# Order of mixing in algebraic systems

Łódź, 24/6/2021

§1: Definitions
(Corners in semi-groups (in blue))
§2: Non-algebraic phenomena
§3: Disconnected groups
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Remark:— the acting groups are abelian. The rich theory of actions of semi-simple Lie groups is dominated by decay properties of representations of the acting group. See, for example, work of Björklund, Einsiedler, Gorodnik on quantitative multiple mixing + references there.

| Semi-simple | Abelian |
|---|---|
| Representation theory of $G$. | Diophantine properties on $X$. |

A natural group to study is $\mathbb{Z}$ actions, but there is a problem...

For $\mathbb{Z}$-actions

$$
\begin{array}{ll}
 & \text{ergodic} \\
 & \Uparrow \\
 & \text{weak-mixing} \\
 & \Uparrow \\
 & \text{mild-mixing} \\
 & \Uparrow \\
\text{Rohklin's classical problem} & \left\{ \begin{array}{l} \text{mixing} \\ \Uparrow ? \\ \text{mixing on 3 sets.} \end{array} \right.
\end{array}
$$

Already for $\mathbb{Z}^2$ we know that the heirarchy of mixing properties is much richer.

## §1: Definitions

Setting:-

G - a countable (semi) group acting on $(X, \mu)$ a probability space.

" $\to \infty$ " - means going to infinity or leaving finite sets.

The action is <u>mixing</u> if

$$\mu(A \cap gB) \longrightarrow \mu(A)\mu(B)$$

as $g \to \infty$

for any measurable sets $A, B$.

The action is <u>mixing on k sets</u> if

$$\mu(A_0 \cap g_1 A_1 \cap \cdots \cap g_{k-1} A_{k-1}) \longrightarrow \prod_{j=0}^{k-1} \mu(A_j)$$

as $g_1, \ldots, g_{k-1} \longrightarrow \infty$

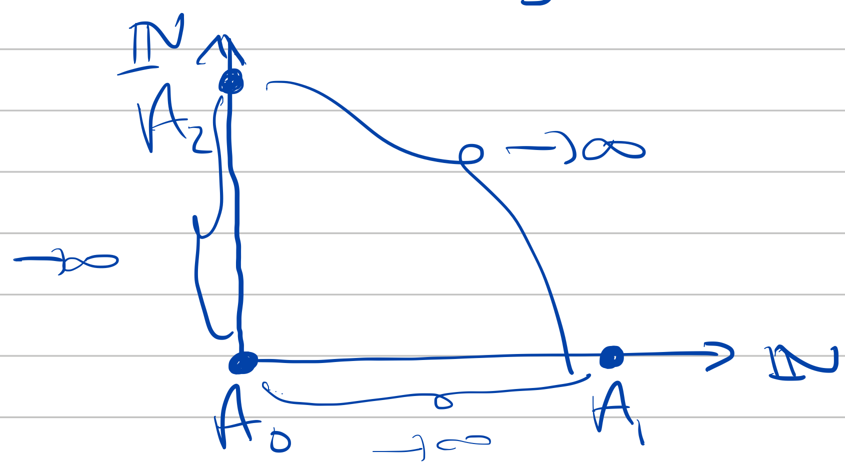
and $g_i g_j^{-1} \longrightarrow \infty$.

**Definition:-** The <u>order of mixing</u> of the $G$-action is the largest $k \in \mathbb{N}$ for which the action is mixing on $k$ sets, and the action is called mixing of all orders if it is mixing on $k$-sets $\forall k$.

**Example:-** The definitions make sense for semigroups, but some care is needed. Let $T:(X,\mu) \to (X,\mu)$ be a mixing transformation. Define an action of $\mathbb{N}^2$ on $X$ by

$$(n,m) : x \longmapsto T^n T^m x$$

This action is not faithful, is mixing, but is not mixing on 3 sets.




$$\mu\left(A_0 \cap (n,0)\cdot A_1 \cap (0,n)\cdot A_2\right)$$
$$= \mu\left(A_0 \cap T^n(A_1 \cap A_2)\right)$$
$$\to \mu(A_0)\,\mu(A_1 \cap A_2).$$

The invertible extension action of $\mathbb{Z}^2$ is not mixing:–

$$\tilde{X} = \{ (x_{n,m}) \in X^{\mathbb{Z}^2} \mid (x_{n+1,m}) = T(x_{n,m}), \\ (x_{n,m+1}) = T(x_{n,m}) \ \forall n,m \in \mathbb{Z} \}$$

has $(-1,1)$ acting as the identity.

The "corner" in $\mathbb{N}^d$ gives the following general picture:–

<u>Theorem</u>:– If an $\mathbb{N}^d$ action $(d \geq 2)$ is mixing on $k$ sets, then its invertible extension is mixing on $(k-1)$ sets.

- Faithful examples are easy to make for $k = 2, 3$.

- This means that

$$\text{order of mixing}(\mathbb{Z}^d \text{ extension}) \\ \geq \text{order of mixing}(\mathbb{N}^d \text{ action}) - 1.$$

<u>Problem</u>:– Construct examples where this is equality for order of mixing $k$, for any given $k \geq 2$.

From now on, assume $G = \mathbb{Z}^d$.

A <u>shape</u> $S \subset \mathbb{Z}^d$ is a finite set.

A shape $S$ is a <u>mixing shape</u> if

$$\mu\Big(\bigcap_{n \in S} (kn) \cdot A_n\Big) \longrightarrow \prod_{n \in S} \mu(A_n)$$

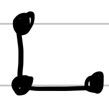
as $k \to \infty$.

That is, the "times" lie along dilates of a single shape.

<u>Example</u> (Ledrappier) A $\mathbb{Z}^2$ action can be mixing and have a shape of 3 elements that is <u>not</u> a mixing shape.

That is, mixing $\not\Rightarrow$ mixing on 3 sets, and that can be witnessed by a shape.

Idea:-

$$X = \{ (x(n,m)) \in \{0,1\}^{\mathbb{Z}^2} \mid$$  sums to 0 mod 2 everywhere $\}$.

Action = shift ; $\mu$ = Haar measure.

<u>Claim</u>:- the action is mixing (I will return to this).

Claim:— the shape $\{(0,0),(1,0),(0,1)\}$ is <u>not</u> a mixing shape.

<u>Proof of second claim</u>:—

Write down an element of $X$:—

$a_0+a_4$

$a_0+a_2 \quad a_1+a_3 \quad - - -$

$a_0+a_1 \quad a_1+a_2 \quad a_2+a_3 \quad a_3+a_4 \quad \ldots$

$- - \; a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 - - -$

and in general:—

$$x_{(0,2^n)} = x_{(0,0)} + x_{(2^n,0)}$$

$x(0,0)$ $\qquad\qquad$ $x(2^n,0)$

The three positions

$(0,0)$, $(2^n,0)$, $(0,2^n)$

are correlated $\forall n$.

Proof of first claim = a dictionary between algebraic dynamics and algebra.

"Fourier analysis"

| | |
|---|---|
| $X$ is a compact abelian group | $\hat{X}$ is a discrete abelian group. |
| $\mathbb{Z}^2$-action by shifting | $\hat{X}$ is a $\mathbb{Z}[\mathbb{Z}^2]$ module, $\mathbb{Z}[\mathbb{Z}^2] = \mathbb{Z}[t_1^{\pm}, t_2^{\pm}]$ |
| Action is mixing | Identity $m_0 + t_1^{n_1} t_2^{n_2} m_1 = 0$ for non-zero $m_0, m_1$ <u>bounds</u> $(n_1, n_2) \in \mathbb{Z}^2$. |

This is a machine that also expresses higher-order mixing.

For Ledrappier example,

$$\hat{X} \cong \underbrace{\mathbb{F}_2}_{\text{defines the } \{0,1\} \text{ alphabet}}[\underbrace{t_1^{\pm}, t_2^{\pm}}_{\text{defines the shift action}}] / \underbrace{\langle 1 + t_1 + t_2 \rangle}_{\text{imposes the sum to zero condition.}}$$

Non-zero elements :-

= Laurent polynomials not divisible by $(1+t_1+t_2)$.

Then check that

$$m_0 + t_1^{n_1} t_2^{n_2} m_1 = 0$$

for $m_0, m_1 \neq 0 \Rightarrow (n_1, n_2)$ bounded.

<u>Theorem (Schmidt)</u>: For $\mathbb{Z}^d$-actions by compact group automorphisms, <u>failure</u> of mixing of all orders can only happen in two essential ways :-

- degeneracy (like an element of the action having finite order)
- cancellation phenomena of Ledrappier's sort.

## §2: Non-algebraic phenomena

**Theorem** (Arenas-Carmona, Berend, Bergelson, 2008)
If you stay away from dilates of the bad shape, Ledrappier's example looks mixing of all orders.

(and in 2019 the same authors gave a similar result for $\mathbb{Z}^2$ algebraic actions of much more general type)

**Problem** :- What is the general picture for $\mathbb{Z}^d$ algebraic systems that have a non-mixing shape?

**Example** (Ferenczi, Kaminski, 1995; Ward, 1997)
A $\mathbb{Z}^d$-action can have all shapes mixing but be rigid in the sense that there is a sequence $(\underline{n}_j)$ in $\mathbb{Z}^d$ with $n_j \to \infty$ as $j \to \infty$ and

$$\mu(A \cap n_j \cdot A) \to \mu(A) \quad \text{as } j \to \infty.$$

## §3.1 Disconnected groups

Ledrappier's example is a $\mathbb{Z}^2$ action by automorphisms of a totally disconnected group.

A shape $S$ is called admissible if it does not lie in a line, does contain $0$, and $\frac{1}{k}S$ contains non-integer points for any $k \geq 2$.

**Theorem** (M. 1997) $(d \geq 2)$

(1) For any admissible $S$, there is an algebraic $\mathbb{Z}^d$-action for which $S$ is a minimal non-mixing shape.

(2) If every shape is mixing for an algebraic $\mathbb{Z}^d$ action, then it is mixing of all orders.

(3) If $S$ and $T$ are admissible shapes, then there is an algebraic $\mathbb{Z}^d$-action that is mixing on $S$ and not mixing on $T$ unless that is obviously false (i.e. unless some translate of $T$ is a subset of $S$).

Answering a question of Schmidt, Masser gave an effective statement on order of mixing.

<u>Theorem</u> (Masser, 2004)
For $\mathbb{Z}^d$-actions by automorphisms of compact groups,

$$\left\{ \begin{array}{c} \text{order of mixing} \\ \text{detected using} \\ \underline{\text{shapes}}\text{ only} \end{array} \right\} = \left\{ \begin{array}{c} \text{order of} \\ \text{mixing} \end{array} \right\}$$

in principle this is an <u>algebraic</u> calculation (but potentially an extremely difficult one)

in principle this is a <u>Diophantine</u> problem in positive characteristic.

# §41- Connected groups

Similar questions for $\mathbb{Z}^d$-actions by automorphisms of <u>connected</u> groups are different.

Fourier analysis dictionary continued.

| | |
|---|---|
| $X$ = Compact totally disconnected group | $\hat{X}$ is a <u>torsion</u> group ($\Rightarrow$ questions become algebraic geometry over finite fields). |
| $X$ = Compact connected group | $\hat{X}$ has <u>no additive torsion</u> ($\Rightarrow$ algebraic geometry over $\mathbb{Q}$). |

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SS These type of questions are not susceptible to Ax-Grothendieck arguments.

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<u>Theorem</u> (Schmidt, 1989) A mixing $\mathbb{Z}^d$-action by automorphisms of a connected group has all shapes mixing.

**Theorem** (Schmidt + W., 1993) A mixing $\mathbb{Z}^d$-action by automorphisms of a connected group is mixing of all orders.

Both results have no dynamics at all — they are number theory.

**Ideas in proof of mixing of all orders**

**Failure** to be mixing on $k$ sets

$\Longrightarrow$ there are non-zero elements
Fourier analysis
$m_1, m_2, \ldots, m_k$ of the module $\hat{X}$ and a sequence of solutions $(n_1^{(j)}, \ldots, n_k^{(j)}) \in (\mathbb{Z}^d)^k$ with $n_\ell^{(j)} \to \infty$ as $j \to \infty$ and $n_\ell^{(j)} - n_{\ell'}^{(j)} \to \infty$ as $j \to \infty$ such that

$$n_1^{(j)} \cdot m_1 + n_2^{(j)} \cdot m_2 + \cdots + n_k^{(j)} \cdot m_k = 0$$

for all $j$.

dual action of the original $\mathbb{Z}^d$-action.

$\Longrightarrow$ a Diophantine equation of
Algebra
"S-unit" type

in a field of characteristic zero with too many (infinitely many) solutions.

because group was connected

**Meaning:—**

In a field $K$ with zero characteristic, elements $a_1, \ldots, a_k \in K \setminus \{0\}$, $\Gamma$ a finitely-generated subgroup of $K^\times$ (multiplicative), an equation like

$\mathbb{Z}^d$-action

failure of $k$-mixing

$$a_1 x_1 + \cdots + a_k x_k = 0$$

with $x_1, \ldots, x_k \in \Gamma$ and no vanishing subsums has finitely many solutions (a deep result due to Schlickewei).

our only way out

$\Longrightarrow$ a subsum that vanishes infinitely often

$\Longrightarrow$ a witness to failure to mix on $k'$ sets for some $k' < k$.

Reverse Fourier analysis

Contrapositive:— mixing $\Rightarrow$ mixing of all orders. $\blacksquare$

Schlickewei's S-unit theorem is a corollary of the so-called "subspace theorem" of W. Schmidt, as generalized by Schlickewei.

<u>Optimist</u>:- prove the mixing of all orders result using dynamics.

<u>Realist</u>:- don't try this, just be thankful the subspace theorem exists.

(Aside:- the subspace theorems all generalize/improve/make more effective statements like this:-
if $L_1, \ldots, L_n$ are linearly independent linear forms in $n$ variables with algebraic coefficients, and $\varepsilon > 0$ is fixed, then the non-zero integer points $x$ with

$$|L_1(x) \cdots L_n(x)| < |x|^{-\varepsilon}$$

lie in a finite number of proper subspaces of $\mathbb{Q}^n$.)

## §5:- Actions of "the rationals"

"The rationals" have two identities:-

$(\mathbb{Q},+)$ : a rather small group, but bigger than $\mathbb{Z}$.

$(\mathbb{Q}^\times,\times)$ : a <u>huge</u> group.

and this is where the arguments above break down.

<u>Theorem</u> (Miles, W., 2006) An action on connected groups of a countable, torsion-free abelian group with finite torsion-free rank (subgroups of $\mathbb{Q}^d$) also has mixing $\Rightarrow$ mixing of all orders.

<u>Proof</u>:- Same idea, but a bigger Diophantine gun is needed:- a more uniform S-unit theorem due to Evertse, Schlickewei, W. Schmidt.

<u>Example</u> (Miles + W., 2006) There are mixing actions of $\mathbb{Q}^{\times}$ by automorphisms of compact connected groups that are not mixing of all orders.

## §6: "Effectiveness" and "Rigidity"

Effective mixing results flow from stronger Diophantine results.

<u>Theorem</u> (Miles + W., 2015) For a mixing $\mathbb{Z}^d$ action of "entropy rank one" (+ some mild conditions) there is a class $S$ of smooth functions and a decay function $\phi = o(1)$ so that

$$\left| \int f(x) g(T_{\underline{n}} x) \, d\mu(x) - \int f d\mu \int g d\mu \right| \leq C(f,g) \phi(\|n\|),$$

for $f, g \in S$.

(uses Yu's p-adic version of Baker's theorem).

__Theorem__ (Gorodnik + Spatzier, 2015)

For a $\mathbb{Z}^d$-action on a nilmanifold $X$ by automorphisms (+ mild conditions) there is a function $\theta \longmapsto \eta(\theta) > 0$ s.t. if $C^\theta(X)$ denotes the class of Hölder functions with exponent $\theta$ and Hölder norm $\|\cdot\|_\theta$, then there is a constant $C > 0$ such that

$$\left| \int f(T_{\underline{\ell}}x) g(T_{\underline{m}}x) h(T_{\underline{n}}x) \, d\mu(x) - \int f \, d\mu \int g \, d\mu \int h \, d\mu \right|$$

$$\leq C \left( \exp\left(\min\{\|\underline{m}-\underline{\ell}\|, \|\underline{\ell}-\underline{n}\|, \|\underline{n}-\underline{m}\|\}\right)\right)^{-\eta(\theta)} \times \|f\|_\theta \|g\|_\theta \|h\|_\theta ;$$

for all $f, g, h \in C^\theta$.

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The Diophantine input for mixing on four sets is not yet available.

# "Rigidity"

**Theorem** (Schmidt, 1995).
For systems like Ledrappier's, invariant measures that have the same minimal non-mixing shapes must be Haar measure.

**Problem:** For $\times 2, \times 3$ system on the circle, can the mixing of all orders property be used to force the same rigidity?

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