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BOOK REVIEW

Ergodic Theory: with a view towards Number Theory

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Ergodic theory, which evolved out of problems from statistical mechanics in the early decades of the twentieth century, has in the last half century had an intense influence on number theory. Furstenberg's proof of Szemerédi's theorem (1977), Margulis' proof of the Oppenheim conjecture (1987) and the recent works of M. Einsiedler, A. Katok and E. Lindenstrauss on the Littlewood conjecture may be recalled as some of the dramatic episodes in successful applications of ergodic theory in addressing problems in combinatorial number theory, diophantine approximation and other topics in number theory—in many cases the ergodic-theoretic solutions remain so far the only available ones. The tremendous amount of work on the general theme of multiple recurrence following Furstenberg's approach to the Szemerédi theorem, Marina Ratner's work on classification of invariant measures and orbit closures of actions of subgroups generated by unipotent elements, establishing the Raghunathan conjecture, and the subsequent work on orbits of diagonalizable subgroups towards the Margulis conjecture, with implications for the Littlewood conjecture, have lent a robust stature to the theory and have exhibited the vast potential that remains to be harnessed towards further fruitful applications of ideas in ergodic theory to problems in number theory (as also in certain other areas).

This stage of development of the topic, almost into an independent branch stemming from basic ergodic theory, evidently warrants systematization of the accumulated knowledge and its diffusion to the wider mathematical world, and especially the student community. There have been many surveys and expositions, at various stages, towards giving a consolidated picture of the developments and serving as convenient references for researchers in the area. There have also been some nice books of a topical nature around the themes on which there has been marked progress. Furstenberg's book [**2**] especially comes to mind in respect of the topic of multiple recurrence. In relation to diophantine approximation, the book of Dave Morris [**3**] giving an exposition around the work of Ratner, and that of M. Bachir Bekka and M. Mayer [**1**] leading up to a proof of the Oppenheim conjecture may be especially recalled. These books have done a yeoman service in making certain crucial parts of the area accessible to a broad mathematical community, through gentle introductions to the topic at a beginner's level, with minimal prerequisites. Even so, work on diffusion at a broader level may be said to be at a fledgling stage. In particular, a book with a wider perspective on ergodic theory, and yet with a focus on the interaction with number theory, remained a glaring need in the overall context of the development of the subject.

The book under review goes a long way in fulfilling this need. The title itself, unusual as it is, seems to convey the role carved out for the book in this respect. While it covers a good deal of conventional ground in ergodic theory, it also motivates the study in terms of problems of combinatorial number theory and diophantine approximation. Furstenberg's proof of Szemerédi's theorem is one of the highlights of the book. The proof is described in detail, and indeed an alternative proof that appeared in a survey article of Furstenberg, Katznelson and Ornstein, not involving van der Waerden's theorem, is also presented. On the front of applications to diophantine approximation the coverage in the book is rather limited in the overall context of the current status of the area, but nevertheless conveys successfully the flavour of the topic. Part of the issue here is the prerequisites involved in providing a full-blown exposition of the topic. The authors commit themselves to not bringing in the theory of Lie groups. This policy does indeed have some merit in terms of keeping the contents accessible to a wider readership, given that in spite of its wide use in many areas of mathematics (and physics) the topic of Lie groups still remains unfamiliar territory to large sections of the mathematical community. To meet the constraint the authors largely confine themselves to discussing flows on homogeneous spaces of $\mathrm{SL}(2, \mathbb{R})$ (with finite invariant measure). This special case does carry many features of what is involved in general, and indeed forms a delightful part of the theory. For these flows the authors have endeavoured to give a very thorough treatment. In particular, the flows involved being the geodesic and horocycle flows associated with surfaces of constant negative curvature and finite area, their significance in terms of hyperbolic geometry has been well brought out.

On the other hand, the proofs in the special case do not involve many crucial issues in the general theory, and confinement to it inevitably becomes an impediment to quite an extent in conveying a fuller sense of the salient points in the developments that stand out as pillars of the recent progress in the area, or even some of the earlier results along the way. This reviewer feels that an intermediate path, dealing with $\mathrm{SL}(n, \mathbb{R})$ and its familiar subgroups more freely, yet without going deep into Lie theory, could have been adopted, enabling dissemination of some issues in the general case. However the authors perhaps wished to keep the contents more focused by adopting a well-set, though restricted, framework. The authors have promised a sequel to the present volume, where they propose to give 'some of the deeper applications' (as stated in the Preface to the book), and one hopes that with it the underlying conflict would be suitably resolved.

To be sure the interrelation between ergodic theory and number theory did not begin with the dramatic episodes recalled above. It is manifest already in the relation between the simplest dynamical systems consisting of rotations of the circle and values of linear forms $\alpha x + y$, where $\alpha \in \mathbb{R}$, over integral pairs (x, y), or values of αn modulo 1 for integral n. Furstenberg's proof of Weyl's theorem on equidistribution modulo 1, of values of a polynomial $p(n) = \sum_{k=0}^{d} a_k n^k$, with a_k irrational for at least one $k \neq 0$, via the unique ergodicity of certain affine automorphisms of tori may with hindsight be viewed as a curtain-raiser for the applications in diophantine approximation in the ensuing decades. Another, rather less well-known interaction is seen in the application of ergodicity of the Gauss map to the statistical features of the partial quotients in the continued fraction expansion, and its relation with the geodesic flow associated with the modular surface. The authors have successfully brought out the significance of these in the overall context. They also introduce the reader to nilrotations to which the phenomenon of unique ergodicity extends in a natural way.

Here are some overall features of the book. The material presented is copiously supplemented with notes at the end of each chapter, dealing with various interrelations, historical aspects etc. The details given here are impressive in many respects. Needless to say, there is a rich bibliography, with 395 titles! At the end, apart from the general index, an author index is included. At the beginning, the Preface is followed by a section titled 'Leitfaden' (a German word meaning 'Guide'). It gives, together with brief comments on the individual chapters, a diagram (flow chart) showing interrelations between various chapters in the form of a graph, in which some (directed) edges are solid lines indicating logical dependency, and others are dotted indicating partial or motivational links.

The main body of the book begins with a chapter on motivation, where the major problems in number theory to which techniques from ergodic theory have been successfully applied, as well as those on the horizon which it is hoped to address via ergodic theory, are introduced. This is followed by a second chapter which gives basic grounding in ergodic theory. Two proofs of Birkhoff's pointwise ergodic theorem are included. Notions of mixing, weak mixing, the associated unitary operator and various interrelations are developed. In the last section the authors also introduce the reader to induced transformations, the Kakutani skyscraper construction and the Kakutani–Rohlin lemma. The third chapter gives an introduction to continued fractions. According to the authors' flow chart this chapter has only a motivational connection with the rest of the chapters. It may be mentioned that the introduction here differs from that in books on number theory, reflecting the spirit of ergodic theory. The fourth chapter is on invariant measures, equidistribution (uniform distribution), etc., and includes Furstenberg's proof of Weyl's theorem on equidistribution for irrational polynomials. Chapters 5, 6 and 7 form a continuity, with the last one devoted to the proof of Szemerédi's theorem by ergodic theoretic methods; Chapter 5 deals with probabilistic background and Chapter 6 with factors and joinings which are crucial to the proof. Chapter 8 discusses ergodic theory in the framework of actions of locally compact groups, especially actions of amenable groups. The main purpose here is erudition in the ergodic theory culture. Chapter 10 is on the nilrotations mentioned earlier. Chapters 9 and 11 are concerned with the other main theme of the book, namely ergodic theory as applied in recent years to problems of diophantine approximation, via study of flows on homogeneous spaces. Chapter 9 is on the geodesic flows on quotients of the hyperbolic plane, which corresponds to action of the diagonal one-parameter subgroup of $SL(2, \mathbb{R})$ on a homogeneous space of the latter. After introducing the geometric framework and noting the correspondence, ergodicity of the geodesic flow in the finite-volume case is proved via Hopf's argument. The authors deduce from it the ergodicity of the Gauss map, highlighting the connection between the two. The last chapter is in many ways the meaty part of the exposition of ideas related to applications to problems in diophantine approximation. Various important results, including the Mautner phenomenon, the Howe–Moore mixing theorem, and mixing of higher orders are described for actions of $SL(2, \mathbb{R})$. Rigidity of the finite invariant measures of the horocycle flow, and the equidistribution of its non-periodic orbits in the finite-measure case, namely the case for

SL(2, $\mathbb{R}$) of Ratner's theorems on classification of invariant measures and equidistribution of orbits of unipotent flows, are covered.

The book is a very welcome addition and would no doubt inspire interest in the area among researchers as well as students, and cater to it successfully. One keenly looks forward to the arrival of the sequel promised by the authors.

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