Chapter 7
(Weak) Containment and the Regular Representation

For compact groups, we have seen in Chapter 5 that every irreducible representation is contained in the regular representation. In Chapter 2 we have seen that for abelian non-compact groups no irreducible representation can be contained in the regular representation.† However, as we will see in Section 8.3 for non-abelian non-compact groups it is possible for certain irreducible unitary representations to be contained in the regular representation. In this chapter we classify such representations from an abstract point of view, and relate weak containment in the regular representation to almost square integrability of diagonal matrix coefficients.

7.1 Discrete Series Representations

Definition 7.1. An irreducible unitary representation π of the group G is called a *discrete series representation* if π is contained in the regular representation of G.

Theorem 7.2 (Characterization of discrete series). *Suppose the group G is unimodular, and let π be an irreducible unitary representation of G. Then the following are equivalent.*(11)

(1) *π is a discrete series representation (that is, $\pi < \lambda$).*
(2) *There exist vectors $u, v \in \mathcal{H}_\pi \smallsetminus \{0\}$ such that the matrix coefficient $\varphi^\pi_{u,v}$ is square integrable (that is, $\varphi^\pi_{u,v} \in L^2(G)$).*
(3) *The matrix coefficient $\varphi^\pi_{u,v}$ is square integrable for any pair of vectors $u, v \in \mathcal{H}_\pi$.*

Proof. We first show that (1) implies (2). Suppose that $\pi < \lambda$ and so $\mathcal{H}_\pi = \mathcal{V}$ for an invariant supspace $\mathcal{V} \subseteq L^2(G)$ as in (1). Let $v \in \mathcal{V} \smallsetminus \{0\}$ and let u be the

† For this, see Exercise 1.10 and its hint on p. 445, Exercise 2.12, Lemma 2.15, and the characterization of containment in Proposition 2.49.

projection $P(f)$ where $P\colon L^2(G) \to \mathcal{V} \subseteq L^2(G)$ is the orthogonal projection operator and f is a function in $C_c(G)$. We also define $\widetilde{f}(g) = f(g^{-1})$ for $f \in C_c(G)$ and $g \in G$. Since $\mathcal{V}$ is invariant, P is equivariant and so

$$\begin{aligned}\varphi^\pi_{u,v}(g) = \langle \lambda_g Pf, v\rangle = \langle \lambda_g f, v\rangle &= \int f(g^{-1}h)v(h)\,\mathrm{d}m(h)\\ &= \int v(h)\widetilde{f}(h^{-1}g)\,\mathrm{d}m(h) = v * \widetilde{f}(g)\end{aligned}$$

for all $g \in G$. Now recall that $v * \widetilde{f} \in L^2(G)$ (see [16, Lem. 3.75] or generalize Lemma 2.14), and note that density of $C_c(G) \subseteq L^2(G)$ implies that there exists some $f \in C_c(G)$ such that $u = Pf \in \mathcal{V}\smallsetminus\{0\}$. This shows (2).

Notice that (3) clearly implies (2), since an irreducible representation π by definition satisfies $\mathcal{H}_\pi \neq \{0\}$. Hence the following step will finish the proof of the theorem.

We will now show that (2) implies (1) and (3). So suppose that u_0, v_0 in $\mathcal{H}_\pi\smallsetminus\{0\}$ have $\varphi^\pi_{u_0,v_0} \in L^2(G)$. We fix u_0 and consider the subspace

$$D_{u_0} = \{v \in \mathcal{H}_\pi \mid \varphi^\pi_{u_0,v_0} \in L^2(G)\}$$

as the domain of the (possibly unbounded) operator T from $\mathcal{H}_\pi$ into $L^2(G)$ defined by

$$T(v) = \overline{\varphi^\pi_{u_0,v}} \in L^2(G)$$

for $v \in D_{u_0}$. As the main step of the remaining proof, we claim that T is a densely defined closed equivariant operator.

Equivariance follows from the properties of the matrix coefficients. Indeed, for $v \in D_{u_0}$ and $g_0 \in G$ we have

$$\begin{aligned}\lambda_{g_0}\overline{\varphi^\pi_{u_0,v}}(g) = \overline{\varphi^\pi_{u_0,v}(g_0^{-1}g)} &= \overline{\langle \pi(g_0^{-1}g)u_0, v\rangle}\\ &= \overline{\langle \pi(g)u_0, \pi(g_0)v\rangle} = \overline{\varphi^\pi_{u_0,\pi(g_0)v}}(g)\end{aligned}$$

for every $g \in G$, equivalently $\lambda_{g_0}T(v) = T(\pi_{g_0}(v)) \in L^2(G)$ and $\pi_{g_0}v \in D_{u_0}$.

Since $v_0 \in D_{u_0}\smallsetminus\{0\}$ and D_{u_0} is invariant under π by the above, we see that $\overline{D_{u_0}} = \mathcal{H}_\pi$ by irreducibility of π. In other words, T is densely defined. To complete the proof of the claim, it remains to show that T is a closed operator. Suppose therefore that (v_n) is a sequence in D_{u_0} with

$$(v_n, Tv_n) \longrightarrow (v, f) \in \mathcal{H}_\pi \times L^2(G)$$

as $n \to \infty$. Choosing a subsequence if necessary we may upgrade the L^2 convergence and suppose without loss of generality that $Tv_n \to f$ almost everywhere (see e.g. [15, Cor. A.12] for this argument). Furthermore, notice that $v_n \to v \in \mathcal{H}_\pi$ implies

$$(Tv_n)(g) = \overline{\varphi^\pi_{u_0,v_n}}(g) = \overline{\langle \pi(g)u_0, v_n\rangle} \longrightarrow \overline{\langle \pi(g)u_0, v\rangle}$$

as $n \to \infty$ for all $g \in G$. Together we see that

$$f(g) = \lim_{n\to\infty} (Tv_n)(g) = \overline{\langle \pi(g)u_0, v\rangle}$$

for almost every $g \in G$. However, since $f \in L^2(G)$ we see that $v \in D_{u_0}$ belongs to the domain of T and $f = Tv$. In particular, $(v, f) \in \mathrm{Graph}(T)$ as required.

The claim above, the assumed irreducibility of π, and Schur's lemma in the form of Corollary 1.30 together imply that $D_{u_0} = \mathcal{H}_\pi$ and either $T = 0$ or T is a scalar multiple of a unitary isomorphism. Using $u_0 \in D_{u_0} = \mathcal{H}_\pi$ we see that $Tu_0 = \overline{\varphi^\pi_{u_0,u_0}} \neq 0$ and obtain $\pi < \lambda$ as in (1).

To prove (3), let $u_1, v_1 \in \mathcal{H}_\pi$ be any vectors. By the argument above we already have $\varphi_{u_0,v_1} \in L^2(G)$ and repeat the argument as follows. Define

$$\widetilde{D}_{v_1} = \{u \in \mathcal{H}_\pi \mid \varphi^\pi_{u,v_1} \in L^2(G)\}$$

and $\widetilde{T}(u) = \varphi^\pi_{u,v_1}$ for $u \in \widetilde{D}_{v_1}$. Now recall that

$$\rho_{g_0}(\widetilde{T}(u))(g) = \langle \pi(gg_0)u, v_1\rangle = \langle \pi(g)\pi(g_0)u, v_1\rangle = \widetilde{T}(\pi(g_0)u)(g)$$

for $g, g_0 \in G$. Since G is assumed to be unimodular, we may apply the argument above to deduce that $\widetilde{T}$ is a densely defined closed equivariant operator from $\mathcal{H}_\pi$ to $L^2(G)$ with the latter equipped with the right-regular representation. As above, this implies by Schur's lemma (Corollary 1.30) that $D_{v_1} = \mathcal{H}_\pi$, and in particular $\varphi^\pi_{u_1,v_1} \in L^2(G)$. As $u_1, v_1 \in \mathcal{H}_\pi$ were arbitrary, (3) follows. □

To see an example of a discrete series representation, the reader may continue with Section 8.3 (which only builds on this section and some preparations concerning hyperbolic geometry in Section 8.2). As one of our motivations for this volume is to better understand decay of matrix coefficients, Theorem 7.2 is of interest to us. Indeed, the requirement to belong to $L^2(G)$ is a requirement on the average decay of the matrix coefficients. This observation will become very important for us after replacing the strong condition of containment by weak containment as in the next section.

7.2 Almost Square Integrable Matrix Coefficients

We now present a theorem of Cowling, Haagerup and Howe [9] that will allow us to characterize in the next chapter the so called *tempered representations* for the group $\mathrm{SL}_2(\mathbb{R})$.

Definition 7.3 (Tempered representations). Let π be a unitary representation of G. We say that π is *tempered* if it is weakly contained in the regular representation λ of G.

Our interest in tempered representations comes from our wish to understand decay of matrix coefficients. We will see the connection explicitly in the next chapter, where the following theorem will be a crucial ingredient.

The reader may also wonder whether non-tempered representations actually exist, since so far we have only encountered tempered representations (without pointing this out). However, we will see in the next chapter that (for example) the trivial representation is not tempered for non-compact semi-simple groups.

Definition 7.4 (Almost square integrability). Let π be a unitary representation of G. We say that π is *almost square integrable* if there exists a dense subset $\mathcal{V} \subseteq \mathcal{H}_\pi$ such that $\varphi_v^\pi \in L^{2+\varepsilon}(G)$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$.

Theorem 7.5 (Almost square integrable matrix coefficients [9]). *Let G be a locally compact group and π a unitary representation. Suppose that $v \in \mathcal{H}_\pi$ has the property that the diagonal matrix coefficient*

$$\varphi_v^\pi(g) = \langle \pi_g v, v \rangle$$

for $g \in G$ belongs to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. Then the cyclic representation generated by v is weakly contained in the regular representation of G. Moreover, if π is almost square integrable, then π is tempered.

Along the way, and as a warm-up for the proof of the theorem, we will obtain the following result which is of independent interest.

Proposition 7.6 (Sub-exponential growth). *Suppose that G has sub-exponential growth, meaning that for every compact subset $K \subseteq G$ we have*

$$\lim_{n\to\infty} m\left(K^n\right)^{\frac{1}{n}} = 1. \tag{7.1}$$

Then every unitary representation is tempered.

For the proof of the theorem and the proposition we are going to use Lemma 4.23 and its twin for the regular representation below and the equivalent definition $(\pi \prec_{\mathrm{op}} \rho)$ of weak containment from Theorem 4.21.

Lemma 7.7 (2-norm formula for operator norm). *Let $f \in C_c(G)$. Then for the left-regular representation λ of G we have*

$$\|\lambda_*(f)\|_{\mathrm{op}} = \lim_{n\to\infty} \|(f^* * f)^{*n}\|_2^{\frac{1}{2n}}.$$

Proof. By Lemma 4.23 we have

$$\|\lambda_*(f)\| = \sup_{f_0 \in L^2(G)} \lim_{n\to\infty} \left(\int_G (f^* * f)^{*n} \varphi_{f_0} \, \mathrm{d}m \right)^{\frac{1}{2n}}$$

where $\varphi_{f_0}(g) = \langle \lambda_g f_0, f_0 \rangle$ for $f_0 \in L^2(G)$. Also, by Lemma 4.23, we may also restrict to functions $f_0 \in C_c(G)$ in which case $\varphi_{f_0} \in C_c(G)$. Noticing that for $f_0 \in C_c(G)$

$$\int_G (f^* * f)^{*n} \varphi_{f_0} \, \mathrm{d}m \leqslant \|(f^* * f)^{*n}\|_2 \|\varphi_{f_0}\|_2,$$

we get

$$\|\lambda_*(f)\|_{\mathrm{op}} \leqslant \liminf_{n\to\infty} \|(f^* * f)^{*n}\|_2^{\frac{1}{2n}}. \tag{7.2}$$

On the other hand, setting $f_0 = f^* * f \in C_c(G)$ gives

$$\begin{aligned}(f^* * f)^{*n}(g) &= (f^* * f)^{*(n-1)} * f_0(g) \\ &= \int_G (f^* * f)^{*(n-1)}(h) f_0(h^{-1}g) \, \mathrm{d}m(h) \\ &= \left(\lambda_* \left((f^* * f)^{*(n-1)} \right) f_0 \right)(g)\end{aligned}$$

for all $g \in G$, which leads to

$$\begin{aligned}\|(f^* * f)^{*n}\|_2 &\leqslant \|\lambda_*(f^* * f)^{*(n-1)}\|_{\mathrm{op}} \|f_0\|_2 \\ &\leqslant \|\lambda_*(f)\|_{\mathrm{op}}^{2(n-1)} \|f_0\|_2\end{aligned}$$

and in turn to

$$\limsup_{n\to\infty} \|(f^* * f)^{*n}\|_2^{\frac{1}{2n}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}.$$

Together with (7.2), this gives the lemma. □

PROOF OF PROPOSITION 7.6. Let G have sub-exponential growth, let π be a unitary representation of G, and f a function in $C_c(G)$. We wish to show that $\|\pi_*(f)\|_{\mathrm{op}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}$, and will use Lemmas 4.23 and 7.7 to prove this. So set $f_0 = f^* * f$ and $K = \operatorname{supp} f_0$. Then $\operatorname{supp} f_0^{*n} \subseteq K^n$, and we may use Cauchy–Schwarz for any $v \in \mathcal{H}_\pi$ to obtain

$$\left| \int_G f_0^{*n} \varphi_v^\pi \, \mathrm{d}m \right|^2 \leqslant \int_{K^n} |f_0^{*n}|^2 \, \mathrm{d}m \int_{K^n} |\varphi_v^\pi|^2 \, \mathrm{d}m.$$

Taking the $4n$th root and using $\|\varphi_v^\pi\|_\infty \leqslant \|v\|^2$, this gives

$$\lim_{n\to\infty} \left| \int_G f_0^{*n} \varphi_v^\pi \, \mathrm{d}m \right|^{\frac{1}{2n}} \leqslant \lim_{n\to\infty} \|f_0^{*n}\|_2^{\frac{1}{2n}} \lim_{n\to\infty} \left(\|v\|^4 m(K^n) \right)^{\frac{1}{4n}} = \|\lambda_*(f)\|_{\mathrm{op}}$$

by (7.1) and Lemma 7.7. Taking the supremum over $v \in \mathcal{H}_\pi$, Lemma 4.23 shows that $\|\pi_*(f)\|_{\mathrm{op}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}$. As $f \in C_c(G)$ was arbitrary, Theorem 4.21 shows that $\pi \prec \lambda$. □

In general, groups may not satisfy sub-exponential growth (see Exercise 7.9 below), but the following is a general estimate.

Lemma 7.8 (At most exponential growth). *Let $K \subseteq G$ be a compact subset with non-empty interior. Then there exist constants $c > 0$, $M > 1$ (depending on K) with $m_G(K^n) \leqslant cM^n$ for all $n \geqslant 1$.*

Proof. By compactness, there exists a finite collection $x_1, \dots, x_M \in G$ with the property that

$$K^2 \subseteq \bigcup_{j=1}^{M} x_j K.$$

By induction, this implies that

$$K^n \subseteq \bigcup_{j_1,\dots,j_{n-1}=1}^{M} x_{j_1} \cdots x_{j_{n-1}} K,$$

which gives the lemma with $c = \frac{m_G(K)}{M}$. □

We are now ready to prove the main result of this section.

Proof of Theorem 7.5. Suppose that π is a unitary representation of G and $v \in \mathcal{H}_\pi$ has the property that the matrix coefficient $\varphi_v^\pi(g) = \langle \pi(g)v, v\rangle$ belongs to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. Let us now bound the expressions appearing in Lemma 4.23. To do this, fix $\varepsilon > 0$, let $f \in C_c(G)$, write $f_0 = f^* * f$ and set $K = \operatorname{supp} f_0$. This implies that $\operatorname{supp} f_0^{*n} \subseteq K^n$, and so

$$\begin{aligned}\left| \int_{K^n} f_0^{*n}(g) \varphi_v^\pi \,\mathrm{d}m \right|^2 &\leqslant \int_{K^n} |f_0^{*n}|^2 \,\mathrm{d}m \int_{K^n} |\varphi_v|^2 \,\mathrm{d}m \\ &\leqslant \|f_0^{*n}\|_2^2 \biggl(\int_{K^n} |\varphi_v|^{2+\varepsilon} \,\mathrm{d}m_G \biggr)^{\frac{2}{2+\varepsilon}} \biggl(\int_{K^n} \mathrm{d}m_G \biggr)^{\frac{\varepsilon}{2+\varepsilon}} \\ &\leqslant \|f_0^{*n}\|_2^2 \|\varphi_v\|_{2+\varepsilon}^2 (cM^n)^{\frac{\varepsilon}{2+\varepsilon}},\end{aligned}$$

where we have used the Cauchy–Schwarz and Hölder inequalities with the conjugate pair of exponents $\left(\frac{2+\varepsilon}{2}, \frac{2+\varepsilon}{\varepsilon}\right)$. Taking the $4n$th root and letting n go to infinity, it follows that

$$\begin{aligned}\lim_{n\to\infty} \biggl(\int_G (f^* * f)^{*n} \varphi_v^\pi \,\mathrm{d}m \biggr)^{\frac{1}{2n}} &\leqslant \lim_{n\to\infty} \|(f^* * f)^{*n}\|_2^{\frac{1}{2n}} M^{\frac{\varepsilon}{4(2+\varepsilon)}} \\ &= \|\lambda_*(f)\|_{\mathrm{op}} M^{\frac{\varepsilon}{4(2+\varepsilon)}}\end{aligned}$$

by Lemma 7.7. Since this holds for all $\varepsilon > 0$, we also obtain

$$\lim_{n\to\infty}\left(\int_G (f^* * f)^{*n}\varphi_v^\pi \,\mathrm{d}m\right)^{\frac{1}{2n}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}.$$

If there is a dense set of vectors $v \in \mathcal{H}_\pi$ for which the matrix coefficients are almost square integrable, then Lemma 4.23 gives $\|\pi(f)\|_{\mathrm{op}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}$ for all $f \in C_c(G)$ and Theorem 4.21 gives $\pi \prec \lambda$, proving the last part of the theorem.

Now suppose that $v \in \mathcal{H}_\pi$ has almost square integrable matrix coefficients. To prove the first part of the theorem assume without loss of generality that $\mathcal{H}_\pi$ is a cyclic representation with generator v. Hence vectors of the form $w = \sum_{j=1}^n \alpha_j \pi_{g_j} v$ for $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $g_1, \dots, g_n \in G$ are dense in $\mathcal{H}_\pi$. We claim that the matrix coefficients of such vectors also belong to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. To see this let w be as above and calculate

$$\begin{aligned}\varphi_w^\pi(g) = \langle \pi_g w, w\rangle &= \sum_{j,k=1}^n \alpha_j \overline{\alpha_k} \left\langle \pi_g \pi_{g_j} v, \pi_{g_k} v\right\rangle \\ &= \sum_{j,k=1}^n \alpha_j \overline{\alpha_k} \left\langle \pi_{g_k^{-1} g g_j} v, v\right\rangle \\ &= \sum_{j,k=1}^n \alpha_j \overline{\alpha_k} \varphi_v^\pi(g_k^{-1} g g_j).\end{aligned}$$

Clearly the left-regular representation preserves the p-norm for $p \in [1,\infty]$ and so $\|\varphi_v^\pi(g_k^{-1}\cdot g_j)\|_p = \|\varphi_v^\pi(\cdot g_j)\|_p$. Furthermore,

$$\|\varphi_v^\pi(\cdot g_j)\|_p^p = \int_G |\varphi_v^\pi(g g_j)|^p \,\mathrm{d}m(g) = \int |\varphi_v^\pi(g)|^p \,\mathrm{d}m(g) \Delta(g_j)^{-1} < \infty$$

by Lemma 1.8. It follows that $\varphi_w^\pi \in L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$, and the theorem follows from the case considered above. □

It is clear that, unlike the statement for discrete series representations in Theorem 7.2, the condition of almost square integrability is not a characterization for temperedness (for example, for $G = \mathbb{R}$). However, as we will see in the next chapter it is a characterization for some groups (for example, for $G = \mathrm{SL}_2(\mathbb{R})$)

Exercise 7.9. Show that the '$ax + b$'-group from Section 3.3.2 does not have subexponential growth, but that all its irreducible unitary representations (as in Proposition 3.18) are tempered.