varepsilon|\lambda_i^{-n+1}|$ if $|\lambda_i| > 1$. It follows that

$$m\left(D_A(0,n,\varepsilon)\right) = (2\varepsilon)^r \prod_{|\lambda_i|>1} \lambda_i^{-n+1},$$

which shows that in this case

$$\mathrm{vol}_m(A) = \sum_{\lambda} \log^+ |\lambda|,$$

giving the claimed formula by Theorem 6.7 as the Lebesgue measure m is an A-homogeneous measure. This also proves the final claim in Proposition 6.10 (strictly speaking, only in the case where A is diagonalizable over $\mathbb{R}$ and the metric is carefully chosen, but the general case is similar and is contained in the following discussion).

This simple case above should be kept in mind for the general case below. There are two difficulties to overcome. First, the matrix A may have complex eigenvalues. Second, these eigenvalues may give rise to non-trivial Jordan blocks (that is, the matrix might not be diagonalizable). The first problem is easily dealt with by using a complex vector space, identifying[(32)] for example the action of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ (or any of its conjugates) on $\mathbb{R}^2$ with multiplication by $a + \mathrm{i}b$ on $\mathbb{C}$. The second involves an important principle which will arise several times: Jordan blocks distort the exponentially decaying parallelepiped by an amount that is polynomially bounded. In the limit it is the exponential rate that determines the volume decay, and that is the essence of the formula in Proposition 6.10.

Proof of Proposition 6.10. First notice that $x \mapsto Ax$ is uniformly continuous with respect to d, and Lebesgue measure m on $\mathbb{R}^r$ is A-homogeneous, since both the metric and the measure are translation invariant. Thus by Theorem 6.7

$$h_{\mathsf{d}}(A) = \mathrm{vol}_m(A).$$

We also note that we can calculate the volume decay entropy with respect to any metric derived from a norm, since the resulting metrics would be uniformly equivalent (see Exercise 6.1.1).

By choosing a suitable basis in $\mathbb{R}^r$ we may assume that the matrix A has the Jordan form[†]

[†] If the reader has not seen the Jordan normal form over $\mathbb{R}$ this can be avoided as follows. Extend $A\colon \mathbb{R}^r \to \mathbb{R}^r$ to a $\mathbb{C}$-linear map $A_{\mathbb{C}}\colon \mathbb{C}^r \to \mathbb{C}^r$ and use Exercise 6.1.3 to get $h_{\mathsf{d}}(A_{\mathbb{C}}) = 2h_{\mathsf{d}}(A)$. Now calculate $h_{\mathsf{d}}(A_{\mathbb{C}})$ as done here but only relying on the complex version of the Jordan decomposition.

$$A = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{pmatrix}$$

where each block J_k corresponds to an eigenvalue λ_k of A. In the case $\lambda_k \in \mathbb{R}$, the corresponding block has the form

$$J = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & \ddots \\ & & & \lambda_k & 1 \end{pmatrix} \tag{6.5}$$

and in the case of a pair of eigenvalues $\lambda_k, \overline{\lambda_k} \in \mathbb{C}$, the corresponding block has the form

$$J = \begin{pmatrix} \Lambda_k & I_2 & & \\ & \Lambda_k & I_2 & \\ & & \ddots & \ddots \\ & & & \Lambda_k & I_2 \end{pmatrix}, \tag{6.6}$$

where $\lambda_k = a + \mathrm{i}b$, $\Lambda_k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Since the Lebesgue measure in $\mathbb{R}^r$ is the product of the Lebesgue measures on the subspace corresponding to each block, it is sufficient to prove (6.4) on each block separately, so long as we also establish that the limit

$$\lim_{n\to\infty} -\frac{1}{n} \log m \left(D_{J_k}(0, n, \varepsilon)\right)$$

exists, so that the entropy adds over the product by Corollary 6.8.

Thus we are reduced to a single Jordan block $J = J_k$. If it corresponds to a complex conjugate pair of eigenvalues as in (6.6), then by making the identification $(x, y) \mapsto x + \mathrm{i}y$ between $\mathbb{R}^2$ and $\mathbb{C}$ we may assume that the block always has the form in (6.5), with $\lambda_k \in \mathbb{R}$ in the real case and $\lambda_k \in \mathbb{C}$ in the complex case. Below we will write $\mathbb{K}$ for $\mathbb{R}$ or $\mathbb{C}$ depending on whether λ_k is real or complex and set $\dim \mathbb{R} = 1$ and $\dim \mathbb{C} = 2$.

So assume that

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda & 1 \end{pmatrix} = \lambda I_\ell + N$$

is an $\ell \times \ell$ matrix corresponding to one Jordan block, acting on $\mathbb{K}^\ell$. Recall that

$$J^n = (\lambda I + N)^n = \sum_{k=0}^{\ell-1} \binom{n}{k} \lambda^{n-k} N^k,$$

for any $n \in \mathbb{Z}$, where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ for all $n \in \mathbb{Z}$ and $k \geqslant 0$, and note that

$$(N^m)_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in \{(1, m+1), (2, m+2), \dots, (\ell - m, \ell)\}; \\ 0 & \text{if not;} \end{cases}$$

and N^m is the zero matrix if $m \geqslant \ell$.

Fix $\varepsilon > 0$ and $t_1 \in \mathbb{K}$ with $|t_1| > |\lambda|$. Then every entry of the matrix $t_1^{-n} J^n$ has the form $t_1^{-n} \lambda^n$ multiplied by a polynomial in n and hence goes to zero as $n \to \infty$. Therefore there is some constant $C > 0$ such that

$$\|t_1^{-n} J^n x\| \leqslant C \|x\|$$

for all $n \geqslant 0$ and $x \in \mathbb{K}^\ell$. Given a metric d on $\mathbb{K}^\ell$ we see that if $\mathsf{d}(t_1^j x, 0) < \frac{\varepsilon}{C}$ for some $j \geqslant 0$, then $\mathsf{d}(J^j x, 0) < \varepsilon$. Thus

$$D_{t_1}\left(0, n, \tfrac{\varepsilon}{C}\right) \subseteq D_J\left(0, n, \varepsilon\right),$$

for all $n \geqslant 1$ and

$$D_{t_1}\left(0, n, \tfrac{\varepsilon}{C}\right) = \begin{cases} t_1^{-n+1} B_{\varepsilon/C}(0) & \text{if } t \geqslant 1, \\ B_{\varepsilon/C}(0) & \text{if } t < 1. \end{cases}$$

It follows that

$$\ell \dim \mathbb{K} \log^+ t_1 \geqslant \limsup_{n \to \infty} -\frac{1}{n} \log m\left(D_J(0, n, \varepsilon)\right) \tag{6.7}$$

where m is Lebesgue measure on $\mathbb{K}^\ell$.

Now let $t_2 \in \mathbb{K}$ satisfy $|t_2| < |\lambda|$, so that the entries of the matrix $t_2^n J^{-n}$ are of the form $t_2^n \lambda^{-n}$ multiplied by a polynomial in n for all $n \geqslant 1$, so that we again have some $C > 0$ with

$$\|t_2^n J^{-n} x\| \leqslant C \|x\|$$

for all $n \geqslant 0$ and $x \in \mathbb{K}^\ell$. As before, it follows that

$$D_J\left(0, n, \varepsilon\right) \subseteq D_{t_2}\left(0, n, C\varepsilon\right)$$

and therefore

$$\liminf_{n \to \infty} -\frac{1}{n} \log m\left(D_J(0, n, \varepsilon)\right) \geqslant \ell \dim \mathbb{K} \log^+ |t_2|.$$

Together with (6.7), this shows (recalling that $|t_1|$ and $|t_2|$ can be chosen arbitrarily close to $|\lambda|$) that

$$\lim_{n\to\infty} -\frac{1}{n}\log m\left(D_J(0,n,\varepsilon)\right) = \ell \dim \mathbb{K} \log^+ |\lambda|.$$

Recalling that in the case of $\dim \mathbb{K} = 2$ we are actually dealing simultaneously with a pair of complex eigenvalues this concludes the proof of Proposition 6.10. □

Exercises for Section 6.3

Exercise 6.3.1. Show that the volume decay entropy for a linear map can be defined using any bounded set $Q \subseteq \mathbb{R}^d$ that contains the origin in its interior, by showing that for such a set

$$\mathrm{vol}_\mu(A) = \lim_{n\to\infty} -\frac{1}{n}\log m\left(\bigcap_{k=0}^{n-1} A^{-k}Q\right).$$

Exercise 6.3.2. Show that Lemma 5.20 is false without the assumption that X is compact.

6.4 Entropy for Flows on Compact Homogeneous Spaces

In this section† G will denote a unimodular Lie group. A *uniform lattice*‡ is a discrete subgroup $\Gamma \leqslant G$ for which $X = \Gamma\backslash G$ is compact. Recall (see, for example, [46, Sect. 9.3.3], or the more general Lemma 8.26) that it is possible to define a left-invariant metric $\mathsf{d}_G(\cdot,\cdot)$ on G which then via

$$\mathsf{d}_X(\Gamma g_1, \Gamma g_2) = \min_{\gamma\in\Gamma} \mathsf{d}_G(g_1, \gamma g_2)$$

defines a metric on any quotient $X = \Gamma\backslash G$ by a discrete subgroup $\Gamma \leqslant G$. Moreover, for every compact subset $K \subseteq X$ there exists some $r > 0$ (called an *injectivity radius*) with the property that

$$B_r^G \ni h \longmapsto xh \in B_r^X(x)$$

† In this section we follow Bowen [25] closely. We would like to point out that the material here and in Section 6.3 gives an easy route to a 'formula' for the entropy of certain maps at the expense of masking the detailed dynamics.

‡ A lattice is a discrete subgroup $\Gamma \leqslant G$ for which the quotient space $\Gamma\backslash G$ has finite volume but is not required to be compact. Unfortunately, Theorem 6.5 does not apply in the non-compact setting, hence the need to restrict to uniform lattices.

is an isometry for every $x \in K$. Hence, as we are assuming that $X = \Gamma\backslash G$ is compact, this implies the assumptions regarding the metric in Theorem 6.5 and the canonical map $\pi\colon (G, \mathsf{d}_G) \to (X, \mathsf{d}_X)$ defined by $\pi(g) = \Gamma g$.

We also recall that the Haar measure m_G on a unimodular group G gives rise to a right-invariant Haar measure m_X on the quotient $X = \Gamma\backslash G$ of G by a discrete subgroup Γ such that $m_X(\pi(B)) = m(B)$ for any Borel set $B \subseteq G$ on which π is injective. As we assume here, in addition, that X is compact we may normalize the Haar measures so that $m_X(X) = 1$.

Now fix some $a \in G$ and define

$$T_G\colon g \longmapsto ga^{-1}$$

for $g \in G$, respectively

$$T_X\colon x \longmapsto xa^{-1}$$

for $x \in X$. Then we also have the commutative diagram in Theorem 6.5, which implies that $h_{\mathsf{d}_G}(T_G) = h_{\mathsf{d}_X}(T_X)$. These quantities can be calculated as in the following theorem. For this we need another notion, that of the *adjoint* Ad_a of the element $a \in G$, which is defined to be the derivative of the map $g \mapsto aga^{-1}$ at the identity $e \in G$. Recall that the tangent space of G at the identity is the Lie algebra $\mathfrak{g}$ of G. In the case of a closed linear Lie group $G < \mathrm{GL}_d(\mathbb{R})$, the Lie algebra is a subspace of $\mathrm{Mat}_{d,d}(\mathbb{R})$ and the adjoint of a is simply conjugation by a on the Lie algebra.

Theorem 6.11. *Let $X = \Gamma\backslash G$ be a compact quotient of a unimodular*[†] *Lie group by a discrete subgroup. Let $a \in G$, and define $T_X(x) = xa^{-1}$ for $x \in X$. Then the topological entropy of T_X is given by*

$$h_{\mathrm{top}}(T_X) = h_{m_X}(T_X) = \mathrm{vol}_{m_G}(T_G) = \mathrm{vol}_{m_{\mathfrak{g}}}(\mathrm{Ad}_a) = \sum_{i=1}^{\dim(G)} \log^+ |\lambda_i|,$$

where $\lambda_1, \dots, \lambda_{\dim(G)}$ are the eigenvalues (listed with algebraic multiplicity) of the linear map Ad_a, and $m_{\mathfrak{g}}$ is the Lebesgue measure on $\mathfrak{g} \cong \mathbb{R}^{\dim(G)}$.

If Ad_a is diagonalizable then there exists a constant $c \geqslant 1$ (depending only on a and d) such that

$$c^{-1}\varepsilon^{\dim G} e^{-h_{\mathrm{top}}(T_X)n} \leqslant m(D_{T_X}(x, n, \varepsilon)) \leqslant c\varepsilon^{\dim G} e^{-h_{\mathrm{top}}(T_X)n}$$

for all $n \geqslant 1$ and $\varepsilon \in (0, \delta_{T_X})$.

Proof. By the discussion before the statement of the theorem, we have

$$h_{\mathrm{top}}(T_X) = h_{\mathsf{d}_X}(T_X) = h_{\mathsf{d}_G}(T_G).$$

Let $\theta_a\colon g \mapsto aga^{-1}$ denote conjugation by a as a map on G. Since d_G is invariant under left multiplication, it follows that

[†] The existence of a lattice in G implies that G is unimodular (see [46, Prop. 9.20]).

$$\mathsf{d}_G\left(\theta_a^k(y),\theta_a^k(x)\right)<\varepsilon$$

if and only if

$$\mathsf{d}_G\left(T_G^k(y),T_G^k(x)\right)<\varepsilon,$$

for any $k\in\mathbb{Z}$. This implies that $D_{T_G}(x,n,\varepsilon)=D_{\theta_a}(x,n,\varepsilon)$ is the Bowen ball for T_G and for θ_a. Similarly, $D_{T_G}(hx,n,\varepsilon)=hD_{T_G}(x,n,\varepsilon)$ for any $h\in G$, which implies that the bi-invariant Haar measure m_G is homogeneous (in the sense of Definition 6.6) both for T_G and for θ_a.

We also note the Bowen balls in X for T_X are, by Theorem 6.5, (and for sufficiently small ε) simply the images of the Bowen balls in G for T_G, so that m_X is a T_X-homogeneous and invariant measure on X. This implies

$$h_{\rm top}(T_X)=h_{\mathsf{d}_X}(T_X)=h_{\mathsf{d}_G}(T_G)=\mathrm{vol}_{m_G}(T_G)=\mathrm{vol}_{m_G}(\theta_a)=\mathrm{vol}_{m_X}(T_X).$$

On the other hand $\mathrm{Ad}_a\colon\mathfrak{g}\to\mathfrak{g}$ is linear, and so Proposition 6.10 shows that

$$h_{\mathsf{d}_\mathfrak{g}}(\mathrm{Ad}_a)=\mathrm{vol}_{m_\mathfrak{g}}(\mathrm{Ad}_a)=\sum_{i=1}^{\dim G}\log^+|\lambda_i|.$$

Thus it remains only to show that $\mathrm{vol}_{m_G}(\theta_a)=\mathrm{vol}_{m_\mathfrak{g}}(\mathrm{Ad}_a)$, which will follow by analyzing how the (measures of the) Bowen balls $D_{\theta_a}(I,n,\varepsilon)$ at the identity in the Lie group, and $D_{\mathrm{Ad}_a}(0,n,\delta)$ at 0 in the Lie algebra relate to each other. This will also imply the last statement in the theorem by using the last statement in Proposition 6.10.

Recall that $\exp\colon\mathfrak{g}\to G$ is a diffeomorphism when restricted to be a map from some open neighbourhood of $0\in\mathfrak{g}$ to some open neighbourhood of $e\in G$. Moreover, $\exp(\mathrm{Ad}_a(v))=\theta_a(\exp(v))$ for any $v\in\mathfrak{g}$. This clearly implies that there exists an ε_0 and a constant $c>1$ so that for every $\varepsilon\in(0,\varepsilon_0)$ we have

$$\exp\left(D_{\mathrm{Ad}_a}(0,n,\varepsilon)\right)\subseteq D_{\theta_a}(I,n,c\varepsilon),\tag{6.8}$$

and

$$\exp\left(D_{\mathrm{Ad}_a}(0,n,c\varepsilon)\right)\supseteq D_{\theta_a}(I,n,\varepsilon).\tag{6.9}$$

Finally, the Haar measure m_G is a smooth measure, see Lemma 6.12 below. In particular, there is a neighbourhood $B_{\delta_0}^{\mathfrak{g}}$ of $0\in\mathfrak{g}$, and some constant $C\geqslant 1$ such that

$$\frac{m_G(\exp(B))}{m_\mathfrak{g}(B)}\in[\tfrac{1}{C},C]\tag{6.10}$$

for any Borel subset $B\subseteq B_{\delta_0}^{\mathfrak{g}}$. Now (6.8)–(6.10) and the definition in (6.3) together imply that $\mathrm{vol}_{m_G}(\theta_a)=\mathrm{vol}_{m_\mathfrak{g}}(\mathrm{Ad}_a)$ as required. □

Suppose a manifold G is written as a countable union of open sets $U_k\subseteq G$ with the property that for each k there is a diffeomorphism

$$\psi_k\colon B_k\longrightarrow U_k$$

from a bounded open subset $B_k \subseteq \mathbb{R}^d$ to U_k, where $d = \dim M$. A measure μ on G is said to be *smooth* if there are non-negative smooth functions ρ_k on B_k such that for any bounded measurable function $f\colon G \to \mathbb{R}$ with $\mathrm{Supp}(f) \subseteq U_k$ we have

$$\int_{U_k} f \,\mathrm{d}\mu = \int_{B_k} f(\psi_k(x))\rho_k(x)\,\mathrm{d}x,$$

where the integral on the right is taken with respect to Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^n$.

Lemma 6.12. *Let m_G be a left-invariant Haar measure on a σ-compact*[†] *Lie group G. Then m_G is a smooth measure on G.*

Proof. Let G be a σ-compact Lie group with Lie algebra $\mathfrak{g}$ and suppose that $\delta_0 > 0$ is small enough to ensure that $v \mapsto \exp(v)$ is a diffeomorphism from the neighbourhood $B^{\mathfrak{g}}_{4\delta_0}$ of $0 \in \mathfrak{g}$ to a neighbourhood U' of $I \in G$. Write

$$\log\colon U' \longrightarrow B^{\mathfrak{g}}_{4\delta_0}$$

for the smooth inverse of the map $\exp|_{B^{\mathfrak{g}}_{4\delta_0}}$, and set $U = \exp B^{\mathfrak{g}}_{\delta_0}$. Since G is σ-compact we may write

$$G = \bigcup_{k=1}^{\infty} g_k U.$$

We define $U_k = g_k U$ and $\psi_k\colon B^{\mathfrak{g}}_{\delta_0} \to U_k$ by $\psi_k(v) = g_k \exp(v)$ for $k \geqslant 1$.

Fix a Lebesgue measure on $\mathfrak{g}$, denoted $\mathrm{d}v$. We claim that there exists a smooth function $\rho\colon B^{\mathfrak{g}}_{\delta_0} \to \mathbb{R}_{>0}$ such that the left Haar measure m_G on G satisfies

$$\int_{U_k} f(g)\,\mathrm{d}m_G(g) = \int_U f(g_k u)\,\mathrm{d}m_G(u) = \int_{B^{\mathfrak{g}}_{\delta_0}} f\left(g_k \exp(v)\right)\rho(v)\,\mathrm{d}v$$

for any bounded measurable function f defined on U_k and for all $k \geqslant 1$.

To define ρ we first recall that $(v,w) \mapsto v * w = \log(\exp(v)\exp(w))$ is a smooth map wherever it is defined.[‡] Fix v, define the map $\phi_v\colon w \mapsto v * w$, and let $\mathrm{D}\,\phi_v(w)$ denote the total derivative of ϕ_v at w. We define

$$\rho(v) = \left(\det\left(\mathrm{D}\,\phi_v(0)\right)\right)^{-1}, \tag{6.11}$$

which is smooth on $B^{\mathfrak{g}}_{4\delta_0}$ by construction. Also notice that the derivative of the map $(v,w) \mapsto v * w$ at 0 is simply $(v,w) \mapsto v + w$, so we may assume (by choosing δ_0 smaller if necessary) that

---

[†] We note that connectedness implies σ-compactness: If $U < G$ is a symmetric neighbourhood of the identity with compact closure, then $\bigcup_{n=1}^{\infty} U^n$ is an open subgroup. If G is connected, this forces $G = \bigcup_{n=1}^{\infty} U^n$ and so G is σ-compact.

[‡] For example, for v lying in the set $B^{\mathfrak{g}}_{4\delta_0}$ and any w chosen small enough to ensure that $\exp(v)\exp(w) \in \exp B^{\mathfrak{g}}_{4\delta_0}$.

$$B^{\mathfrak{g}}_{\delta_0} * B^{\mathfrak{g}}_{\delta_0} \subseteq B^{\mathfrak{g}}_{3\delta_0}.$$

Now let f be a measurable bounded function on G that vanishes outside of U, and fix $v_0 \in B^{\mathfrak{g}}_{\delta_0}$. Then

$$\begin{aligned}\int_{(-v_0)*B^{\mathfrak{g}}_{\delta_0}} & f(\exp(v_0)\exp(v))\rho(v)\,\mathrm{d}v \\ &= \int_{(-v_0)*B^{\mathfrak{g}}_{\delta_0}} f(\exp(v_0 * v))\rho(v_0 * v)\frac{\det(\mathrm{D}\,\phi_{v_0 * v}(0))}{\det(\mathrm{D}\,\phi_v(0))}\,\mathrm{d}v \qquad (6.12)\end{aligned}$$

by the definition of ρ in (6.11). Since

$$\phi_{v_0 * v}(w) = \log\left(\exp(v_0)\exp(v)\exp(w)\right) = \phi_{v_0}(\phi_v(w))$$

for all $v, v_0 \in B^{\mathfrak{g}}_{\delta_0}$ and all sufficiently small $w \in \mathfrak{g}$, we may apply the chain rule to obtain

$$\det\left(\mathrm{D}\,\phi_{v_0 * v}(0)\right) = \det\left(\mathrm{D}\,\phi_{v_0}(v)\right)\det\left(\mathrm{D}\,\phi_v(0)\right),$$

and deduce that the Jacobian of $\phi_{v_0}(v) = v_0 * v$ at v takes the form

$$J_{\phi_{v_0}}(v) = \frac{\det(\mathrm{D}\,\phi_{v_0 * v}(0))}{\det(\mathrm{D}\,\phi_v(0))}.$$

Applying the substitution $v' = v_0 * v$ in (6.12) gives

$$\int_{(-v_0)*B^{\mathfrak{g}}_{\delta_0}} f(\exp(v_0)\exp(v))\rho(v)\,\mathrm{d}v = \int_{B^{\mathfrak{g}}_{\delta_0}} f(\exp(v'))\rho(v')\,\mathrm{d}v'. \qquad (6.13)$$

This 'translation invariance' for functions with support near I is the key step in proving the claim and the lemma.

Choose a partition $\{P_1, P_2, \dots\}$ of G with $P_k \subseteq U_k = g_k U$ for all $k \geqslant 1$, and define a measure m on G by the formula

$$m(B) = \sum_{k=1}^{\infty} \int \mathbb{1}_{B \cap P_k}\big(g_k \exp(v)\big)\rho(v)\,\mathrm{d}v$$

for any measurable set $B \subseteq G$. First note that this clearly defines a measure because each summand defines a measure. We prove first that this measure m is independent of the choice of the points g_k and the partition $\{P_1, P_2, \dots\}$. Suppose therefore that

$$G = \bigcup_{\ell=1}^{\infty} U'_\ell$$

with $U'_\ell = g'_\ell U$ for all $\ell \geqslant 1$ and $\{P'_1, P'_2, \dots\}$ is a partition of G with $P'_\ell \subseteq U'_\ell$ for all $\ell \geqslant 1$. Taking the common refinement of the two partitions we have

$$m(B) = \sum_{k,\ell : P_k \cap P'_\ell \neq \varnothing} \int \mathbb{1}_{B \cap P_k \cap P'_\ell}(g_k \exp(v))\rho(v)\,\mathrm{d}v$$

for all measurable $B \subseteq G$. Fix a pair (k, ℓ) with $P_k \cap P'_\ell \neq \varnothing$. Choose a point $v_{k,\ell} \in B^{\mathfrak{g}}_{\delta_0}$ so that

$$g_{k,\ell} = g_k \exp(v_{k,\ell}) \in P_k \cap P'_\ell$$

and define $f(u) = \mathbb{1}_{B \cap P_k \cap P'_\ell}(g_k u)$ so that f vanishes outside of U. Applying (6.13) with $v_0 = v_{k,\ell}$ we obtain

$$\begin{aligned}\int_{B^{\mathfrak{g}}_{\delta_0}} \mathbb{1}_{B \cap P_k \cap P'_\ell}(g_k \exp(v))\rho(v)\,\mathrm{d}v &= \int_{(-v_{k,\ell})*B^{\mathfrak{g}}_{\delta_0}} \mathbb{1}_{B \cap P_k \cap P'_\ell}(g_{k,\ell} \exp(v))\rho(v)\,\mathrm{d}v \\ &= \int_{B^{\mathfrak{g}}_{3\delta_0}} \mathbb{1}_{B \cap P_k \cap P'_\ell}(g_{k,\ell} \exp(v))\rho(v)\,\mathrm{d}v\end{aligned}$$

and so

$$m(B) = \sum_{k,\ell : P_k \cap P'_\ell \neq \varnothing} \int_{B^{\mathfrak{g}}_{3\delta_0}} \mathbb{1}_{B \cap P_k \cap P'_\ell}(g_{k,\ell} \exp(v))\rho(v)\,\mathrm{d}v,$$

where the partitions $\{P_1, P_2, \dots\}$ and $\{P'_1, P'_2, \dots\}$ appear symmetrically, which implies the claimed independence.

Now let $B \subseteq G$ be measurable and $g \in G$. Then

$$\begin{aligned}m(gB) &= \sum_{k=1}^{\infty} \int \mathbb{1}_{gB \cap P_k}(g_k \exp(v))\rho(v)\,\mathrm{d}v \\ &= \sum_{k=1}^{\infty} \int \mathbb{1}_{B \cap g^{-1}P_k}\left(g^{-1}g_k \exp(v)\right)\rho(v)\,\mathrm{d}v = m(B)\end{aligned}$$

since $\{g^{-1}P_1, g^{-1}P_2, \dots\}$ is a partition with $g^{-1}P_k \subseteq g^{-1}U_k = U'_k = g^{-1}g_k U$ for all $k \geqslant 1$ and this translated partition can also be used to define the measure m.

Using the independence of m from the choice of the partition it is easy to see that m is locally finite and positive on non-empty open sets. Thus m is smooth and is a left Haar measure on G, giving the lemma. □

Notes to Chapter 6

(29)(Page 162) The following example to illustrate this is taken from Walters [206, Sect. 7.2]. Let d_1 be the usual metric on $X = (0,\infty)$; define another metric d_2 by setting $\mathsf{d}_2(x,y) = |\log x - \log y|$. Notice that d_1 and d_2 define the same topology on X. Since $T\colon (X,\mathsf{d}_2) \to (X,\mathsf{d}_2)$ is an isometry, $h_{\mathsf{d}_2}(T) = 0$. On the other hand,

$$\max_{0\leqslant j\leqslant n-1}\{\mathsf{d}_1(T^j x, T^j y)\} = 2^{n-1}|x-y|,$$

so the cardinality of an $(n, \frac{1}{2^k})$-spanning set for any compact sub-interval with non-empty interior is $\gg 2^n$. It follows that $h_{\mathsf{d}_1}(T) \geqslant \log 2$, showing that the topological entropy depends on the uniform equivalence class of the metric, not just the equivalence class.

(30)(Page 163) Lemma 5.21 is shown by Adler, Konheim and McAndrew [4]; Lemma 5.22 is shown by Goodwyn [77] (both have a purely topological argument, without the use of a metric: as pointed out by Goodwyn, even for compact topological spaces it is necessary to assume that the spaces are Hausdorff to know that $N(\mathscr{U}\times\mathscr{V}) = N(\mathscr{U})\times N(\mathscr{V})$). Lemma 6.4 is taken from Bowen [25]; the possible non-additivity over products is missed in [25] and corrected in [24]. The only obstacle to additivity is that we cannot pass from the inequality $a_n \geqslant b_n + c_n$ to the inequality $\limsup_{n\to\infty} a_n \geqslant \limsup_{n\to\infty} b_n + \limsup_{n\to\infty} c_n$ without additional hypotheses.

(31)(Page 170) This idea was developed by Bowen [25], [24]; he used it to compute the topological entropy of affine maps of Lie groups and other homogeneous spaces, and to show that Haar measure is maximal for affine maps. Much of the material in Section 6.3 comes from [25]. Theorem 6.9 (the generalization of Theorem 1.32 to automorphisms of the r-torus) was shown for $r = 2$ by Sinaĭ [190]; the general case was stated in [190] and in a paper of Genis [70]. Arov [9] gave a proof as part of his calculation of the entropy of endomorphisms of solenoids (these are generalizations of the torus). Berg [11] gave an independent proof using different methods. Finally Yuzvinskiĭ [216] computed the entropy of any compact group endomorphism. For modern treatments, see Walters [206, Chap. 7] for toral automorphisms; Lind and Ward [125] for automorphisms of the solenoid, as discussed in Section C.4.

(32)(Page 173) There are several ways to do this; the path chosen here is done so in order to relate the calculation for each block directly to a real part of the space on which the matrix acts, rather than a different complexified space.