Chapter 3
Normal Abelian Subgroups and some Metabelian Groups

We discuss in this chapter the unitary duals of a few solvable non-compact non-abelian groups G. In some cases we will find a complete description of the unitary dual $\widehat{G}$. This will give us a chance to see how the results of Chapter 2 can be of importance for other groups, to see concrete instances of infinite-dimensional irreducible unitary representations, and to see that the Fell topology on $\widehat{G}$ (to be introduced in Section 4.4) can have exotic properties. Moreover, we will use here (in an *ad hoc* manner) a new type of construction that lifts a unitary representation from a subgroup of G to a unitary representation of G. This construction will also appear in other forms of increasing complexity in the following chapters. The first cases considered here should help the reader to become more familiar with this 'induced representation' construction.

However, we will also see that for some solvable groups G a complete description of the unitary dual is not a reasonable goal. As a result our focus will be on giving examples of both behaviours instead of a complete abstract discussion. Consequently, the only section of this chapter that will be important for later core developments (the above mentioned insights arising from examples notwithstanding) is Section 3.1.

3.1 Normal Abelian Subgroups

The results of Chapter 2 will be important in this chapter and also in later discussions concerning a non-abelian group G. In fact, if G has a closed normal abelian subgroup $H \lhd G$ then we will often be able to obtain useful information about a unitary representation π of G by restricting π to H, applying the results of Chapter 2, and using information about how the action of G on $H \lhd G$ (or rather, on its dual group $\widehat{H}$) and the spectral theory of $\pi|_H$ interact (as discussed in Section 1.3.6 and in Exercise 1.90 in simple instances). Here $\pi|_H$ denotes the unitary representation of H on $\mathcal{H}_\pi$ by restricting the homomorphism $\pi\colon G \to \mathrm{B}(\mathcal{H}_\pi)$ to H. We will see applications of this approach even for simple groups G, where one first restricts to subgroups of G that themselves have a normal abelian subgroup.

So suppose that G is a locally compact, σ-compact, metric group, and that $H \lhd G$ is a closed normal abelian subgroup of G. In this case any $g \in G$ induces an automorphism $\theta_g\colon H \to H$ defined by $\theta_g(h) = ghg^{-1}$. Using Lemma 2.29 for the abelian group H we obtain for each $g \in G$ an automorphism $\widehat{\theta}_g\colon \widehat{H} \to \widehat{H}$. We recall that $\widehat{\theta}_{g_1 g_2} = \widehat{\theta_{g_1}\theta_{g_2}} = \widehat{\theta}_{g_2}\widehat{\theta}_{g_1}$ for $g_1, g_2 \in G$.

For a unitary representation π of G, we will simply write μ_v (or $\mu_{v,w}$) for the spectral measure $\mu_v^{\pi|_H}$ on $\widehat{H}$ obtained from Corollary 2.11 for $\pi|_H$ and $v \in \mathcal{H}_\pi$ (resp. $\mu_{v,w}^{\pi|_H}$ obtained from Proposition 2.52 for $v, w \in \mathcal{H}_\pi$).

Proposition 3.1 (Spectral measures for normal subgroups). *Let G be a locally compact σ-compact metric group, and suppose that $H \lhd G$ is a closed normal abelian subgroup. Let π be a unitary representation of G. Then for $v, w \in \mathcal{H}_\pi$ and $g \in G$ we have*

$$\mu_{\pi_g v} = \big(\widehat{\theta}_g^{-1}\big)_* \mu_v$$

and

$$\mu_{\pi_g v, \pi_g w} = \big(\widehat{\theta}_g^{-1}\big)_* \mu_{v,w}.$$

Proof. Let $g \in G$ and $h \in H$. Then

$$\begin{aligned}\int_{\widehat{H}} \langle h, t\rangle \,\mathrm{d}\mu_{\pi_g v, \pi_g w}(t) &= \big\langle \pi_h \pi_g v, \pi_g w\big\rangle = \big\langle \pi_{g^{-1}hg} v, w\big\rangle \\ &= \int_{\widehat{H}} \big\langle \theta_{g^{-1}}(h), t\big\rangle \,\mathrm{d}\mu_{v,w}(t) \\ &= \int_{\widehat{H}} \Big\langle h, \widehat{\theta}_g^{-1} t\Big\rangle \,\mathrm{d}\mu_{v,w}(t) = \int_{\widehat{H}} \langle h, s\rangle \,\mathrm{d}\big(\widehat{\theta}_g^{-1}\big)_* \mu_{v,w}(s)\end{aligned}$$

for all $v, w \in \mathcal{H}_\pi$, which proves the proposition by uniqueness of the non-diagonal spectral measure in Proposition 2.52. For $v = w \in \mathcal{H}_\pi$ we also obtain the case of the principal matrix coefficient in Corollary 2.11. □

We will now study the functional calculus for $\pi|_H$ in Proposition 2.56, which gives the Hilbert space $\mathcal{H}_\pi$ for a unitary representation π of G a module structure over $\mathscr{L}^\infty(\widehat{H})$ by letting $F \in \mathscr{L}^\infty(\widehat{H})$ act via $\pi_{\mathrm{FC}}(F) = (\pi|_H)_{\mathrm{FC}}(F)$.

Corollary 3.2 (Functional calculus for normal subgroups). *Using the same assumptions as in Proposition 3.1, we have*

$$\pi_g \pi_{\mathrm{FC}}(F) \pi_g^{-1} = \pi_{\mathrm{FC}}(F \circ \widehat{\theta}_g)$$

for all $F \in \mathscr{L}^\infty(\widehat{H})$ and $g \in G$.

Proof. For $v, w \in \mathcal{H}_\pi$ we have by the definition of the functional calculus for $\pi|_H$ in Proposition 2.56, Proposition 3.1 and the definition again that

$$\begin{aligned}\left\langle \pi_{\mathrm{FC}}(F \circ \widehat{\theta}_g)v, w \right\rangle &= \int_{\widehat{H}} F \circ \widehat{\theta}_g \,\mathrm{d}\mu_{v,w} = \int_{\widehat{H}} F \,\mathrm{d}(\widehat{\theta}_g)_* \mu_{v,w} \\ &= \int_{\widehat{H}} F \,\mathrm{d}\mu_{\pi_g^{-1} v, \pi_g^{-1} w} = \left\langle \pi_{\mathrm{FC}}(F) \pi_g^{-1} v, \pi_g^{-1} w \right\rangle \\ &= \left\langle \pi_g \pi_{\mathrm{FC}}(F) \pi_g^{-1} v, w \right\rangle,\end{aligned}$$

which proves the corollary. □

It will also be convenient to use the projection-valued measures

$$\mathit{\Pi}_B = \pi_{\mathrm{FC}}(\mathbb{1}_B)$$

for measurable subsets $B \subseteq \widehat{H}$ as introduced in Section 2.6.1. Recall that these are orthogonal projections satisfying

$$\mathit{\Pi}_{B_1} \mathit{\Pi}_{B_2} = \mathit{\Pi}_{B_2} \mathit{\Pi}_{B_1} = \mathit{\Pi}_{B_1 \cap B_2}$$

for all measurable $B_1, B_2 \subseteq \widehat{H}$, and by Exercise 2.61 (see also the hint on p. 507) we also have

$$\mathit{\Pi}_{\bigsqcup_{n=1}^\infty B_n} = \sum_{n=1}^\infty \mathit{\Pi}_{B_n},$$

with the sum converging in the strong operator topology whenever $B_1, B_2, \dots$ are measurable mutually disjoint subsets of $\widehat{H}$.

We will be using the abbreviation

$$G \times \widehat{H} \ni (g, t) \longmapsto g \bullet t = \widehat{\theta}_g^{-1}(t) \in \widehat{H} \tag{3.1}$$

for the natural action of $g \in G$ on $t \in \widehat{H}$. With this Corollary 3.2 applied to $F = \mathbb{1}_B$ for a measurable subset $B \subseteq \widehat{H}$ gives the conjugacy formula

$$\mathit{\Pi}_{g \bullet B} = \pi_g \mathit{\Pi}_B \pi_g^{-1} \tag{3.2}$$

since

$$\mathit{\Pi}_{g \bullet B} = \pi_{\mathrm{FC}}\big(\mathbb{1}_{\widehat{\theta}_g^{-1} B}\big) = \pi_{\mathrm{FC}}\big(\mathbb{1}_B \circ \widehat{\theta}_g\big) = \pi_g \pi_{\mathrm{FC}}\big(\mathbb{1}_B\big) \pi_g^{-1} = \pi_g \mathit{\Pi}_B \pi_g^{-1}$$

by definition. We will use this identity frequently. It may help to remember that $\mathit{\Pi}_B$ is the projection to the subspace 'corresponding to generalized eigenvalues in B'. Hence (3.2) states how generalized eigenvalues for $\pi|_H$ behave when π_g is applied.

We leave the continuity of the action of G on $\widehat{H}$ as an exercise, since it will be quite obvious in the concrete cases we wish to study.

Exercise 3.3. Assume as above that G is a locally compact, σ-compact, metric group and that $H \lhd G$ is a closed normal abelian subgroup. Show that the action of G on $\widehat{H}$ defined in (3.1) is continuous.

We conclude this section with another general observation, this time concerning the maximal spectral measure obtained in Proposition 2.62 for the restriction $\pi|_H$.

Lemma 3.4 (Quasi-invariance). *Using the same assumptions as in Proposition* 3.1*, we have that a maximal measure is quasi-invariant under the action of G on $\widehat{H}$. That is, if $\mu_{\max}$ is a maximal measure on $\widehat{H}$ then for any $g \in G$ the push-forward measure $g_*\mu_{\max}$ under the action of g on $\widehat{H}$ lies in the same measure class as $\mu_{\max}$.*

Proof. Let $v_{\max} \in \mathcal{H}_\pi$ be a vector of maximal spectral type, $\mu_{\max} = \mu_{v_{\max}}$, and $g \in G$. Then $\pi_g v_{\max} \in \mathcal{H}_\pi$ and Proposition 3.1 implies that

$$g_*\mu_{\max} = \left(\widehat{\theta}_g^{-1}\right)_* \mu_{v_{\max}} = \mu_{\pi_g v_{\max}} \ll \mu_{\max},$$

since $\mu_{\max}$ is a maximal spectral measure (see Proposition 2.62). Applying this to g^{-1} gives $g_*^{-1}\mu_{\max} \ll \mu_{\max}$, hence $\mu_{\max} \ll g_*\mu_{\max}$, and the lemma follows. □

3.2 A Construction Using Invariant Measures

We assume now that G is a locally compact, σ-compact, metric group, that $H \lhd G$ is a closed normal abelian subgroup, that $A < G$ is a closed subgroup with $H \cap A = \{e\}$, and that

$$H \times A \ni (h, a) \longmapsto ha \in G$$

is a homeomorphism of topological spaces. In other words, G is the semi-direct product $A \ltimes H$ (which may also be written $H \rtimes A$) of its subgroups H and A. We will refer to H as the normal abelian subgroup, and to A as the complementary subgroup. We note that if in addition A is abelian, then G is called *metabelian*. We also note that Lemma 3.4 already shows a connection between unitary representations and quasi-invariant measures.

Making our life a bit easier, we assume in the following that μ is an A-invariant σ-finite measure on $\widehat{H}$ with respect to the action introduced in (3.1). In order to have the standing assumption of Section 1.1.2 satisfied, we let X be a measurable subset of $\widehat{H}$ with $\mu(\widehat{H} \smallsetminus X) = 0$, and assume that one of the following two conditions holds:

- $X = A{\bullet}t_0$ is a single A-orbit for some $t_0 \in \widehat{H}$, and we will equip X with the quotient topology induced by the orbit map $A \ni a \mapsto a{\bullet}t_0 \in X$. This makes X into a locally compact, σ-compact, metric space, and the Borel σ-algebra of X agrees with the Borel σ-algebra of X viewed as a subset of $\widehat{H}$. We assume that μ is locally finite on X, which then makes it into the Haar measure on
$$X = A{\bullet}t_0 \cong A/\operatorname{Stab}_A(t_0).$$
- X has the property that the induced topology of X as a subset of $\widehat{H}$ is locally compact, σ-compact, and μ is locally finite when considered as a measure on X.

We will see that these assumptions on $G \cong A \ltimes H$, $X \subseteq \widehat{H}$, and the measure μ are a good compromise. They will allow us to give a quite general framework for the construction of (irreducible) unitary representations that is still relatively easy to work with. Moreover, in some cases we will obtain from this construction all irreducible unitary representations of G.

Lemma 3.5 (Representations arising from invariant measures). *Every A-invariant σ-finite measure μ on $\widehat{H}$ as above gives rise to a unitary representation π^μ of G on $\mathcal{H}_\mu = L^2_\mu(X)$ by letting the elements of the normal abelian subgroup H act as multiplication operators and elements of the complementary subgroup A act via Koopman operators. More formally, we define*
$$\begin{aligned}\bigl(\pi^\mu_h f\bigr)(t) &= \langle h,t\rangle f(t),\\ \bigl(\pi^\mu_a f\bigr)(t) &= f(a^{-1}{\bullet}t) = f\bigl(\widehat{\theta}_a t\bigr), \text{ and}\\ \bigl(\pi^\mu_{ha} f\bigr)(t) &= \bigl(\pi^\mu_h \pi^\mu_a f\bigr)(t) = \langle h,t\rangle f(\widehat{\theta}_a t)\end{aligned}$$
for all $h \in H$, $a \in A$, $f \in L^2(\widehat{H},\mu)$, and $t \in \widehat{H}$.

Proof. Since $\langle h,t\rangle \in \mathbb{S}^1$ for every $h \in H$ and $t \in \widehat{H}$, it is clear that π^μ_h defines a unitary operator on $L^2(\widehat{H},\mu) = L^2_\mu(X)$. Moreover, for a convergent sequence (h_n) in H with $h_n \to h \in H$ as $n \to \infty$ and $f \in L^2_\mu(X)$ we also have
$$\|\pi^\mu_{h_n} f - \pi^\mu_h f\|_2^2 = \int |\langle h_n,t\rangle - \langle h,t\rangle|^2 |f(t)|^2 \,\mathrm{d}\mu(t) \longrightarrow 0$$
as $n \to \infty$, by dominated convergence.

For elements $a \in A$ we have that $A \ni a \mapsto \pi^\mu_a$ defines a unitary representation of A by Proposition 1.3 (or by the more general Proposition 1.5).

For $h \in H$, $a \in A$ and $f \in L^2_\mu(X)$, we define $\pi^\mu_{ha} f = \pi^\mu_h \pi^\mu_a f$. For $t \in \widehat{H}$ this means that
$$\pi^\mu_h\bigl(\pi^\mu_a f\bigr)(t) = \langle h,t\rangle\bigl(\pi^\mu_a f\bigr)(t) = \langle h,t\rangle f\bigl(\widehat{\theta}_a t\bigr).$$

To see that π^μ is indeed a representation of $G \cong A \ltimes H$, we use the above for h and a^{-1}, and calculate

$$\begin{aligned}
\big(\pi_a^\mu \pi_h^\mu \pi_{a^{-1}}^\mu f\big)(t) &= \big(\pi_h^\mu \pi_{a^{-1}}^\mu f\big)\big(\widehat{\theta}_a t\big) \\
&= \Big\langle h, \widehat{\theta}_a t\Big\rangle f\big(\widehat{\theta}_a^{-1}\widehat{\theta}_a t\big) \\
&= \langle \theta_a h, t\rangle f\big(\widehat{\theta}_{a^{-1}}\widehat{\theta}_a t\big) \\
&= \big\langle aha^{-1}, t\big\rangle f(t) = \big(\pi_{aha^{-1}}^\mu f\big)(t).
\end{aligned}$$

For $a_1, a_2 \in A$ and $h_1, h_2 \in H$ this now implies

$$\begin{aligned}
\pi_{h_1 a_1}^\mu \pi_{h_2 a_2}^\mu &= \pi_{h_1}^\mu \pi_{a_1}^\mu \pi_{h_2}^\mu \pi_{a_2}^\mu \\
&= \pi_{h_1}^\mu \underbrace{\pi_{a_1}^\mu \pi_{h_2}^\mu \pi_{a_1^{-1}}^\mu}_{=\pi^\mu_{a_1 h_2 a_1^{-1}}} \pi_{a_1}^\mu \pi_{a_2}^\mu \\
&= \pi_{h_1 a_1 h_2 a_1^{-1}}^\mu \pi_{a_1 a_2}^\mu = \pi_{h_1 a_1 h_2 a_2}^\mu
\end{aligned}$$

by definition, as required.

To see continuity of the representation π^μ of G, let $f \in L^2_\mu(X)$ and suppose that (h_n) and (a_n) are sequences in H and A, with $h_n \to h$ and $a_n \to a$ as $n \to \infty$. Then

$$\|\pi_{h_n a_n}^\mu f - \pi_{ha}^\mu f\| \leqslant \|\pi_{h_n}^\mu\big(\pi_{a_n}^\mu f - \pi_a^\mu f\big)\| + \|\pi_{h_n}^\mu \pi_a^\mu f - \pi_h^\mu \pi_a^\mu f\| \longrightarrow 0$$

as $n \to \infty$, as required. □

We recall that a non-zero A-invariant measure μ on $\widehat{H}$ is called A-ergodic if any measurable subset $B \subseteq \widehat{H}$ with $\mu(B \triangle a{\bullet}B) = 0$ for all $a \in A$ must satisfy $\mu(B) = 0$ or $\mu(\widehat{H} \smallsetminus B) = 0$. We note that the Haar measure on a single orbit $A{\bullet}t_0$ for some $t_0 \in \widehat{A}$ is always A-ergodic.†

Lemma 3.6 (Ergodicity implies irreducibility). *If μ is an A-invariant ergodic locally finite measure on $X \subseteq \widehat{H}$ as above, then the unitary representation π^μ is irreducible.*

Proof. Suppose that $\mathcal{V} \subseteq \mathcal{H}_\mu$ is a non-trivial π^μ-invariant closed subspace. Let $v_{\max}$ be a vector in $\mathcal{V}$ with maximal spectral type as in Proposition 2.62 for π^μ restricted to H and restricted to $\mathcal{V}$. We note that the spectral measure $\mu_{\max}$ of $v_{\max}$ is given by $|v_{\max}|^2\,\mathrm{d}\mu$. Let $B = \{t \in \widehat{H} \mid v_{\max}(t) \neq 0\}$. We claim that B is (up to sets of measure zero) invariant under A. Indeed,

$$\pi_a^\mu v_{\max} = v_{\max} \circ \widehat{\theta}_a \in \mathcal{V}$$

† It is tempting to think that this is tautological, but in fact it is an elementary consequence of properties of Haar measures (see [24, Lem. 10.3] or [22, Prop. 8.6] in the case of the Haar measure on A).

has an absolutely continuous spectral measure $|v_{\max}|^2 \circ \widehat{\theta}_a \,\mathrm{d}\mu$ with respect to the spectral measure $\mathrm{d}\mu_{\max} = |v_{\max}|^2 \,\mathrm{d}\mu$. This implies that almost every $t \in \widehat{\theta}_a^{-1} B$ belongs to B. Using $\pi^{\mu}_{a^{-1}} v_{\max} \in \mathcal{V}$ in the same way, we see that $\mu(B \triangle \widehat{\theta}_a^{-1} B) = 0$ as claimed.

By ergodicity of μ, this implies that $\mu(B) = 0$ or $\mu(\widehat{H} \smallsetminus B) = 0$. As $\mathcal{V}$ is a non-trivial subspace we have $v_{\max} \neq 0$, and so $\mu(\widehat{H} \smallsetminus B) = 0$. Applying the measurable functional calculus for $\pi^{\mu}|_H$, we see that $\mathscr{L}^{\infty}(\widehat{H}) v_{\max} \subseteq \mathcal{V}$. However, since $v_{\max}(t) \neq 0$ for μ-almost every $t \in X$ the subspace

$$\mathscr{L}^{\infty}(\widehat{H}) v_{\max} \subseteq L^2_{\mu}(X)$$

is dense. Hence $\mathcal{V} = \mathcal{H}_{\mu}$ is irreducible, and the lemma follows. □

Lemma 3.7 (No isomorphisms). *Let μ_1 and μ_2 be two A-invariant and ergodic measures on $\widehat{H}$ as above that are not multiples of each other. Then π^{μ_1} and π^{μ_2} are not isomorphic.*

We note that in the concrete cases of the above framework that we will consider, the proof of Lemma 3.7 will be significantly easier than in the general case considered here.

Proof of Lemma 3.7. By assumption, both measures are σ-finite. Hence we can apply the Lebesgue decomposition theorem to μ_1 and μ_2, which allows us to write $\mu_2 = \mu_{\mathrm{abs}} + \mu_{\mathrm{sing}}$ for two uniquely determined measures

$$\mu_{\mathrm{abs}} = \mu_2|_{B_{\mathrm{abs}}} \ll \mu_1,$$
$$\mu_{\mathrm{sing}} = \mu_2|_{B_{\mathrm{sing}}} \perp \mu_1,$$

and a measurable partition $\widehat{H} = B_{\mathrm{abs}} \sqcup B_{\mathrm{sing}}$. Applying the push-forward by some $a \in A$, invariance of μ_1 and μ_2, and the uniqueness of the Lebesgue decomposition, we see that μ_{abs} and μ_{sing} are also invariant under A.

We claim that the Radon–Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ of an A-invariant measure ν that is absolutely continuous with respect to another A-invariant and ergodic measure μ is almost surely constant. Applying the claim to

$$\nu = \mu_{\mathrm{abs}} \ll \mu = \mu_1$$

(and $\mu_{\mathrm{abs}} \ll \mu_2$), we deduce that μ_{abs} is a multiple of μ_1 (and of μ_2). Thus if $\mu_{\mathrm{abs}} \neq 0$, then μ_1 and μ_2 are multiples of each other and so by our assumption (and the claim) we have $\mu_1 \perp \mu_2$.

To prove the claim, we define $F = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ and we let $B \subseteq \widehat{H}$. Then

$$\begin{aligned}\int_B F \,\mathrm{d}\mu = \nu(B) = \nu(a^{-1}B) &= \int_{a^{-1}B} F \,\mathrm{d}\mu \\ &= \int_{a^{-1}B} (F \circ a^{-1}) \circ a \,\mathrm{d}\mu = \int_B F \circ a^{-1} \,\mathrm{d}\mu.\end{aligned}$$

As $B \subseteq \widehat{H}$ was arbitrary, this shows that $F = F \circ a^{-1}$ almost everywhere with respect to μ. Ergodicity of μ now implies that F must be constant μ-almost everywhere, which proves the claim.

Returning to the unitary representations, we note that the spectral measure of $v \in \mathcal{H}_{\mu_1} = L^2(\widehat{H}, \mu_1)$ for $\pi^{\mu_1}|_H$ is given by $|v|^2 \,\mathrm{d}\mu_1$, which applies similarly to elements of $\mathcal{H}_{\mu_2}$. Since $\mu_1 \perp \mu_2$, the spectral measures for elements of $\mathcal{H}_{\mu_1}$ and $\mathcal{H}_{\mu_2}$ are mutually singular, and hence the lemma follows from the claim. □

To summarize, we have shown (under mild technical assumptions) that a σ-finite A-invariant and ergodic measure on $\widehat{H}$ gives rise to an irreducible unitary representation of $G = A \ltimes H$, and that these unitary representations are non-isomorphic for any two non-proportional ergodic measures. There might be other irreducible representations (see, for example, Sections 3.2.1 and 3.6).

Exercise 3.8. Let $G = A \ltimes H$ and let μ_1, μ_2 be two A-invariant σ-finite measures on $\widehat{H}$ as above. Show that if $\mu_1 \ll \mu_2$ then $\pi^{\mu_1} < \pi^{\mu_2}$.

3.2.1 An Impossibly Complicated Dual

We wish to apply the above results to a concrete solvable group to see that sometimes it is more or less impossible to classify all irreducible unitary representations for a given group. For these negative results we will rely on some background in ergodic theory (see [22], for example). We define the normal abelian subgroup as

$$H = \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{2,2}(R) \;\middle|\; h \in R \right\}$$

where $R = \mathbb{F}_2[T, T^{-1}]$, and the complementary subgroup as the diagonal subgroup $A \cong \mathbb{Z}$ in

$$G = \mathbb{Z} \ltimes R = \left\{ \begin{pmatrix} T^n & h \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{2,2}(R) \;\middle|\; n \in \mathbb{Z}, h \in R \right\}$$

to simplify some discussions. This group can also be described using the wreath product as $\mathbb{F}_2 \wr \mathbb{Z}$ and is called the *lamplighter group*.† With some

† The name of this group comes from the following image. Envision an infinite street with lamps at each integer coordinate, with a lamplighter moving along the street. In this picture elements of G can be interpreted as instructions to the lamplighter to move along the street $\mathbb{Z}$ by distance one (corresponding to $\begin{pmatrix} T & 0 \\ & 1 \end{pmatrix}$) and to light or extinguish the lamp at that position (corresponding to $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$).

modifications, the discussion also holds for

$$G_2 = \mathbb{Z} \ltimes \mathbb{Z}[\tfrac{1}{2}] = \left\{ \begin{pmatrix} 2^n & h \\ 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{2,2}(\mathbb{Z}[\tfrac{1}{2}]) \;\middle|\; n \in \mathbb{Z}, h \in \mathbb{Z}[\tfrac{1}{2}] \right\}$$

and

$$G_M = \mathbb{Z} \ltimes \mathbb{Z}^2 = \left\{ \begin{pmatrix} M^n & h \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}) \;\middle|\; n \in \mathbb{Z}, h \in \mathbb{Z}^2 \right\}$$

for some fixed hyperbolic $M \in \mathrm{GL}_2(\mathbb{Z})$ and many other groups.

For

$$H \cong R = \mathbb{F}_2[T, T^{-1}] \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{F}_2$$

the Pontryagin dual is given by $\widehat{H} \cong \mathbb{F}_2^{\mathbb{Z}}$ (see Proposition 2.27). In other words, in the language of ergodic theory $\widehat{H}$ is the full shift on 2 symbols. The automorphism θ of H defined by $\theta(h) = Th$ featuring in the definition of G can be written as

$$\theta\big((c_n)_{n \in \mathbb{Z}}\big) = (c_{n-1})_{n \in \mathbb{Z}}$$

if we identify the polynomial $h = \sum_{n \in \mathbb{Z}} c_n T^n \in H$ with the finitely supported sequence $(c_n) \in \bigoplus_{n \in \mathbb{Z}} \mathbb{F}_2$ of its coefficients. The dual automorphism is therefore given by the left shift, so

$$\widehat{\theta}\big((t_n)_{n \in \mathbb{Z}}\big) = (t_{n+1})_{n \in \mathbb{Z}}$$

for $(t_n)_{n \in \mathbb{Z}} \in \widehat{H}$.

We will now show that the space of ergodic measures for the full shift is ridiculously large and that such measures cannot be meaningfully classified. We indicate a few ways to find invariant ergodic probability measures, and ask the reader to take on faith the fact that these are very far from exhaustive:

- (Periodic points and orbits) The full shift has many periodic points which may be obtained by taking any finite block of 0s and 1s and concatenating it infinitely often, and each such point gives rise to a finitely supported ergodic invariant measure simply by averaging along its orbit. Precisely, if $x \in \mathbb{F}_2^{\mathbb{Z}}$ has $\widehat{\theta}^n x = x$ for some $n \geqslant 1$ then $\frac{1}{n} \sum_{k=0}^{n-1} \delta_{\widehat{\theta}^k x}$ is such a measure, where δ_x denotes the point mass at x. Moreover, if the orbit of $x \in \mathbb{F}_2^{\mathbb{Z}}$ is infinite, then we can also use the σ-finite measure $\mu = \sum_{k \in \mathbb{Z}} \delta_{\widehat{\theta}^k x}$ for the construction of the irreducible unitary representation.
- (Bernoulli measures) For any $p \in (0, 1)$ we can define the product measure using probabilities p and $1 - p$ for the symbols $0 \in \mathbb{F}_2$ and $1 \in \mathbb{F}_2$ respectively. The resulting shift-invariant ergodic measure is called a *Bernoulli measure*.
- (Markov and Parry measures) For any finite set S of finite words (or blocks) in the alphabet $\mathbb{F}_2 = \{0, 1\}$, one can define a closed $\widehat{\theta}$-invariant subset $X_S \subseteq \widehat{H}$ of all sequences that do not contain any word from S. Under mild conditions on S one can find a natural ergodic invariant prob-

ability measure on X_S, called the *Parry measure*, which is in some sense the most uniformly distributed measure on X_S. Varying the set S gives another collection of ergodic invariant probability measures, a countable collection in this case. Allowing more general constructions gives uncountably many different ergodic invariant probability measures called *Markov measures*.

- (Coding a measure-preserving system) Given an ergodic measure-preserving system $T\colon X \to X$ on a probability space $(X, \mathcal{B}, \nu)$ we can define, for any measurable set $A \subseteq X$, a map
$$\Phi_A\colon X \to \mathbb{F}_2^{\mathbb{Z}}$$
by sending $x \in X$ to the sequence $\big(\mathbb{1}_A(T^n x)\big)_{n\in\mathbb{Z}}$. Then $\mu_{T,A} = (\Phi_A)_*\nu$ is an ergodic invariant probability measure on $\mathbb{F}_2^{\mathbb{Z}}$. For this rather general construction it is very hard to see when precisely one gets different ergodic measures (though this is certainly going to happen often). So let us specialize this setup in the following construction.
- (Torus rotations) Let $d \geqslant 1$, $X = \mathbb{T}$ and $T = R_\alpha\colon x \mapsto x+\alpha$ be a rotation defined by some $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$. Assuming that $1, \alpha_1, \dots, \alpha_d$ are linearly independent over $\mathbb{Q}$, this defines an ergodic measure-preserving system with respect to Lebesgue measure m. For any measurable set A in $\mathbb{T}^d$ we now obtain a map Φ_A and an ergodic invariant probability measure $\mu_{\alpha,A}$ on $\mathbb{F}_2^{\mathbb{Z}}$. As T has 'very few factors', we can find a variety of ergodic measures by varying α and A. Using the spectral theory of the associated Koopman operators or entropy theory (see [21, Sec. 1.3]), we also see that these measures are all different from the Bernoulli, Parry, or Markov measures above, and it is clear that they cannot coincide with any periodic point measure unless $m(A) \in \{0, 1\}$. Using $X = \mathbb{T}^2$, from any 'picture with a frame' $A \subseteq \mathbb{T}^2$, we can construct a new irreducible unitary representation of the lamplighter group in such a way that different pictures (when considered modulo the Lebesgue measure) give rise to different irreducible unitary representations; see Figure 3.1 and Exercise 3.9.

The above selection of examples is influenced by the mathematical interests of the authors. Many other examples and constructions of ergodic measures or irreducible unitary representations are possible (see also Section 3.6.1).

In the following exercise, we invite the reader to prove the mentioned 'constructions of irreducible unitary representations via pictures'.

Exercise 3.9. Let $X = \mathbb{T}^2$, $\nu = m_{\mathbb{T}^2}$ be the Haar measure, and define $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ by $T\colon \mathbb{T}^2 \ni x \mapsto x + \alpha \in \mathbb{T}^2$ for some $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. We suppose throughout that $1, \alpha_1, \alpha_2$ are linearly independent over $\mathbb{Q}$.
(a) Prove that $m_{\mathbb{T}^2}$ is an ergodic invariant measure for T. Show that any orbit
$$\{x + n\alpha \mid n \in \mathbb{Z}\}$$
is dense in $\mathbb{T}^2$.

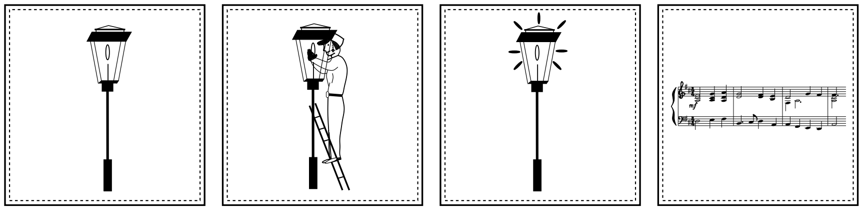

Fig. 3.1: For the purpose of Exercise 3.9, we say that a subset A of $\mathbb{T}^2$ 'has a frame' is there exists some $\varepsilon > 0$ with the property that $[0,1] \times [-\varepsilon, \varepsilon] + \mathbb{Z}^2 \subseteq A$ and $[-\varepsilon, \varepsilon] \times [0,1] + \mathbb{Z}^2 \subseteq A$ but the sets $(\varepsilon, 1-\varepsilon) \times \big((\varepsilon, 2\varepsilon) \cup (1-2\varepsilon, 1-\varepsilon)\big) + \mathbb{Z}^2$ and $\big((\varepsilon, 2\varepsilon) \cup (1-2\varepsilon, 1-\varepsilon)\big) \times (\varepsilon, 1-\varepsilon) + \mathbb{Z}^2$ are disjoint from A. These pictures define irreducible unitary representations of the lamplighter group that have no better description than that given by Exercise 3.9 and the pictures themselves.

(b) Now suppose that $A \subseteq \mathbb{T}^2$ is an arbitrary measurable subset with a 'frame' as illustrated in Figure 3.1. Use A to define a factor map $\Phi_A \colon \mathbb{T}^2 \to \mathbb{F}_2^{\mathbb{Z}}$ sending $x \in \mathbb{T}^2$ to the sequence $\big(\mathbb{1}_A(x + n\alpha)\big)_{n\in\mathbb{Z}}$. Show that Φ_A is injective.
(c) Show that for two measurable subsets $A_1, A_2 \subseteq \mathbb{T}^2$ with frames, we have $\mu_{T,A_1} = \mu_{T,A_2}$ if and only if $m(A_1 \triangle A_2) = 0$.
(d) Now conclude, together with the results of this section, that any black and white picture on $[0,1]^2$ with a frame gives rise to an irreducible unitary representation of the lamplighter group uniquely associated to that picture.

3.3 Characterization of some Unitary Duals

We show in this section in a couple of concrete examples how a complete description of the unitary duals of metabelian groups can look like.

3.3.1 The Isometry Group of the Plane

We define the orientation-preserving isometry group $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ of the plane as the semi-direct product of $\mathrm{SO}_2(\mathbb{R})$ and $\mathbb{R}^2$. More concretely, we define

$$G = \left\{ \begin{pmatrix} k & h \\ 0 & 1 \end{pmatrix} \;\middle|\; k \in \mathrm{SO}_2(\mathbb{R}), h \in \mathbb{R}^2 \right\}$$

and will consider the subgroups

$$K = \left\{ \begin{pmatrix} k_\phi & 0 \\ & 1 \end{pmatrix} \;\middle|\; k_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \text{ with } \phi \in \mathbb{R} \right\} = \mathrm{SO}_2(\mathbb{R})$$

and

$$H = \left\{ \begin{pmatrix} I & h \\ & 1 \end{pmatrix} \,\middle|\, h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathbb{R}^2 \right\} \cong \mathbb{R}^2,$$

satisfying $G = HK$, $H \lhd G$, and $K \cap H = \{I\}$. We will identify the complementary subgroup K with $\mathrm{SO}_2(\mathbb{R})$ and the normal abelian subgroup H with $\mathbb{R}^2$. We note that conjugation of $h \in H$ by $k \in K$ simply corresponds to application of $k \in \mathrm{SO}_2(\mathbb{R})$ to $h \in \mathbb{R}^2$. By the discussion in Section 2.4.2, the homomorphism dual to the conjugation map θ_k is therefore given by $\widehat{\theta_k} = k^{\mathrm{t}} = k^{-1}$. Adapting the notation of Section 3.1 and (3.1), we have $k{\bullet}t = kt$ for $k \in \mathrm{SO}_2(\mathbb{R})$ and $t \in \widehat{H} \cong \mathbb{R}^2$.

The Irreducible Representations

We will roughly classify the unitary representations depending on whether H acts trivially or not.

(Old) Every irreducible unitary representation of

$$K \cong G/H$$

gives rise to a unitary representation of G, which we will refer to as an *old representation*. Since $G/H \cong \mathrm{SO}_2(\mathbb{R}) \cong \mathbb{T}$ we see that for every $n \in \mathbb{Z}$ there is an associated old irreducible unitary representation defined by the character

$$\chi_n \colon G \ni \begin{pmatrix} \cos\phi & -\sin\phi & h_1 \\ \sin\phi & \cos\phi & h_2 \\ & & 1 \end{pmatrix} \longmapsto \mathrm{e}^{\mathrm{i}n\phi}.$$

(New) We will refer to an irreducible unitary representation π of G for which $\pi|_H$ is a non-trivial representation of H as a *new representation*. As we have seen in Corollary 1.79, such representations must exist.

We now define for every $r > 0$ a new representation π^r on $\mathcal{H}_r = L^2(\mathbb{R}^2, \mu_r)$, where μ_r is the normalized arc length measure on the circle

$$r\mathbb{S}^1 = \{t \in \mathbb{R}^2 \mid \|t\| = r\}$$

of radius r. We will think of $r\mathbb{S}^1$ as a subset of the dual group $\widehat{H} \cong \widehat{\mathbb{R}^2} \cong \mathbb{R}^2$. In fact μ_r is a K-invariant and ergodic probability measure on

$$X = r\mathbb{S}^1 = K{\bullet}\begin{pmatrix} r \\ 0 \end{pmatrix}.$$

As discussed in Section 3.2, we can now define $\pi^r|_H$ simply by the multiplication representation

$$\left(\pi^r_h f\right)(t) = \langle h, t\rangle f(t)$$

for $h \in H$, $f \in L^2(\widehat{H}, \mu_r)$, and $t \in \widehat{H}$. On the subgroup K we define

$$\big(\pi_k^r f\big)(t) = f\big(\widehat{\theta}_k t\big) = f(k^{-1}t)$$

for $k \in K$, $f \in L^2(\widehat{H}, \mu_r)$, and $t \in \widehat{H}$. Lemmas 3.5 and 3.6 show that this gives rise to an irreducible unitary representation π^r on $\mathcal{H}_r = L^2(\mathbb{R}^2, \mu_r)$.

Lemma 3.10 (Non-isomorphic). *Let r and s be different positive real numbers. Then the new representations π^r and π^s are not isomorphic.*

Proof. This follows from Lemma 3.7, but as the case at hand is so much easier we also give a direct argument. For any $f \in \mathcal{H}_r$ the spectral measure of f with respect to $\pi^r|_H$ has support in $r\mathbb{S}^1$, while for $f \in \mathcal{H}_s$ the spectral measure of f with respect to $\pi^s|_H$ has support in $s\mathbb{S}^1$ (see Proposition 2.51(2)). The lemma follows since $r\mathbb{S}^1 \cap s\mathbb{S}^1 = \emptyset$. □

The reader may benefit from analysing alongside our discussion a different but slightly easier semi-direct product. Hence we fix for the following exercises an integer $d \geqslant 2$, and define

$$K_d = \left\{ \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ & & 1 \end{pmatrix} \in K \;\middle|\; \phi \in \tfrac{2\pi}{d}\mathbb{Z} \right\} \cong \mathbb{Z}/d\mathbb{Z}$$

and $G_d = K_d H < G$. As before, we will distinguish between old representations (which correspond to elements of $\widehat{K_d} \cong \mathbb{Z}/d\mathbb{Z}$) and new representations.

Exercise 3.11. (a) Construct for every $t_0 \in \widehat{H} \smallsetminus \{0\}$ a new representation π of G_d on $\mathbb{C}^d$ such that for some $v \in \mathbb{C}^d$ we have $\pi_h v = \langle h, t_0 \rangle v$ for all $h \in H$.
(b) Show that the new representation defined in (a) is indeed a unitary representation of G_d. Also show directly that the representation is irreducible.
(c) Characterize the pairs $t_0, t_1 \in \widehat{H} \smallsetminus \{0\}$ with the property that the new unitary representations from (a) associated to t_0 and t_1 are isomorphic.

Exercise 3.12. Prove the analogue to Proposition 3.13 for the group G_d.

Classification of the Unitary Dual

Proposition 3.13 (Description of $\widehat{G}$). *Let $G = K \ltimes H = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the isometry group of the plane as above. Then the set of old representations χ_n for $n \in \mathbb{Z}$ and of new irreducible representations π^r for $r \in (0, \infty)$ constructed above comprise the complete set of irreducible representations.*

We note that this proposition also follows quite easily from our main technical result of the chapter, namely Theorem 3.20, but we believe it is worthwhile to prove the case at hand directly as a warm up.

Proof of Proposition 3.13. Let π be an irreducible representation of G. If $\pi|_H$ is trivial, then π induces a representation of $G/H \cong K = \mathrm{SO}_2(\mathbb{R}) \cong \mathbb{T}$ and so equals an old representation defined by some weight $n \in \mathbb{Z} = \widehat{\mathbb{T}}$.

So let us assume that $\pi|_H$ is non-trivial. Let $\mu_{\max} = \mu_{v_{\max}}$ with $v_{\max} \in \mathcal{H}_\pi$ be a maximal spectral measure as in Proposition 2.62 for $\pi|_H$, and let

$$S = \operatorname{supp} \mu_{\max} \subseteq \widehat{H}$$

be its support. We claim that $S = r\mathbb{S}^1$ for some $r > 0$.

For the proof of the claim we first show that $S \subseteq \widehat{H}$ is invariant under the compact subgroup $K = \mathrm{SO}_2(\mathbb{R})$. Indeed, applying π_k for some $k \in K$ to $v_{\max}$ we obtain by Lemma 3.4 that $\mu_{\pi_k v_{\max}} = k_* \mu_{v_{\max}}$ defines the same measure class as $\mu_{v_{\max}}$. This implies that $kS = S$ for all $k \in K$ for all $k \in K$. To prove the claim, we define the function $R\colon \widehat{H} \cong \mathbb{R}^2 \ni t \mapsto \|t\|$ and note that $R(kt) = R(t)$ for all $k \in K$ and $t \in \widehat{H}$. Together with Corollary 3.2, this shows that $B = \pi_{\mathrm{FC}}(R)$ is equivariant for π restricted to K. Since B is equivariant for π restricted to H by construction of the measurable functional calculus in Proposition 2.56, B is equivariant for π. By irreducibility of π and Schur's lemma (Theorem 1.29), it follows that $B = \pi_{\mathrm{FC}}(R) = rI$ for some constant r. By the spectral theorem (Corollary 2.63) and the measurable functional calculus (Proposition 2.56(6)), this amounts to the claim that

$$S = \operatorname{supp} \mu_{\max} = r\mathbb{S}^1.$$

Moreover, we must have $r > 0$ since $\pi|_H$ is assumed to be non-trivial.

Having found the right candidate for r, we still need to prove that π is isomorphic to π^r. For this we first notice that there exists some weight $n \in \mathbb{Z}$ and some eigenvector $v \in \mathcal{H}_\pi$ for $\pi|_K$ of weight n (since $K \cong \mathbb{T}$ is compact and abelian with $\widehat{K} \cong \mathbb{Z}$).

We define the measurable map $\mathrm{D} \in \mathscr{L}^\infty(\widehat{H})$ by taking the direction of the argument; that is,

$$\widehat{H} \ni t \longmapsto \mathrm{D}(t) = \begin{cases} 1 & \text{if } t = 0, \\ \mathrm{e}^{\mathrm{i}\phi} & \text{if } t = \|t\| k_\phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for some } k_\phi \in K. \end{cases}$$

We will use D together with the functional calculus in Proposition 2.56 for the restriction $\pi|_H$. By Corollary 3.2, we have

$$\pi_{\mathrm{FC}}(\mathrm{D} \circ k_\psi^{\mathrm{t}}) = \pi_{k_\psi} \pi_{\mathrm{FC}}(\mathrm{D}) \pi_{k_\psi}^{-1}$$

for $k_\psi \in K$, where

$$\mathrm{D} \circ k_\psi^{\mathrm{t}}(t) = \mathrm{D}\left(\|t\| k_{\phi-\psi} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) = \mathrm{e}^{\mathrm{i}(\phi-\psi)} = \mathrm{e}^{-\mathrm{i}\psi}\, \mathrm{D}(t)$$

for all

$$t = \|t\| k_\phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \widehat{H}. \tag{3.3}$$

This now implies for $\pi_{\text{FC}}(\mathrm{D})v$ that w is an eigenvector for $\pi|_K$ of weight $(n-1)$. Indeed,

$$\pi_{k_\psi}\pi_{\text{FC}}(\mathrm{D})v = \pi_{k_\psi}\pi_{\text{FC}}(\mathrm{D})\pi_{k_\psi}^{-1}\pi_{k_\psi}v = \pi_{\text{FC}}\big(\mathrm{e}^{-\mathrm{i}\psi}\,\mathrm{D}\big)\mathrm{e}^{\mathrm{i}n\psi}v = \mathrm{e}^{\mathrm{i}(n-1)\psi}\pi_{\text{FC}}(\mathrm{D})v$$

for all $k_\psi \in K$. Similarly, $\pi_{\text{FC}}(\overline{\mathrm{D}})v$ is an eigenvector of weight $(n+1)$. In other words, $\pi_{\text{FC}}(\mathrm{D})$ lowers, and $\pi_{\text{FC}}(\overline{\mathrm{D}})$ raises, the K-weight of any eigenvector. We also note that $\pi_{\text{FC}}(\mathrm{D})$ and $\pi_{\text{FC}}(\overline{\mathrm{D}}) = \pi_{\text{FC}}(\mathrm{D})^*$ are unitary, since $|\,\mathrm{D}(t)| = 1$ for all $t \in \widehat{H}$. In particular, we have

$$\|v\| = \|\pi_{\text{FC}}(\mathrm{D})v\| = \|\pi_{\text{FC}}(\overline{\mathrm{D}})v\|.$$

Hence we may and will assume that $v = v_0 \in \mathcal{H}_\pi$ is an eigenvector for $\pi|_K$ of weight 0 with $\|v\| = 1$. For every $m \in \mathbb{Z}$ we also define $v_m = \pi_{\text{FC}}(\mathrm{D}^m)v_0$ and notice that this implies that $\|v_m\| = 1$, $\pi_{\text{FC}}(\mathrm{D}^n)v_m = v_{m+n}$, v_m has weight $-m$, and hence $\langle v_m, v_n\rangle = \delta_{m,n}$ for all $m, n \in \mathbb{Z}$. We define the closed subspace

$$\mathcal{V} = \langle v_m \mid m \in \mathbb{Z}\rangle \subseteq \mathcal{H}_\pi$$

and

$$U\colon \mathcal{V} \ni \sum_{m\in\mathbb{Z}} a_m v_m \longmapsto \sum_{m\in\mathbb{Z}} a_m\,\mathrm{D}^m \in \mathcal{H}_r = L^2(r\mathbb{S}^1, \mu_r).$$

We claim that $\mathcal{V} = \mathcal{H}_\pi$ and that U is an equivariant isomorphism from $\mathcal{H}_\pi$ to $\mathcal{H}_r$, which will imply the proposition.

Since μ_r is the normalized arc length measure on $r\mathbb{S}^1$, it follows that the functions D^m for $m \in \mathbb{Z}$ make up an orthonormal basis of $\mathcal{H}_r$. In particular, U is a unitary isomorphism between $\mathcal{V}$ and $\mathcal{H}_r$. Moreover,

$$\begin{aligned}\pi^r_{k_\psi}\,\mathrm{D}^m(t) &= \pi^r_{k_\psi}\,\mathrm{D}^m\left(\|t\|k_\phi\begin{pmatrix}1\\0\end{pmatrix}\right)\\ &= \mathrm{D}^m\left(k_\psi^{-1}\|t\|k_\phi\begin{pmatrix}1\\0\end{pmatrix}\right) = \mathrm{e}^{\mathrm{i}m(\phi-\psi)} = \mathrm{e}^{-\mathrm{i}m\psi}\,\mathrm{D}^m(t)\end{aligned}$$

for all t as in (3.3) shows that D^m is also an eigenvector for $\pi^r|_K$ of weight $-m$. Therefore

$$U\colon \mathcal{V} \longrightarrow \mathcal{H}_r$$

is already equivariant for $\pi|_K$ as well as for the operators $\pi_{\text{FC}}(\mathrm{D}^n)$ on $\mathcal{H}_\pi$, respectively $M_{\mathrm{D}^n} = \pi^r_{\text{FC}}(\mathrm{D}^n)$ on $\mathcal{H}_r$ for all $n \in \mathbb{Z}$.

For the proof that $\mathcal{V}$ is invariant under $\pi|_H$ and that U is also equivariant for $\pi|_H$ (which are both unclear at this point) we will again use the functional calculus for $\pi|_H$. In fact, using the fact that $\pi|_H$ and $\pi^r|_H$ have their spectral measures supported on the same circle $r\mathbb{S}^1$, we will show that π_h for $h \in H$

can be expressed in terms of the operators $\pi_{\mathrm{FC}}(\mathrm{D}^n)$ for $n \in \mathbb{Z}$ which we used to define $\mathcal{V}$.

More precisely, on $r\mathbb{S}^1$ we have the identities

$$\begin{aligned}
t_1 &= \tfrac{r}{2}\big(\mathrm{D}(t) + \mathrm{D}^{-1}(t)\big), \\
t_2 &= \tfrac{r}{2}\big(\mathrm{D}(t) - \mathrm{D}^{-1}(t)\big), \\
\mathrm{e}^{2\pi \mathrm{i} h_1 t_1} &= \sum_{n=0}^{\infty} \tfrac{1}{n!}\Big(2\pi \mathrm{i} h_1 \tfrac{r}{2}\big(\mathrm{D}(t) + \mathrm{D}^{-1}(t)\big)\Big)^n, \\
\mathrm{e}^{2\pi \mathrm{i} h_2 t_2} &= \sum_{n=0}^{\infty} \tfrac{1}{n!}\Big(2\pi \mathrm{i} h_2 \tfrac{r}{2}\big(\mathrm{D}(t) - \mathrm{D}^{-1}(t)\big)\Big)^n,
\end{aligned}$$

where the two sums converge uniformly on $r\mathbb{S}^1$ for every

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in H.$$

By the properties of the functional calculus in Proposition 2.56, this implies

$$\begin{aligned}
\pi_{h_1 e_1} &= \pi_{\mathrm{FC}}\big(t \longmapsto \mathrm{e}^{2\pi \mathrm{i} h_1 t_1}\big) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\big(2\pi \mathrm{i} h_1 \tfrac{r}{2}\big)^n \sum_{\ell=0}^{n} \tbinom{n}{\ell} \pi_{\mathrm{FC}}\big(\mathrm{D}^{n-2\ell}\big) \\
\pi_{h_2 e_2} &= \pi_{\mathrm{FC}}\big(t \longmapsto \mathrm{e}^{2\pi \mathrm{i} h_2 t_2}\big) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\big(2\pi \mathrm{i} h_2 \tfrac{r}{2}\big)^n \sum_{\ell=0}^{n} \tbinom{n}{\ell} (-1)^\ell \pi_{\mathrm{FC}}\big(\mathrm{D}^{n-2\ell}\big)
\end{aligned}$$

for all $h = h_1 e_1 + h_2 e_2 \in H$. This shows that we can express π_h for any $h \in H$ in terms of h, the number r, and $\pi_{\mathrm{FC}}(\mathrm{D}^n)$ for $n \in \mathbb{Z}$. Since $\mathcal{V}$ is invariant under the latter and U is equivariant for the latter, it follows that $\mathcal{V}$ is invariant under $\pi|_H$ and U is equivariant for $\pi|_H$. As $\mathcal{V}$ is closed and invariant, and π is assumed to be irreducible, it follows that $\mathcal{V} = \mathcal{H}_\pi$. This concludes the proof. □

One may wonder why we had to work quite so hard to obtain the isomorphism above, and in particular to obtain spectral multiplicity one for the restriction to the normal abelian subgroup $H \lhd G$. The following exercise shows that the latter is not automatic for cyclic representations.

Exercise 3.14 (Infinite spectral multiplicity of cyclic representations). Let G be the group $\mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be as above and $r > 0$. Show that $(\pi^r)^\infty$ is cyclic.

The reader may wonder why we restricted our attention in this section to the two-dimensional case. Indeed it is possible to extend the above to higher dimensions, but the complete description of irreducible representations in these cases is a bit more involved. We will discuss this briefly in Section 3.5.1.

The Full Isometry Group of the Plane

We will now discuss, through a series of exercises, the unitary dual $\widehat{G}$ of the full isometry group

$$G = \mathrm{O}_2(\mathbb{R}) \ltimes \mathbb{R}^2 = \left\{ \begin{pmatrix} k & h \\ & 1 \end{pmatrix} \;\middle|\; k \in \mathrm{O}_2(\mathbb{R}), h \in \mathbb{R}^2 \right\}.$$

Exercise 3.15 (Old representations). Show that the unitary dual of $\mathrm{O}_2(\mathbb{R})$ is given by

$$\widehat{\mathrm{O}_2(\mathbb{R})} = \{\mathbb{1}, \det\} \cup \{\delta^n \mid n \in \mathbb{N}\},$$

where $\mathbb{1}$ is the trivial representation of $\mathrm{O}_2(\mathbb{R})$, det is the unitary character defined by the determinant map, and δ^n for each $n \in \mathbb{N}$ is an irreducible unitary representation of $\mathrm{O}_2(\mathbb{R})$ on $\mathbb{C}^2$ so that $\delta^n|_{\mathrm{SO}_2(\mathbb{R})}$ has eigenspaces for weights $n, -n$.

Exercise 3.16 (New representations). Let $r > 0$. Show that the new representation π^r of $\mathrm{SO}_2(\mathbb{R})$ extends, by the same formulas, to a new representation $\pi^{r,+}$ of $\mathrm{O}_2(\mathbb{R})$ on $\mathcal{H}_r$. Show that $\pi^{r,-} = \det \otimes \pi^{r,+}$ defined by

$$G \ni g \longmapsto \pi^{r,-}(g) = \det(g)\pi^{r,+}(g)$$

defines an irreducible new representation $\pi^{r,-}$ that is not unitarily equivalent to $\pi^{r,+}$.

Exercise 3.17 (Completeness). Prove that

$$\widehat{\mathrm{O}_2(\mathbb{R}) \ltimes \mathbb{R}^2} = \widehat{\mathrm{O}_2(\mathbb{R})} \sqcup \{\pi^{r,+}, \pi^{r,-} \mid r \in (0, \infty)\}.$$

3.3.2 The Affine Group in One Dimension

As our next solvable group, we wish to study the affine group in one dimension, known colloquially as the '$ax + b$' group. In fact we have a choice of either studying the connected group

$$G_{>0} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{R}_{>0}, b \in \mathbb{R} \right\} \cong \mathbb{R}_{>0} \ltimes \mathbb{R}$$

or the full affine group

$$G_{\times} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{R}^{\times}, b \in \mathbb{R} \right\} \cong \mathbb{R}^{\times} \ltimes \mathbb{R}.$$

We will start with $G = G_{>0}$ and explain below how to adopt the results to $G_{\times}$. We will use the letter a for elements of $\mathbb{R}^{\times}$, and the letter b for elements of $\mathbb{R}$. For convenience, we will use the abbreviations

$$g_a = \begin{pmatrix} a & 0 \\ & 1 \end{pmatrix}, h_b = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix},$$

and define the abelian complementary subgroup $A = \{g_a \mid a \in \mathbb{R}_{>0}\}$ and the normal abelian subgroup $H = \{h_b \mid b \in \mathbb{R}\}$ so that $G = AH = HA$ and every element

$$g = \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \in G$$

can be written in a unique way as the product $g = h_b g_a$.

The Irreducible Representations

We will use the same terminology as was used in Section 3.3.1.

(Old) Every irreducible unitary representation of $\mathbb{R}_{>0} \cong A \cong G/H$ gives rise to an irreducible unitary representation of $G = G_{>0}$ by letting the normal subgroup H act trivially. Since $\mathbb{R}_{>0} \ni a \mapsto \log a \in \mathbb{R}$ is an isomorphism, we see that every such old representation is given by a unitary character

$$\chi_\alpha \colon G \ni h_b g_a \longmapsto a \in \mathbb{R}_{>0} \longmapsto \mathrm{e}^{2\pi \mathrm{i}\alpha \log a} = a^{2\pi \mathrm{i}\alpha} \in \mathbb{S}^1$$

for some (uniquely determined) $\alpha \in \mathbb{R}$.

(New) We now define two new representations π^+ and π^- for which the normal subgroup $\mathbb{R} \cong H \lhd G$ acts non-trivially.

To define π^+, we will use the measure μ_+ on $\widehat{H}$ defined by

$$\mathrm{d}\mu_+ = \mathbb{1}_{(0,\infty)} \frac{\mathrm{d}t}{t},$$

where we use the isomorphism $\widehat{H} \cong \mathbb{R}$, the variable t (or s) to denote elements of $\widehat{H}$, and $\mathrm{d}t$ to denote the Lebesgue measure on $\widehat{H}$. Along with μ_+ we define the Hilbert space

$$\mathcal{H}_+ = L^2(\widehat{H}, \mu_+).$$

We note that the action of $g_a \in A$ on $\widehat{H}$ defined by

$$\widehat{H} \ni t \longmapsto g_a \bullet t = a^{-1} t \in \widehat{H}$$

preserves μ_+, since for any measurable $B \subseteq \mathbb{R}_{>0}$ we have

$$\mu_+(a^{-1}B) = \int_0^\infty \mathbb{1}_{a^{-1}B}(t) \frac{\mathrm{d}t}{t} = \int_0^\infty \mathbb{1}_B(at) \frac{\mathrm{d}t}{t} = \int_0^\infty \mathbb{1}_B(s) \frac{\mathrm{d}s}{s} = \mu_+(B)$$

via the substitution $s = at$. As in Section 3.2, we now define the unitary representation on H by using the multiplication representation

$$\left(\pi^+_{h_b} f\right)(t) = \mathrm{e}^{2\pi \mathrm{i} bt} f(t)$$

for $h_b \in H$, $f \in \mathcal{H}_+$, and $t \in \mathbb{R}$. On A we use the Koopman representation to define

$$\left(\pi^+_{g_a} f\right)(t) = f(at)$$

for $a \in \mathbb{R}_{>0}$, $f \in \mathcal{H}_+$, and $t \in \widehat{H}$. Putting these definitions together, we define once more

$$\pi^+_g f = \pi^+_{h_b g_a} f = \pi^+_{h_b} \pi^+_{g_a} f$$

for $g = h_b g_a = \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \in G$.

Similarly, we define π^- on $\mathcal{H}_- = L^2(\widehat{\mathbb{R}}, \mu_-)$ using the measure μ_- defined by $\mathrm{d}\mu_- = \mathbb{1}_{(-\infty,0)} \frac{\mathrm{d}t}{|t|}$ on $\widehat{H} \cong \mathbb{R}$.

Lemmas 3.5 and 3.6 now show that π^+ and π^- are both irreducible unitary representations of G, and Lemma 3.7 shows that these are not isomorphic to each other (this also follows from the fact that $\operatorname{supp}(\pi^+|_H) \neq \operatorname{supp}(\pi^-|_H)$).

Classification of the Unitary Dual

Proposition 3.18 (Description of $\widehat{G}$). *Let*

$$G = G_{>0} = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{R}_{>0}, b \in \mathbb{R} \right\}$$

be as above. Then the set of old representations defined by characters on A and the two new representations π^+ and π^- comprise the complete set of irreducible representations of G.

Proof. Let π be an irreducible unitary representation of G and let $\mu_{\max}$ be a maximal spectral measure on $\widehat{H}$ for $\pi|_H$.

We note that

$$\widehat{H} \cong \mathbb{R} = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty)$$

and that each of the sets $B = (-\infty, 0)$, $B = \{0\}$, or $B = (0, \infty)$ satisfies the property $a \bullet B = B$ for all $a > 0$. Hence (3.2) implies that Π_B is equivariant for A. As it is defined by the functional calculus for $\pi|_H$ it is also equivariant for $\pi|_H$. By Schur's lemma (Theorem 1.29) and since Π_B is an orthogonal projection, we therefore obtain $\Pi_B = I$ or $\Pi_B = 0$. For $\mu_{\max}$ this implies that $\mu_{\max}(\widehat{H} \smallsetminus B) = 0$ or $\mu_{\max}(B) = 0$.

If $\mu_{\max}$ is supported on $\{0\}$, then $\pi|_H$ is trivial and π can be thought of as a representation of $G/H \cong A$, and hence is defined by a character of A by Schur's lemma (Theorem 1.29).

Suppose now that

$$\mu_{\max}\left(\widehat{H} \smallsetminus A \bullet t_0\right) = 0$$

for $t_0 = 1$. Applying Theorem 3.20 below now shows that π is isomorphic to π^+.

If, on the other hand $\mu_{\max}\big(\widehat{H}\smallsetminus A{\boldsymbol{\cdot}}t_0\big) = 0$ for $t_0 = -1$, then applying Theorem 3.20 shows that π is isomorphic to π^-. □

The Full Affine Group

For the full affine group $G = G_\times$ we define

$$H = \left\{ h_b = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\}$$

and

$$A_\times = \left\{ g_a = \begin{pmatrix} a & 0 \\ & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{R}^\times \right\}.$$

The 'old representations' are in this case given by characters of the abelian group $G/H \cong A_\times \cong \mathbb{R} \times \mathrm{C}_2$. The 'new representation' $\pi^\times$ is in this case defined by the Haar measure $\mu^+ + \mu^-$ on the orbit $A_\times{\boldsymbol{\cdot}}1 = \mathbb{R}^\times$ inside $\widehat{H} \cong \mathbb{R}$.

In fact Lemmas 3.5 and 3.6 show that $\pi^\times$ is an irreducible unitary representation. Moreover, applying Theorem 3.20 below shows that any irreducible representation π of $G_\times$ for which $\pi|_H$ is non-trivial is isomorphic to $\pi^\times$.

Exercise 3.19. Fill in the details of the extension sketched above.

3.3.3 Characterization of some Irreducible Representations

We will prove in this section the technical result needed for classifying the irreducible representations of the affine group, and of the Heisenberg group, in the next section. We would like to apply the same method of proof as in Section 3.3.1 for the isometry group of the plane, but are faced with the technical complication that the group $A = \mathbb{R}$ appearing in the affine group is not compact, while the group $K = \mathrm{SO}_2(\mathbb{R})$ appearing in the isometry group of the plane was compact. In fact we used the latter in Section 3.3.1 to produce an eigenvector for K, and for a unitary representation of A this vector may not exist. To handle this we will use below the fact that $\mathbb{R}$ has a compact quotient $\mathbb{R}/\mathbb{Z} = \mathbb{T}$.

To make the result more generally applicable to semi-direct products of abelian groups we make the following assumptions on $G = A \ltimes H$:

- G is a locally compact, σ-compact, metric group.
- $H \lhd G$ is a closed normal abelian subgroup.
- $A < G$ is a closed abelian subgroup.
- The map $H \times A \ni (h, a) \mapsto ha \in G$ is a homeomorphism of topological spaces.

- A has a discrete co-compact subgroup.

We note that the last assumption is satisfied by $\mathbb{R}^d$, $\mathbb{Z}^d$, $\mathbb{T}^d$, and products of these.

Theorem 3.20 (Characterizing an irreducible representation). *Let G be as above. Assume that π is an irreducible unitary representation of G such that the maximal spectral measure $\mu_{\max}$ for $\pi|_H$ has $\mu_{\max}\big(\widehat{H}\smallsetminus(A\bullet t_0)\big) = 0$ for a single A-orbit $A\bullet t_0$ with $t_0 \in \widehat{H}$ for which the orbit map $A \ni a \mapsto a\bullet t_0$ is injective. Then π is isomorphic to the representation π^μ constructed in Lemma 3.5 for the push-forward image μ of the Haar measure m_A on $A\bullet t_0$ under the orbit map.*

The proof of the proposition will again rely heavily on spectral measures and the functional calculus for $\pi|_H$. We note that the technical assumption on $A_0 < A$ will be used in the form of the conclusion of the next lemma.

Lemma 3.21 (Fundamental domain). *Let A be a locally compact σ-compact metric abelian group, let $\Lambda < A$ be a discrete co-compact subgroup, and let μ be a finite measure on A. Then there exists a measurable fundamental domain $B_A \subseteq A$ for Λ with non-empty interior, compact closure, and a μ-null set as boundary.*

Proof. We will use additive notation in the abelian group A throughout the proof of this lemma. Recall that a measurable fundamental domain in A for Λ is a measurable subset $B \subseteq A$ with the property that $|B \cap (a+\Lambda)| = 1$ for every $a \in A$. By discreteness of Λ and local compactness of A there exists, for every $a_0 \in A$, some $\varepsilon_0 > 0$ for which the map

$$B_{\varepsilon_0}(a_0) \ni a \longmapsto a + \Lambda \in A/\Lambda$$

is injective, and $\overline{B_{\varepsilon_0}(a_0)}$ is compact. Since $\mu\big(\partial B_\varepsilon(a_0)\big) > 0$ can only take place for countably many values of $\varepsilon > 0$, we may choose for each a_0 some $\varepsilon_0 > 0$ as above so that we also have $\mu\big(\partial B_{\varepsilon_0}(a_0)\big) = 0$.

By compactness of A/Λ, there exist $a_1, \ldots, a_n \in A$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ with these properties for which

$$A = \bigcup_{k=1}^{n} \big(B_{\varepsilon_k}(a_k) + \Lambda\big).$$

A fundamental domain with the desired properties is now given by

$$B_A = B_{\varepsilon_1}(a_1) \sqcup \big(B_{\varepsilon_2}(a_2) \smallsetminus \big(B_{\varepsilon_1}(a_1) + \Lambda\big)\big) \sqcup \cdots \sqcup \Big(B_{\varepsilon_n}(a_n) \smallsetminus \Big(\bigcup_{k=1}^{n-1} B_{\varepsilon_k}(a_k) + \Lambda\Big)\Big).$$

□

Proof of Theorem 3.20. We let $\Lambda < A$ be a discrete co-compact subgroup of A and $B_A \subseteq A$ a fundamental domain for Λ as in Lemma 3.21. By assumption, the maximal spectral measure $\mu_{\max}$ for $\pi|_H$ satisfies

$$\mu_{\max}\big(\widehat{H} \smallsetminus A\centerdot t_0\big) = 0$$

for a single A-orbit $A\centerdot t_0$ of some $t_0 \in \widehat{H}$. Moreover, the map $A \ni a \mapsto a\centerdot t_0$ is assumed to be bijective, so we may move this $\mu_{\max}$ to a measure $\nu_{\max}$ on A. By Lemma 3.21 we may also assume that $\nu_{\max}\big(\partial B_A\big) = 0$.

Decomposing the Hilbert space. We define

$$B_0 = B_A\centerdot t_0 = \{a\centerdot t_0 \mid a \in B_A\}$$

and will also frequently use the sets $\lambda\centerdot B_0 = (\lambda B_A)\centerdot t_0$ for $\lambda \in \Lambda$. By our assumptions, this shows

$$\widehat{H} = \bigsqcup_{\lambda \in \Lambda} \lambda\centerdot B_0 \sqcup \big(\widehat{H} \smallsetminus A\centerdot t_0\big),$$

where $\widehat{H} \smallsetminus A\centerdot t_0$ is a null set with respect to $\mu_{\max}$. Notice that the latter implies that

$$\Pi_{\widehat{H} \smallsetminus A\centerdot t_0} = 0,$$

and hence

$$I = \Pi_{A\centerdot t_0} = \sum_{\lambda \in \Lambda} \Pi_{\lambda\centerdot B_0}. \tag{3.4}$$

We also define

$$\mathcal{H}_\rho = \Pi_{B_0}\mathcal{H}_\pi$$

and note that

$$\Pi_{\lambda\centerdot B_0}\mathcal{H}_\pi = \pi_\lambda \Pi_{B_0}\mathcal{H}_\pi = \pi_\lambda \mathcal{H}_\rho$$

for all $\lambda \in \Lambda$ by the conjugacy formula in (3.2). This gives

$$\mathcal{H}_\pi = \bigoplus_{\lambda \in \Lambda} \pi_\lambda \mathcal{H}_\rho \tag{3.5}$$

by (3.4).

Defining the representation ρ of $K = A/\Lambda$. Using the decomposition above, we will now define a unitary representation ρ of $K = A/\Lambda$ on the $\pi|_H$-invariant closed subspace $\mathcal{H}_\rho$. This will lead to a K-fixed vector in $\mathcal{H}_\rho$ corresponding to the function $\mathbb{1}_{B_0} \in L^2(\widehat{H}, \mu)$ under the desired isomorphism, and then in turn to the desired equivariant isomorphism between $\mathcal{H}_\pi$ and $L^2(\widehat{H}, \mu)$.

We define $\rho_a v$ for $a \in A$ and $v \in \mathcal{H}_\rho$ by the following steps. For this it may be helpful to compare these steps to the measurable action of A on $B_A = A/\Lambda$ by translation, as depicted in Figure 3.2.

- First apply π_a to v to obtain $\pi_a v \in \mathcal{H}_\pi$.
- Next decompose
$$\pi_a v = \sum_{\lambda \in \Lambda} \Pi_{\lambda \bullet B_0} \pi_a v$$
into the components
$$\Pi_{\lambda \bullet B_0} \pi_a v \in \pi_\lambda \mathcal{H}_\rho$$
for $\lambda \in \Lambda$ corresponding to the orthogonal sum in (3.5).
- Finally, modify the terms of the sum using $\pi_{\lambda^{-1}}$ for $\lambda \in \Lambda$ to obtain the definition
$$\rho_a v = \sum_{\lambda \in \Lambda} \underbrace{\pi_\lambda^{-1} \Pi_{\lambda \bullet B_0} \pi_a v}_{= \Pi_{B_0} \pi_\lambda^{-1} \pi_a v} \in \mathcal{H}_\rho, \tag{3.6}$$
where the second formula for the summands follows from the conjugacy formula (3.2) for λ and B_0.

We will now show that ρ_a is a unitary operator on $\mathcal{H}_\rho$ for all $a \in A$ and ρ indeed defines a unitary representation of $K = A/\Lambda$ on $\mathcal{H}_\rho$.

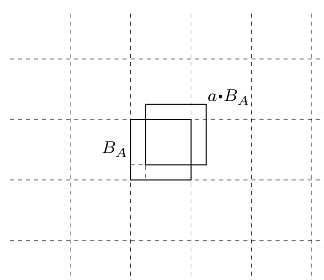

Fig. 3.2: The natural translation action of $a \in A$ on $B_A \cong A/\Lambda$ consists of applying a to an element $b \in B_A$ and finding the unique $\lambda \in \Lambda$ with $\lambda^{-1} ab \in B_A$. In the figure λ is either the identity or one of three other elements of Λ. In the proof of Theorem 3.20 we are 'translating this picture' into subspaces of $\mathcal{H}_\pi$ and a unitary representation of $K = A/\Lambda$.

UNITARITY OF ρ_a. To see that $\rho_a v$ as in (3.4) defines a vector of $\mathcal{H}_\rho$ with $\|\rho_a v\| = \|v\|$, we start by noting that $\Pi_{B_0} v = v$ by definition of $\mathcal{H}_\rho = \Pi_{B_0} \mathcal{H}_\pi \ni v$. Using the conjugacy formula (3.2) for a and B_0, this gives
$$\Pi_{a \cdot B_0} \pi_a v = \pi_a \Pi_{B_0} v = \pi_a v, \tag{3.7}$$

meaning (roughly speaking) that the application of π_a to v shifts the 'H-spectral type' from B_0 to $a{\bullet}B_0$ (see Figure 3.2). Applying (3.7) for $\lambda^{-1}a$ instead of a, and noting that

$$\mathit{\Pi}_{B_0\cap\lambda^{-1}a{\bullet}B_0} = \mathit{\Pi}_{B_0}\mathit{\Pi}_{\lambda^{-1}a{\bullet}B_0}$$

gives

$$\mathit{\Pi}_{B_0\cap\lambda^{-1}a{\bullet}B_0}\pi_\lambda^{-1}\pi_a v = \mathit{\Pi}_{B_0}\pi_\lambda^{-1}\pi_a v \tag{3.8}$$

for all $\lambda \in \mathit{\Lambda}$. We note that compactness of the closure $\overline{B_A}$ of the fundamental domain B_A implies that $B_0 \cap \lambda^{-1}a{\bullet}B_0 \neq \emptyset$ for only finitely many $\lambda \in \mathit{\Lambda}$ (namely for those λ in $\mathit{\Lambda} \cap aB_AB_A^{-1}$). Therefore, the sum in (3.6) has only finitely many non-zero summands. Moreover, for $\lambda_1 \neq \lambda_2$ in $\mathit{\Lambda}$ we have

$$\lambda_1^{-1}a{\bullet}B_0 \cap \lambda_2^{-1}a{\bullet}B_0 = \lambda_1^{-1}a{\bullet}\bigl(B_0 \cap \lambda_1\lambda_2^{-1}{\bullet}B_0\bigr) = \emptyset.$$

Hence (3.8) implies that the vectors appearing in the sum (3.6) defining $\rho_a v$ have disjoint spectral measures, and so are mutually orthogonal. Therefore, unitarity of π_λ for $\lambda \in \mathit{\Lambda}$ and (3.4) give

$$\|\rho_a v\|^2 = \sum_{\lambda\in\mathit{\Lambda}} \|\pi_\lambda^{-1}\mathit{\Pi}_{\lambda{\bullet}B_0}\pi_a v\|^2 = \sum_{\lambda\in\mathit{\Lambda}} \|\mathit{\Pi}_{\lambda{\bullet}B_0}\pi_a v\|^2 = \|\pi_a v\|^2 = \|v\|^2. \tag{3.9}$$

As linearity of ρ_a is clear, we have therefore shown that $\rho_a\colon \mathcal{H}_\rho \to \mathcal{H}_\rho$ is a unitary operator.

Unitary representation of K. Let $a_1, a_2 \in A$. Then, for all $v \in \mathcal{H}_\rho$, we have

$$\begin{aligned}
\rho_{a_1}(\rho_{a_2}v) &= \rho_{a_1}\sum_{\lambda_2\in\mathit{\Lambda}} \mathit{\Pi}_{B_0}\pi_{\lambda_2}^{-1}\pi_{a_2}v \\
&= \sum_{\lambda_1,\lambda_2\in\mathit{\Lambda}} \mathit{\Pi}_{B_0}\pi_{\lambda_1}^{-1}\pi_{a_1}\mathit{\Pi}_{B_0}\pi_{\lambda_2}^{-1}\pi_{a_2}v \\
&= \sum_{\lambda_1,\lambda_2\in\mathit{\Lambda}} \mathit{\Pi}_{B_0}\pi_{\lambda_1\lambda_2}^{-1}\mathit{\Pi}_{\lambda_2a_1{\bullet}B_0}\pi_{a_1a_2}v \\
&= \sum_{\lambda\in\mathit{\Lambda}} \mathit{\Pi}_{B_0}\pi_\lambda^{-1}\sum_{\lambda_2\in\mathit{\Lambda}} \mathit{\Pi}_{a_1\lambda_2{\bullet}B_0}\pi_{a_1a_2}v
\end{aligned}$$

by definition of ρ_{a_2}, resp. ρ_{a_1} in (3.6), the conjugacy formula (3.2) for λ_2a_1 and B_0, commutativity of A, and the substitution $\lambda = \lambda_1\lambda_2$ for λ_1. Next we note that the sets $a_1\lambda_2{\bullet}B_0$ for $\lambda_2 \in \mathit{\Lambda}$ are disjoint with union $A{\bullet}t_0$. Hence

$$\sum_{\lambda_2\in\mathit{\Lambda}} \mathit{\Pi}_{a_1\lambda_2{\bullet}B_0} = I,$$

and we obtain

$$\rho_{a_1}(\rho_{a_2}v) = \sum_{\lambda\in\Lambda} \Pi_{B_0}\pi_\lambda^{-1}\pi_{a_1a_2}v = \rho_{a_1a_2}v$$

by the definition of $\rho_{a_1a_2}$ in (3.6).

We also note that for $a \in \Lambda$ we have

$$\rho_a v = \sum_{\lambda\in\Lambda} \Pi_{B_0}\pi_\lambda^{-1}\pi_a v = v$$

since for $\lambda \in \Lambda\smallsetminus\{a\}$ we have $B_0 \cap \lambda^{-1}aB_0 = \emptyset$ and so $\Pi_{B_0}\pi_\lambda^{-1}\pi_a v = 0$ by (3.8). This shows that ρ defines a representation of $K = A/\Lambda$ on $\mathcal{H}_\rho$.

It remains to see continuity of the representation. So suppose that

$$v \in \mathcal{H}_\rho = \Pi_{B_0}\mathcal{H}_\pi \subseteq \mathcal{H}_\pi.$$

By the properties of the maximal spectral measure and the construction of the fundamental domain B_A we have $\mu_v(\partial B_0) = 0$. Here and below, we use the topology on $A{\bullet}t_0$ for which the bijective orbit map $A \ni a \mapsto a{\bullet}t_0$ is a homeomorphism. Now let $\varepsilon > 0$. Then there exists a compact subset $K \subseteq B_0$ contained in the interior B_0^o of B_0 with $\mu_v(B_0\smallsetminus K) < \varepsilon^2$. Let $U \subseteq A$ be a neighbourhood of the identity so that $a{\bullet}K \subseteq B_0$ and $\|\pi_a v - v\| < \varepsilon$ for all $a \in U$. We now decompose v into the sum

$$v = \Pi_K v + \Pi_{B_0\smallsetminus K}v$$

and note that

$$\|\Pi_{B_0\smallsetminus K}v\| = \mu_v(B_0\smallsetminus K)^{\frac{1}{2}} < \varepsilon.$$

We claim that

$$\rho_a\Pi_K v = \pi_a\Pi_K v, \tag{3.10}$$

which implies

$$\|\rho_a v - v\| \leqslant \|\rho_a\Pi_K v - \Pi_K v\| + 2\varepsilon = \|\pi_a\Pi_K v - \Pi_K v\| + 2\varepsilon < 3\varepsilon$$

for all $a \in U$.

To see (3.10), we first note that by the conjugacy formula (3.2) for a and K we have

$$\Pi_{B_0}\pi_a\Pi_K v = \Pi_{B_0\cap a{\bullet}K}\pi_a v = \Pi_{a{\bullet}K}\pi_a v = \pi_a\Pi_K v.$$

However this implies, by unitarity of ρ_a, that the other terms in the orthogonal sum (3.6) defining $\rho_a v$ must vanish (which can also be checked directly using the conjugacy formula), and therefore (3.10) as claimed. This implies continuity of the representation, and hence ρ is a unitary representation of $K = A/\Lambda$ on $\mathcal{H}_\rho = \Pi_{B_0}\mathcal{H}_\pi$.

Finding a fixed vector for K in $\mathcal{H}_\rho$. We claim that there exists a non-zero vector $v_0 \in \mathcal{H}_\rho$ with $\rho_k v_0 = v_0$ for all $k \in K$. We will assume in addition that v_0 is normalized so that $\|v_0\|^2 = m_A(B_A)$.

Towards the claim, we first note that, since K is a compact abelian group, there exists a character $\chi \in \widehat{K} = \Lambda^\perp < \widehat{A}$ and a non-zero vector $v \in \mathcal{H}_\rho$ with the property that $\rho_k v = \chi(k)v$. Once more using the bijective property of the orbit map $A \ni a \mapsto a\bullet t_0 \in A\bullet t_0$, we can extend χ to a bounded function on $\widehat{H}$. In fact we define $F \in \mathscr{L}^\infty(\widehat{H})$ by

$$F(t) = \begin{cases} 1 & \text{if } t \notin A\bullet t_0, \\ \chi(a) & \text{if } t = a\bullet t_0 \text{ for } a \in A. \end{cases}$$

Using the functional calculus for $\pi|_H$ we see that $\pi_{\mathrm{FC}}(F)$ is a unitary operator, and define $v_0 = \pi_{\mathrm{FC}}(F)v$. With this definition, we now obtain for any $a \in A$ that

$$\begin{aligned} \rho_a v_0 &= \sum_{\lambda\in\Lambda} \pi_\lambda^{-1} \Pi_{\lambda\bullet B_0} \underbrace{\pi_a \pi_{\mathrm{FC}}(F)}_{=\pi_{\mathrm{FC}}(F\circ\widehat{\theta}_a)\pi_a} v \\ &= \sum_{\lambda\in\Lambda} \pi_\lambda^{-1} \pi_{\mathrm{FC}}(F\circ\widehat{\theta}_a) \Pi_{\lambda\bullet B_0} \pi_a v, \end{aligned}$$

where we used Corollary 3.2 and the fact that the functional calculus defines commuting operators. Next we recall that $\chi \in \Lambda^\perp$, which implies that

$$F \circ \widehat{\theta}_a \circ \widehat{\theta}_\lambda^{-1} = F \circ \widehat{\theta}_\lambda^{-1} \circ \widehat{\theta}_a = F \circ \widehat{\theta}_a$$

and, together with Corollary 3.2 and the above, also

$$\rho_a v_0 = \sum_{\lambda\in\Lambda} \pi_{\mathrm{FC}}(F\circ\widehat{\theta}_a)\pi_\lambda^{-1}\Pi_{\lambda\bullet B_0}\pi_a v = \pi_{\mathrm{FC}}(F\circ\widehat{\theta}_a)\rho_a v$$

for all $a \in A$. Finally, we calculate for $t = a'\bullet t_0 \in A\bullet t_0$ that

$$F \circ \widehat{\theta}_a(t) = F(a^{-1}a'\bullet t_0) = \chi(a^{-1}a') = \overline{\chi(a)}F(t).$$

Together with $\rho_a v = \chi(a)v$, this gives $\rho_a v_0 = v_0$ for all $a \in A$ as claimed.

Constructing the Haar measure on A from v_0. We claim that the fixed vector $v_0 \in \mathcal{H}_\rho$ for $K = A/\Lambda$ has spectral measure for $\pi|_H$ given by $\mu|_{B_0}$, where μ is the push-forward to $A\bullet t_0$ of the Haar measure m_A on A. To see this, we will instead use v_0 to construct a Haar measure on A, which by the uniqueness of the Haar measure up to scalar multiples and the normalization $\|v_0\|^2 = m_A(B_A)$ will have to agree with m_A.

In fact, we note that

$$\pi_\eta v_0 \in \pi_\eta \mathcal{H}_\rho = \Pi_{\eta\bullet B_0}\mathcal{H}_\pi$$

for all $\eta \in \Lambda$, and define the measure ν on $\widehat{H}$ by

$$\nu(B) = \sum_{\eta \in \Lambda} \underbrace{\mu_{\pi_\eta v_0}(B \cap \eta \bullet B_0)}_{=\mu_{\pi_\eta v_0}(B)} = \sum_{\eta \in \Lambda} \|\Pi_B \pi_\eta v_0\|^2$$

for all measurable $B \subseteq \widehat{H}$. For $\lambda \in \Lambda$ we then have

$$\nu(\lambda \bullet B) = \sum_{\eta \in \Lambda} \|\Pi_{\lambda \bullet B} \pi_\eta v_0\|^2 = \sum_{\eta \in \Lambda} \underbrace{\|\pi_\lambda \Pi_B \pi_{\lambda^{-1}\eta} v_0\|^2}_{=\|\Pi_B \pi_{\eta'} v_0\|^2} = \nu(B)$$

by the conjugacy formula (3.2) for λ and B and the substitution $\eta' = \lambda^{-1}\eta$. This shows the invariance of ν under Λ.

To see invariance of ν under an arbitrary $a \in A$ we again let $B \subseteq \widehat{H}$ be measurable and will show that $\nu(a \bullet B) = \nu(B)$. Due to the established Λ-invariance and the fact that $\nu(\widehat{H} \smallsetminus A \bullet t_0) = 0$ we may assume $B \subseteq A \bullet t_0$, decompose B into the disjoint union

$$B = \bigsqcup_{\lambda \in \Lambda} (B \cap \lambda \bullet B_0),$$

and reduce to the case of a measurable subset $B \subseteq B_0$. Using $\rho_a v_0 = v_0$ we see that

$$\nu(B) = \mu_{v_0}(B) = \|\Pi_B v_0\|^2 = \|\Pi_B \rho_a v_0\|^2 = \Big\| \sum_{\lambda \in \Lambda} \underbrace{\Pi_{B \cap B_0}}_{=\Pi_B} \pi_\lambda^{-1} \pi_a v_0 \Big\|^2.$$

We recall from the discussion leading to (3.9) that the terms

$$\Pi_{B_0} \pi_\lambda^{-1} \pi_a v_0 = w_\lambda$$

in the definition (3.6) of $\rho_a v_0$ are mutually orthogonal, since

$$\Pi_{\lambda^{-1} a \bullet B_0} w_\lambda = w_\lambda$$

by (3.8), and the sets $\lambda^{-1} a \bullet B_0$ for $\lambda \in \Lambda$ are mutually disjoint. Since the projection operator Π_B commutes with the projection operator $\Pi_{\lambda^{-1} a \bullet B_0}$ for $\lambda \in \Lambda$, it follows that the terms in the sum above are still orthogonal. Therefore, we can take the sum out, apply the conjugacy formula (3.2) for a and $a^{-1} \bullet B$ and the substitution $\eta = \lambda^{-1}$ to obtain

$$\begin{aligned} \nu(B) &= \sum_{\eta \in \Lambda} \| \underbrace{\Pi_B \pi_a}_{=\pi_a \Pi_{a^{-1} \bullet B}} \pi_\eta v_0 \|^2 \\ &= \sum_{\eta \in \Lambda} \|\Pi_{a^{-1} \bullet B} \pi_\eta v_0\|^2 = \nu(a^{-1} \bullet B), \end{aligned}$$

as required.

Using the bijective orbit map $A \ni a \mapsto a\centerdot t_0 \in A\centerdot t_0$, we can move ν to a measure ν_A on A, which by the argument above is translation invariant. Moreover, the fundamental domain B_A has measure

$$\nu_A(B_A) = \nu(B_0) = \|v_0\|^2 = m_A(B_A) < \infty.$$

As $B_A \subseteq A$ has non-empty interior, it follows that any compact subset $M \subseteq A$ can be covered by finitely many Λ-translates of the fundamental domain B_A. This implies $\nu_A(M) < \infty$ and as $M \subseteq A$ was an arbitrary compact subset we see that ν_A is locally finite. Therefore, ν_A is a Haar measure on A satisfying $\nu_A(B_A) = m_A(B_A)$ and hence $\nu_A = m_A$. As μ is defined as the push-forward of m_A under the orbit map, we see that $\nu = \mu$. Therefore we deduce that $\mu_{v_0}(B) = \nu(B) = \mu(B)$ for any measurable $B \subseteq B_0$. As

$$\mu_{v_0}\big(\widehat{H} \smallsetminus B_0\big) = \big\|\Pi_{\widehat{H}\smallsetminus B_0} v_0\big\|^2 = \big\|\Pi_{\widehat{H}\smallsetminus B_0}\Pi_{B_0} v_0\big\|^2 = 0,$$

this shows that $\mu_{v_0} = \mu|_{B_0}$ as claimed.

Defining the isomorphism (on a dense subspace). We will now construct the equivariant unitary isomorphism U between $\mathcal{H}_\pi$ and $L^2_\mu(\widehat{H})$. For this it suffices to define U on a dense π-invariant subspace $\mathcal{V} < \mathcal{H}_\pi$.

To define $\mathcal{V}$ we again use v_0 and the decomposition of $\mathcal{H}_\pi$ in (3.5). In fact we define

$$\mathcal{V}_0 = \pi_{\mathrm{FC}}\big(\mathscr{L}^\infty(\widehat{H})\big)v_0 = \pi_{\mathrm{FC}}\big(\mathscr{L}^\infty(\widehat{H})\mathbb{1}_{B_0}\big)v_0 \subseteq \mathcal{H}_\rho,$$

and obtain $\mathcal{V}$ by applying π_λ for $\lambda \in \Lambda$ and taking the linear hull

$$\mathcal{V} = \sum_{\lambda\in\Lambda} \pi_\lambda \mathcal{V}_0$$

of the resulting subspaces. We note that

$$\pi_\lambda\mathcal{V}_0 = \pi_\lambda\pi_{\mathrm{FC}}\big(\mathscr{L}^\infty(\widehat{H})\big)v_0 = \pi_{\mathrm{FC}}\big(\mathscr{L}^\infty(\widehat{H})\big)\pi_\lambda v_0$$

for any $\lambda \in \Lambda$ by Corollary 3.2 and the conjugacy formula (3.2). We define $\mathscr{L}^\infty_c(A\centerdot t_0)$ to consist of all bounded measurable functions on $\widehat{H}$ that vanish outside of $A\centerdot t_0$ and outside of finitely many of the translates $\lambda\centerdot B_0$ for $\lambda \in \Lambda$. Using this notation, we can also define $\mathcal{V}$ equivalently by

$$\mathcal{V} = \left\{ v_F = \sum_{\lambda\in\Lambda} \pi_{\mathrm{FC}}(F)\pi_\lambda v_0 \;\middle|\; F \in \mathscr{L}^\infty_c(A\centerdot t_0) \right\}.$$

We wish to define $U\colon \mathcal{V} \to L^2_\mu(A\centerdot t_0)$ by sending v_F for $F \in \mathscr{L}^\infty_c(A\centerdot t_0)$ to $F \in L^2_\mu(A\centerdot t_0)$. To see that this gives a well-defined isometry we calculate

$$\begin{aligned}
\|v_F\|^2 &= \sum_{\lambda\in\Lambda} \|\pi_{\mathrm{FC}}(F)\pi_\lambda v_0\|^2 \\
&= \sum_{\lambda\in\Lambda} \int |F|^2 \,\mathrm{d}\mu_{\pi_\lambda v_0} \\
&= \sum_{\lambda\in\Lambda} \int |F|^2 \,\mathrm{d}\lambda_*\mu_{v_0} \\
&= \sum_{\lambda\in\Lambda} \int |F|^2 \,\mathrm{d}\lambda_*\mu|_{B_0} \\
&= \sum_{\lambda\in\Lambda} \int |F|^2 \,\mathrm{d}\mu|_{\lambda\text{.}B_0} = \|F\|^2_{L^2(\widehat{H},\mu)}
\end{aligned}$$

by using in turn the definition of $v_F \in \mathcal{V}$, the properties of the spectral measures, Proposition 3.1, the previous claim that $\mu_{v_0} = \mu|_{B_0}$, and the invariance of μ under the action of $\Lambda < A$. In other words, $U\colon \mathcal{V} \to L^2_\mu(\widehat{H})$ is indeed an isometry.

As $\mathcal{V}$ is defined by an arbitrary function F in $\mathscr{L}^\infty_c(A\text{•}t_0)$, which is an ideal in $\mathscr{L}^\infty(\widehat{H})$, and the unitary representation $\pi|_H$ can be recovered from its functional calculus, we see that $\mathcal{V}$ is invariant under $\pi|_H$. Moreover, for $h \in H$ and $F \in \mathscr{L}^\infty_c(A\text{•}t_0)$ we have

$$U(\pi_h v_F) = U\Bigl(\sum_{\lambda\in\Lambda} \pi_{\mathrm{FC}}(M_h F)\pi_\lambda v_0\Bigr) = M_h F = \pi_\mu(h)F,$$

which shows that U is equivariant for $\pi|_H$.

From the definition of $\mathcal{V}$ it is also clear that $\mathcal{V}$ is invariant under $\pi|_\Lambda$. Moreover, for $\lambda_0 \in \Lambda$ and $F \in \mathscr{L}^\infty_c(A\text{•}t_0)$ we have by Corollary 3.2 that

$$\begin{aligned}
U(\pi_{\lambda_0} v_F) &= U\Bigl(\pi_{\lambda_0} \sum_{\lambda\in\Lambda} \pi_{\mathrm{FC}}(F)\pi_\lambda v_0\Bigr) \\
&= U\Bigl(\sum_{\lambda\in\Lambda} \pi_{\mathrm{FC}}\bigl(F \circ \widehat{\theta}_{\lambda_0}\bigr)\pi_{\lambda_0\lambda} v_0\Bigr) \\
&= F \circ \widehat{\theta}_{\lambda_0} = \pi_\mu(\lambda_0)F,
\end{aligned}$$

which shows that U is equivariant for $\pi|_\Lambda$.

To see the invariance of $\mathcal{V}$ and the equivariance of U under any $a \in A$, we will again use $\rho_a v_0 = v_0$. Indeed, using the Λ-invariance and Λ-equivariance, we restrict ourselves to considering $F \in \mathscr{L}^\infty(B_0) \subseteq \mathscr{L}^\infty_c(A\text{•}t_0)$ and $a \in A$. Then

$$\begin{aligned}\pi_a v_F &= \pi_a \pi_{\mathrm{FC}}(F) v_0\\ &= \pi_{\mathrm{FC}}\bigl(F \circ \widehat{\theta}_a\bigr)\pi_a \rho_a^{-1} v_0\\ &= \sum_{\lambda\in\Lambda} \pi_{\mathrm{FC}}\bigl(F\circ\widehat{\theta}_a\bigr) \underbrace{\pi_a \Pi_{B_0}}_{=\Pi_{a\cdot B_0}\pi_a} \pi_\lambda^{-1}\pi_a^{-1} v_0\\ &= \sum_{\eta\in\Lambda} \pi_{\mathrm{FC}}\bigl(\underbrace{F\circ\widehat{\theta}_a \mathbb{1}_{a\cdot B_0}}_{=F\circ\widehat{\theta}_a}\bigr)\pi_\eta v_0 = v_{F\circ\widehat{\theta}_a} \in \mathcal{V}\end{aligned}$$

by the definition of v_F, the assumption $F \in \mathscr{L}^\infty(B_0)$, Corollary 3.2, the fact that $\rho_a v_0 = v_0$, the definition of ρ_a in (3.6), the conjugacy formula (3.2) for a and B_0, the substitution $\lambda = \eta^{-1}$, and the assumption that $F \in \mathscr{L}^\infty(B_0)$ once more. Applying U now gives again

$$U(\pi_a v_F) = F\circ\widehat{\theta}_a = \pi_a^\mu F.$$

It follows that $\mathcal{V}$ is π-invariant and by irreducibility that $\overline{\mathcal{V}} = \mathcal{H}_\pi$. The equivariant isometric embedding $U\colon \mathcal{V} \to L^2_\mu(\widehat{H})$ now extends uniquely to an equivariant isomorphism between π and π^μ. □

We want to mention that the construction of an irreducible unitary representation arising from an orbit $A{\cdot}t_0$ with trivial centralizer as considered above is a special case of the induced representation. Starting with the unitary representation χ_{t_0} of H corresponding to $t_0 \in \widehat{H}$, the representation π^μ corresponding to the Haar measure μ on the orbit $A{\cdot}t_0$ in the sense of Lemma 3.5 is therefore also denoted by

$$\pi^\mu = \operatorname{Ind}_H^G \chi_{t_0}.$$

We will not pursue this now, but will encounter the notion of induced representation again in Section 8.5.

We will upgrade Theorem 3.33 to a complete classification of unitary duals of certain semi-direct product groups in Section 3.5.

Exercise 3.22 (Other fields). Let $\mathbb{K}$ be a local field (that is, $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$ for some prime p, or a finite extension of $\mathbb{F}_p(\!(t)\!)$ for some prime p). Extend the results of Section 3.3.2 to describe the relationship between the unitary dual of $\mathbb{K}^\times$ and of the affine group

$$G = \left\{ \begin{pmatrix} a & b\\ 0 & 1\end{pmatrix} \;\middle|\; a \in \mathbb{K}^\times, b \in \mathbb{K}\right\}.$$

Exercise 3.23 (The group Sol). Define the group $\mathrm{Sol} = \mathbb{R} \ltimes \mathbb{R}^2$ as the matrix group

$$\mathrm{Sol} = \left\{ \begin{pmatrix} a & 0 & x_1\\ 0 & a^{-1} & x_2\\ 0 & 0 & 1\end{pmatrix} \;\middle|\; a > 0, x = \begin{pmatrix} x_1\\ x_2\end{pmatrix} \in \mathbb{R}^2\right\}.$$

Describe $\widehat{\mathrm{Sol}}$, and prove that your description is complete.

Exercise 3.24. Prove Theorem 3.20, replacing the assumption 'A has a discrete co-compact subgroup' by 'A has a compact open subgroup'.

3.3.4 The Heisenberg Group

We recall that the 3-dimensional Heisenberg group G is defined by

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

and that the multiplication is given by

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+z+ay \\ & 1 & b+y \\ & & 1 \end{pmatrix}$$

for all

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \in G.$$

More generally, for any $d \geqslant 1$ the $(2d+1)$-dimensional Heisenberg group is defined as the set $G = \mathbb{R}^{2d+1}$ whose elements we denote by

$$(a, b, c), (x, y, z) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R},$$

equipped with the multiplication

$$(a, b, c) \cdot (x, y, z) = (a + x, b + y, c + z + \langle a, y \rangle).$$

This coincides with matrix multiplication if (x, y, z) is identified with the matrix

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ & 1 & 0 & \cdots & 0 & y_1 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & 1 & 0 & y_{n-1} \\ & & & & 1 & y_n \\ & & & & & 1 \end{pmatrix} \in \mathrm{SL}_{n+2}(\mathbb{R}).$$

We note that the centre of G is given by the subgroup

$$C(G) = \{(0, 0, z) \mid z \in \mathbb{R}\}.$$

The Irreducible Representations

(Old) We say that a unitary representation is an old representation of G if the centre $C(G)$ acts trivially. Since $G/C(G) \cong \mathbb{R}^{2d}$ is abelian, it follows that any old irreducible representation of G is given by a character of

$$G/C(G) \cong \mathbb{R}^{2d}.$$

(New) We say that a unitary representation is a new representation of G if the centre $C(G)$ acts non-trivially. To construct the irreducible new representations of G, we define the normal subgroup

$$H = \{(0, y, z) \mid y \in \mathbb{R}^d, z \in \mathbb{R}\} \lhd G$$

corresponding to the 'last column' of the matrix group, and the 'complementary group'

$$A = \{(a, 0, 0) \mid a \in \mathbb{R}^d\} < G$$

so that once again $G = AH \cong A \ltimes H$.

To apply the results of Section 3.1 we calculate the effect of the automorphism $\theta_a = \theta_{(a,0,0)}$ of H for all $a \in \mathbb{R}^d$. Indeed, for $(0, y, z) \in H$ we have

$$\begin{aligned} \theta_a\big((0, y, z)\big) &= (a, 0, 0)(0, y, z)(-a, 0, 0) \\ &= (a, y, z + \langle a, y\rangle)(-a, 0, 0) \\ &= (0, y, z + \langle a, y\rangle). \end{aligned}$$

Using the standard basis of $H \cong \mathbb{R}^{d+1}$ with the centre corresponding to the last coordinate, the automorphism θ_a has the matrix representation

$$\begin{pmatrix} I_d & 0 \\ a & 1 \end{pmatrix},$$

so the dual automorphism $\widehat{\theta}_a$ on $\widehat{H} \cong \mathbb{R}^{d+1}$ has the matrix representation

$$\begin{pmatrix} I_d & a^{\mathrm{t}} \\ 0 & 1 \end{pmatrix}.$$

Now let $\xi \in \mathbb{R}^\times$ and define

$$t_0 = \begin{pmatrix} 0 \\ \xi \end{pmatrix} \in \mathbb{R}^{d+1} \cong \widehat{H}.$$

Then the A-orbit of t_0 is given by

$$A{\bullet}t_0 = \left\{ \begin{pmatrix} I_d & a^{\mathrm{t}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \xi \end{pmatrix} \;\middle|\; a \in \mathbb{R}^d \right\} = \mathbb{R}^d \times \{\xi\} \tag{3.11}$$

and gives rise via Lemmas 3.5 and 3.6 to an irreducible representation π^ξ of the Heisenberg group.

Classification of the Unitary Dual

Theorem 3.25 (Stone–von Neumann[(6)]). *Let $d \geqslant 1$. The unitary dual $\widehat{G}$ of the $(2d+1)$-dimensional Heisenberg group consists of the old representations given by unitary characters on $G/C(G) \cong \mathbb{R}^{2d}$ and the infinite-dimensional new irreducible representations π^ξ for $\xi \in \mathbb{R}^\times$. The representation π^ξ for $\xi \in \mathbb{R}^\times$ can also be defined for $(x,y,z) \in G$ by*

$$\bigl(\pi^{\xi}_{(x,y,z)} f\bigr)(t) = \mathrm{e}^{2\pi\mathrm{i}(\langle y,t\rangle + z\xi)} f(t+\xi x) \tag{3.12}$$

for $f \in L^2(\mathbb{R}^d)$ and $t \in \mathbb{R}^d$.

Proof. By the discussion above (relying on Lemmas 3.5 and 3.6), we already constructed the new representations and know that they are irreducible. Moreover, it is clear that different $\xi \in \mathbb{R}^\times$ give rise to inequivalent irreducible representations as ξ is uniquely determined as being the character obtained by restriction to the centre $C(G)$.

Let π now be an arbitrary irreducible unitary representation of G. If $\pi|_{C(G)}$ is trivial, then π induces a representation of $G/C(G) \cong \mathbb{R}^{2d}$ and so it is given by a character of $\mathbb{R}^{2d}$. Hence we may now assume that $\pi|_{C(G)}$ is non-trivial. By Corollary 1.32, we have $\pi|_{C(G)} = \chi I$ for a unitary character χ on $C(G)$. Since $C(G) \cong \mathbb{R}$, we see that χ is represented by a real number $\xi \in \mathbb{R}^\times$. We define the abelian subgroup $H \lhd G$ as above, and consider $\pi|_H$. Since the subgroup $C(G) \subseteq H$ corresponds to the last co-ordinate in $H \cong \mathbb{R}^{d+1}$, we see that the maximal spectral measure $\mu_{\max}$ for $\pi|_H$ satisfies

$$\mu_{\max}\bigl(\widehat{H} \smallsetminus (\mathbb{R}^d \times \{\xi\})\bigr) = 0.$$

However, (3.11) shows that $\mathbb{R}^d \times \{\xi\} = A \boldsymbol{\cdot} t_0$ is a single orbit of the complementary subgroup $A < G$ as above. Moreover, the stabilizer of t_0 in A is trivial. Thus Theorem 3.20 implies that π is isomorphic to π^μ for the A-invariant measure μ on the orbit $A \boldsymbol{\cdot} t_0$.

We now identify the A-orbit $A \boldsymbol{\cdot} t_0 = \mathbb{R}^d \times \{\xi\}$ with $\mathbb{R}^d$ by sending the element $(t,\xi) \in A \boldsymbol{\cdot} t_0$ to t. This implies the formula (3.12) for $x = 0$. The unitary operator $\pi^\mu_{(x,0,0)}$ for $(x,0,0) \in A$ is defined by

$$\bigl(\pi^{\mu}_{(x,0,0)} f\bigr)(t,\xi) = f\bigl(\widehat{\theta}_x(t,\xi)\bigr) = f\bigl((t+x\xi,\xi)\bigr)$$

for $f \in L^2_\mu(\widehat{H})$ and $(t,\xi) \in A \boldsymbol{\cdot} t_0$. Under the isomorphism to $L^2(\mathbb{R}^d)$ this becomes the formula (3.12) for $(y,z) = (0,0)$. Putting these together gives the description of π^ξ in the theorem. □

Exercise 3.26 (Other fields). Let $\mathbb{K}$ be a local field as in Exercise 3.22 and let $d \geqslant 1$. Extend Theorem 3.25 to describe the unitary dual $\widehat{G}$ of the $(2d+1)$-dimensional Heisenberg group G over $\mathbb{K}$ (which is defined by the same formulas, using elements in $\mathbb{K}^{2d+1}$).

3.4 A Plancherel-type Theorem*

We wish to show here that it is quite straightforward to derive a Plancherel type theorem for the concrete groups considered in Section 3.3.(7) Once again we will not try to give the most general result of this type, but instead try to give a convenient framework able to handle the isometry group in Section 3.3.1, the affine group in Section 3.3.2, and the Heisenberg groups in Section 3.3.4.

We continue to assume that $G = A \ltimes H$ is the semi-direct product of a closed normal abelian subgroup $H \lhd G$ and a closed abelian subgroup $A < G$. We assume in addition the following structure for the action of $A < G$ on $\widehat{H}$ introduced in Section 3.1:

- There exists a measurable cross-section $S \subseteq \widehat{H}$ such that

$$A \times S \ni (a, s) \longmapsto a{\bullet}s \in A{\bullet}S \subseteq \widehat{H}$$

  is a measurable isomorphism, and $m_{\widehat{H}}\big(\widehat{H} \smallsetminus A{\bullet}S\big) = 0$.

Exercise 3.27. Describe a choice of S for the isometry group in Section 3.3.1, the affine group in Section 3.3.2, the group Sol from Exercise 3.23, and the Heisenberg group in Section 3.3.4.

For every $s \in S$ we use the isomorphism

$$A \ni a \longmapsto t = a{\bullet}s \in A{\bullet}s \subseteq \widehat{H}$$

to push the Haar measure m_A to an A-invariant measure μ_s on $A{\bullet}s$, which then defines an irreducible unitary representation of G by Lemmas 3.5 and 3.6. For the following it will be more convenient to instead define this representation on $L^2(A)$ by the formulas

$$\begin{aligned}\big(\pi^s_h f\big)(a) &= \langle h, a{\bullet}s\rangle f(a),\\ \big(\pi^s_{a_0} f\big)(a) &= f(a_0^{-1}a)\end{aligned}$$

for all $h \in H$, $a, a_0 \in A$, and $f \in L^2(A)$.

Exercise 3.28. Show that $\pi^s \cong \pi^{\mu_s}$.

Proposition 3.29 (Plancherel). *Let the semi-direct product $G = A \ltimes H$ and the cross-section $S \subseteq \widehat{H}$ be as above. We let $X = S \times A$ and write $s(\cdot)$ for the projection $X \to S$ onto the first coordinate. Then there exists a σ-finite measure ν on X and an equivariant unitary isomorphism between the regular representation λ of G on $L^2(G)$ and the unitary representation π^X defined on $L^2(A \times X, m_A \times \nu)$ by*

$$\begin{aligned}\big(\pi^X_h f\big)(a, x) &= \langle h, a{\bullet}s(x)\rangle f(a, x),\\ \big(\pi^X_{a_0} f\big)(a, x) &= f(a_0^{-1}a, x)\end{aligned}$$

for all $h \in H$, $a, a_0 \in A$, and $f \in L^2_\nu(X)$.

We note that the description above should be thought of as an integral decomposition

$$\lambda^G \cong \int_X \pi^{s(x)} \, \mathrm{d}\nu(x)$$

of the regular representation λ^G into irreducible representations π^s for $s \in S$, where ν plays the role of the spectral measure and the second factor A in X creates multiplicity in the decomposition.

Proof of Proposition 3.29. Let m_G be the left Haar measure on G. Using the coordinate system $ah \in G$ for $a \in A$ and $h \in H$, we may normalize the Haar measures so that $m_G \cong m_A \times m_H$ in these coordinates (see Exercise 3.30).

Using the fact that m_G is a product measure, we can apply the Plancherel theorem (Theorem 2.15) for a given $f \in L^2(G)$ and m_A-almost every $a \in A$ and $m_{\widehat{H}}$-almost every $t \in \widehat{H}$ to define the partial transform $f \mapsto \widetilde{f}$ by

$$\widetilde{f}(a, t) = \int f(ah)\langle h, t\rangle \, \mathrm{d}m_H(h), \tag{3.13}$$

which satisfies

$$\|\widetilde{f}\|_{L^2(A\times\widehat{H}, m_A\times m_{\widehat{H}})} = \|f\|_{L^2(G)}$$

(see Exercise 3.31). Recalling that $m_{\widehat{H}}\big(\widehat{H}\smallsetminus A{\scriptstyle\bullet}S\big) = 0$ and

$$A \times S \ni (a, s) \longmapsto a{\scriptstyle\bullet}s \in A{\scriptstyle\bullet}S$$

is a bijection, we restrict in the following to elements $t \in \widehat{H}$ of the form $t = a{\scriptstyle\bullet}s$ with $a \in A$ and $s \in S$. For $a_1, a_2 \in A$ and $s \in S$ we set

$$U(f)(a_1, s, a_2) = \widetilde{f}(a_1a_2^{-1}, a_2{\scriptstyle\bullet}s)$$

where this is defined. For $h_0 \in H$ we then have

$$\begin{aligned}
U(\lambda^G_{h_0}f)(a_1, s, a_2) &= \widetilde{(\lambda^G_{h_0}f)}(a_1a_2^{-1}, a_2{\scriptstyle\bullet}s)\\
&= \int f(h_0^{-1}a_1a_2^{-1}h)\langle h, a_2{\scriptstyle\bullet}s\rangle \, \mathrm{d}m_H(h)\\
&= \int f(a_1a_2^{-1}\underbrace{\theta_{a_1^{-1}a_2}(h_0^{-1})h}_{=h'})\langle h, a_2{\scriptstyle\bullet}s\rangle \, \mathrm{d}m_H(h)\\
&= \int f(a_1a_2^{-1}h')\langle\theta_{a_1^{-1}a_2}(h_0)h', a_2{\scriptstyle\bullet}s\rangle \, \mathrm{d}m_H(h')\\
&= \langle h_0, a_1{\scriptstyle\bullet}s\rangle \int f(a_1a_2^{-1}h)\langle h, a_2{\scriptstyle\bullet}s\rangle \, \mathrm{d}m_H(h)\\
&= \langle h_0, a_1{\scriptstyle\bullet}s\rangle U(f)(a_1, s, a_2)
\end{aligned}$$

and for $a_0 \in A$ we have

$$\begin{aligned}
U(\lambda^G_{a_0} f)(a_1, s, a_2) &= \widetilde{\lambda^G_{a_0} f}(a_1 a_2^{-1}, a_2 \centerdot s) \\
&= \int f(a_0^{-1} a_1 a_2^{-1} h) \langle h, a_2 \centerdot s \rangle \, \mathrm{d}m_H(h) \\
&= U(f)(a_0^{-1} a_1, s, a_2)
\end{aligned}$$

where defined. Now let $X = S \times A$, define ν to be the pull-back of the Haar measure $m_{\widehat{H}}$ to $S \times A$, and obtain

$$\begin{aligned}
\big\|U(f)\big\|^2_{L^2(A \times S, m_A \times \nu)} &= \int \big|\widetilde{f}(a_1 a_2^{-1}, a_2 \centerdot s)\big|^2 \, \mathrm{d}m_A(a_1) \, \mathrm{d}\nu(s, a_2) \\
&= \int \big|\widetilde{f}(a, a_2 \centerdot s)\big|^2 \, \mathrm{d}m_A(a) \, \mathrm{d}\nu(s, a_2) \\
&= \big\|\widetilde{f}\big\|^2_{L^2(A \times \widehat{H}, m_A \times m_{\widehat{H}})} = \|f\|^2_{L^2(G)}
\end{aligned}$$

for any $f \in L^2(G)$, as required. □

Exercise 3.30. Verify that $m_G = m_A \times m_H$ defines a left Haar measure on $G = A \ltimes H$.

Exercise 3.31. Verify that (3.13) defines a unitary isomorphism.

3.5 Metabelian Groups with Countable Separation of Orbits*

In this section we summarize the framework we have been working with. Moreover, we state and prove a description of the unitary dual of semi-direct products of abelian groups under a necessary technical conditions that will help us to avoid the existence of ergodic measures other than Haar measures on orbits.

We assume once more that:

- G is a locally compact, σ-compact, metric group;
- $H \lhd G$ is a closed normal abelian subgroup;
- $A < G$ is a closed abelian subgroup; and
- the map $H \times A \ni (h, a) \mapsto ha \in G$ is a homeomorphism of topological spaces.

In other words, we assume that the semi-direct product $G = A \ltimes H$ of two abelian groups satisfies our standing assumptions. Moreover, we assume that:

- A and each of its quotients by closed subgroups has compact quotients by discrete subgroups.

As we have seen in many examples before, we have two ways of constructing irreducible unitary representations of G. Indeed, since $G/H \cong A$, we can lift a character $\chi \in \widehat{A}$ to a character $\chi \in \widehat{G}$. On the other hand, we can use any A-invariant and ergodic measure μ on $\widehat{H}$ as in Section 3.2 to define the irreducible unitary representation π^μ of G (see Lemmas 3.5 and 3.6). The correct way to combine these is to allow tensor products $\chi \otimes \pi^\mu$ defined by

$$\chi \otimes \pi^\mu(ha) = \chi(a)\pi^\mu(ha)$$

for $ha \in G$ on $\mathcal{H}_{\chi\otimes\pi^\mu} = L^2_\mu(\widehat{H})$ as in Lemma 1.28.

We now define the necessary technical condition that will help us to avoid ergodic measures on $\widehat{H}$ other than Haar measures on orbits.

Definition 3.32. Let A be a group and X a locally compact, σ-compact, metric space carrying an action of A. We say that the action has *countable separation of orbits* if the σ-algebra[†]

$$\mathcal{B}_X^A = \{B \in \mathcal{B}_X \mid a{\bullet}B = B \text{ for all } a \in A\}$$

is countably generated.

Theorem 3.33 (Semi-direct products with countable separation). *Let $G = A \ltimes H$ be as above. Then every character χ of A and every $t_0 \in \widehat{H}$ gives rise to an irreducible unitary representation $\chi\otimes\pi^\mu$, where μ is the Haar measure on the orbit $A{\bullet}t_0 \subseteq \widehat{H}$. Assume now in addition that the A-action on $\widehat{H}$ has countable separation of orbits. Then any irreducible unitary representation of G is isomorphic to a representation of this form. Finally, two such representations $\chi_1 \otimes \pi^{\mu_1}$ and $\chi_2 \otimes \pi^{\mu_2}$ for $\chi_1, \chi_2 \in \widehat{A}$ and Haar measures μ_1, μ_2 on A-orbits on $\widehat{H}$ are isomorphic if and only if the following two conditions hold true:*

- *μ_1 and μ_2 are multiples of each other, and hence are Haar measures on the same orbit $A{\bullet}t_0$, and*
- *the characters agree when restricted to the stabilizer subgroup*

$$\mathrm{Stab}_A(t_0) = \{a \in A \mid a{\bullet}t_0 = t_0\};$$

in symbols $\chi_1|_{\mathrm{Stab}_A(t_0)} = \chi_2|_{\mathrm{Stab}_A(t_0)}$.

Proof. Let $t_0 \in \widehat{H}$ and $\chi \in \widehat{A}$. We define μ as the push-forward of the Haar measure on $A/\mathrm{Stab}_A(t_0)$ under the orbit map

[†] We note that in ergodic theory this σ-algebra is also important, and is often replaced by a countably generated σ-algebra that is equivalent to $\mathcal{B}_X^A$ modulo an A-invariant measure. As we do not have a preferred measure on $\widehat{H}$ and we wish to use $\mathcal{B}_X^A$ as a tool to exclude ergodic measures other than Haar measures, the definition of $\mathcal{B}_X^A$ has to be taken literally here.

$$a \operatorname{Stab}_A(t_0) \mapsto a{\bullet}t_0 \in X = A{\bullet}t_0,$$

and equip X with the topology on $A/\operatorname{Stab}_A(t_0)$. By [24, Lem. 10.3] or [22, Prop. 8.6], the action of $A/\operatorname{Stab}_A(t_0)$ on $A/\operatorname{Stab}_A(t_0)$ is ergodic with respect to the Haar measure, which implies that μ is A-invariant and ergodic. By Lemmas 3.5 and 3.6 we obtain the irreducible unitary representation π^μ on $G = A \ltimes H$. By Lemma 1.28 it follows that $\chi \otimes \pi^\mu$ is an irreducible unitary representation of G. We also note that the restriction of $\chi \otimes \pi^\mu$ to H agrees with the restriction of π^μ to H, and hence the spectral measure of $v \in L^2_\mu(A{\bullet}t_0)$ is, in either case, given by $|v|^2 \,\mathrm{d}\mu$.

Suppose first that $\mu_1 = \mu_2 = \mu$ is the Haar measure on $X = A{\bullet}t_0$ and

$$\chi_1|_{\operatorname{Stab}_A(t_0)} = \chi_2|_{\operatorname{Stab}_A(t_0)}.$$

We note that $A{\bullet}t_0 \cong A/\operatorname{Stab}_A(t_0)$ and define the well-defined measurable map

$$b(x) = \begin{cases} 1 & \text{for } x \in \widehat{H} \smallsetminus A{\bullet}t_0; \\ \big(\chi_2\overline{\chi_1}\big)(a_x) & \text{for } x = a_x{\bullet}t_0 \in A{\bullet}t_0. \end{cases}$$

This gives

$$\chi_2(a) = b(x)\chi_1(a)b(a^{-1}{\bullet}x)^{-1}$$

for all $a \in A$ and $x \in A{\bullet}t_0$. We use b to define the unitary operator

$$U = M_b \colon L^2_\mu(X) \longrightarrow L^2_\mu(X).$$

It follows now for $v \in L^2_\mu(X)$ that

$$\big(M_b(\chi_1 \otimes \pi^\mu)_a v\big)(x) = b(x)\chi_1(a)v(a^{-1}{\bullet}x)$$

and

$$\big((\chi_2 \otimes \pi^\mu)_a M_b v\big)(x) = \chi_2(a)b(a^{-1}{\bullet}x)v(a^{-1}{\bullet}x)$$

agree for μ-almost every $x \in X$. As M_b commutes with $\pi^\mu|_H$ it follows that M_b gives an equivariant isomorphism between $\chi_1 \otimes \pi^\mu$ and $\chi_2 \otimes \pi^\mu$.

Suppose now that $\chi_1 \otimes \pi^{\mu_1}$ and $\chi_2 \otimes \pi^{\mu_2}$ are isomorphic, where $\chi_1, \chi_2 \in \widehat{A}$ and μ_1, μ_2 are Haar measures on A-orbits inside $\widehat{H}$ as above. The remark above concerning spectral measures implies that the orbits are equal, for otherwise the spectral measures for the restriction to H would be mutually singular. By uniqueness of Haar measures up to scalar multiples, we deduce that μ_1 and μ_2 are proportional. This allows us again to work with a Haar measure $\mu = \mu_1 = \mu_2$ on $A{\bullet}t_0$. Hence we assume that there exists a unitary operator

$$U \colon L^2_\mu(A{\bullet}t_0) \longrightarrow L^2_\mu(A{\bullet}t_0)$$

that is equivariant for $\chi_1 \otimes \pi^\mu$ and $\chi_2 \otimes \pi^\mu$. As the restriction of these agree with the multiplication representation M on $L^2_\mu(A{\bullet}t_0)$, it follows (as a very special case) from Proposition 2.69 that U equals the multiplication operator M_b for some measurable $b\colon \widehat{H} \to \mathbb{S}^1$. For $a \in A$ and $v \in L^2_\mu(A{\bullet}t_0)$, the formula $M_b(\chi_1 \otimes \pi^\mu)_a M_{\overline{b}} = (\chi_2 \otimes \pi^\mu)_a$ shows that

$$b(x)\chi_1(a)\bigl(\pi^\mu_a(\overline{b}v)\bigr)(x) = b(x)\chi_1(a)\overline{b}(a^{-1}{\bullet}x)v(a^{-1}{\bullet}x) = \chi_2(a)v(a^{-1}{\bullet}x)$$

for μ-almost every $x \in A{\bullet}t_0$. As this holds for all $v \in L^2_\mu(A{\bullet}t_0)$ we must have

$$\chi_2(a) = b(x)\chi_1(a)b(a^{-1}{\bullet}x)^{-1} \tag{3.14}$$

for μ-almost every $x \in A{\bullet}t_0$. As A is abelian and μ is the Haar measure on $A{\bullet}t_0$, we know that we have $a^{-1}{\bullet}x = x$ for $a \in \operatorname{Stab}_A(t_0)$ and μ-almost every $x \in \widehat{H}$. Therefore (3.14) implies that $\chi_2(a) = \chi_1(a)$ for all $a \in \operatorname{Stab}_A(t_0)$.

It remains to show that every irreducible unitary representation π of G is isomorphic to a unitary representation of the form $\chi \otimes \pi^\mu$ for a character $\chi \in \widehat{A}$ and the Haar measure μ on a single A-orbit $A{\bullet}t_0$ for some $t_0 \in \widehat{H}$. Here we have to use the assumption of countable separation of orbits for the action of A on $\widehat{H}$.

So let $\mu_{\max}$ be a maximal spectral measure for $\pi|_H$, and let $\mathcal{C} \subseteq \mathcal{B}^A_{\widehat{H}}$ be a countable algebra of A-invariant sets that generates the σ-algebra $\mathcal{B}^A_{\widehat{H}}$ of A-invariant sets on $\widehat{H}$. For any $B \in \mathcal{C}$ we have $a{\bullet}B = B$, and hence by the conjugacy formula (3.2) that Π_B is equivariant for A. As it is defined by the functional calculus for $\pi|_H$, it is also equivariant for H. This implies by Schur's lemma (Theorem 1.29) that Π_B is 0 or I, or equivalently that $\mu_{\max}(B) = 0$ or $\mu_{\max}(\widehat{H} \smallsetminus B) = 0$. Taking the countable intersection

$$X = \bigcap_{\substack{B \in \mathcal{C},\\ \mu_{\max}(\widehat{H} \smallsetminus B) = 0}} B$$

we obtain an A-invariant set with $\mu_{\max}(\widehat{H} \smallsetminus X) = 0$. By the above, we have $X \subseteq B$ or $X \subseteq \widehat{H} \smallsetminus B$ for any $B \in \mathcal{C}$. As $\mathcal{C}$ generates $\mathcal{B}^A_{\widehat{H}}$ by assumption, this also holds for any A-invariant Borel set. For $t_0 \in X$ and $B = A{\bullet}t_0 \in \mathcal{B}^A_{\widehat{H}}$ this implies $X \subseteq B$ and hence $X = A{\bullet}t_0$ by invariance of X.

We define μ as the Haar measure on $A{\bullet}t_0 \cong A/\operatorname{Stab}_A(t_0)$, the closed subgroup

$$T = \overline{\langle A{\bullet}t_0 \rangle}$$

generated by the orbit $X = A{\bullet}t_0 \subseteq \widehat{H}$, and its annihilator $T^\perp < H$. As

$$\operatorname{supp} \mu_{\max} \subseteq \overline{A{\bullet}t_0} \subseteq T,$$

it follows from the spectral theorem (Theorem 2.66) that $\pi|_{T^\perp}$ is trivial. As A normalizes T, the same holds for $T^\perp$ and hence $T^\perp \lhd G$. Therefore we may and will consider π as a unitary representation of

$$G/T^{\perp} \cong A \ltimes H/T^{\perp}.$$

Moreover, as A is abelian, the stabilizer subgroup $S = \mathrm{Stab}_A(t_0)$ acts trivially on $A{\scriptstyle\bullet}t_0$ and hence also on T and on its Pontryagin dual $\widehat{T} \cong H/T^{\perp}$. In other words S belongs to the centre of $G/T^{\perp}$. By Corollary 1.32, there exists a character χ_0 on S such that $\pi|_S = \chi_0 I$. By Pontryagin duality (Proposition 2.31(2)) the character χ_0 on S extends to a character χ on A. We define the representation $\pi^0 = \overline{\chi} \otimes \pi$, which is an irreducible unitary representation by Lemma 1.28. We note that $\pi^0|_S$ is trivial, and so π^0 may be considered as a representation of $G/ST^{\perp} \cong A/S \ltimes H/T^{\perp}$. Writing $A_0 = A/S$, $H_0 = H/T^{\perp}$, and $G_0 = G/ST^{\perp} \cong A_0 \ltimes H_0$, we note that $\widehat{H_0} = T < \widehat{H}$. Hence the unchanged maximal spectral measure $\mu_{\max}$ satisfies $\mu_{\max}(\widehat{H_0} \smallsetminus A_0{\scriptstyle\bullet}t_0) = 0$, and the orbit map $A_0 \ni a \mapsto a{\scriptstyle\bullet}t_0 \in A{\scriptstyle\bullet}t_0 \subseteq T = \widehat{H_0}$ is free. Therefore Theorem 3.20 applies, and we deduce that π^0 is isomorphic to π^{μ}. Since $\pi = \chi \otimes \pi^0$, it follows that π is isomorphic to $\chi \otimes \pi^{\mu}$ as claimed. □

The results in Section 3.3 and this section are special cases of the *Mackey machine* that can be used to calculate the unitary dual of semi-direct products in many more cases.(8)

3.5.1 The Unitary Dual of Isometry Groups

We indicate in this section via another class of examples how the general case of the Mackey machine works. For this we let $K \subseteq \mathrm{SO}_d(\mathbb{R})$ be a compact subgroup, let $H = \mathbb{R}^d$, and define the semi-direct product

$$G = K \ltimes \mathbb{R}^d = \left\{ \begin{pmatrix} k & h \\ 0 & 1 \end{pmatrix} \;\middle|\; k \in K, h \in \mathbb{R}^d \right\}.$$

Extending the case considered in Theorem 3.33, the correct way to construct unitary representations of the semi-direct product G is to use a character $t_0 \in \widehat{H}$, the K-invariant probability measure μ on $K{\scriptstyle\bullet}t_0$ (obtained as the push-forward of the Haar measure on K), and an irreducible unitary representation ρ of $K_0 = \mathrm{Stab}_K(t_0) = \{k \in K \mid k{\scriptstyle\bullet}t_0 = t_0\}$. We then first combine χ_{t_0} and ρ (similarly to Lemma 1.28) to define an irreducible unitary representation $\rho \otimes \chi_{t_0}$ of $K_0 \ltimes H$ on $\mathcal{H}_\rho$ by

$$\rho \otimes \chi_{t_0}(kh) = \langle h, t_0 \rangle \rho(k).$$

Next one defines

$$\pi = \mathrm{Ind}_{K_0 \ltimes H}^{G}(\rho \otimes \chi_{t_0}),$$

where the measure μ is used to define the Hilbert space structure on the space $\mathcal{H}_\pi = \operatorname{Ind}_{K_0 \ltimes H}^{G} \mathcal{H}_\rho$. This leads to a description of all elements of $\widehat{G}$.

To make these objects less mysterious, we recommend assuming that K is finite, as in this case the following exercise applies.

Exercise 3.34. Let $G = A \ltimes H$ be as in the beginning of Section 3.2, and assume in addition that A is discrete. For some $t_0 \in \widehat{H}$ we define

$$A_0 = \operatorname{Stab}_A(t_0) = \{a \in A \mid a{\bullet}t_0 = t_0\}$$

and suppose that $\rho \in \widehat{A_0}$. We define $\pi = \operatorname{Ind}_{A_0 \ltimes H}^{G}(\rho \otimes \chi_{t_0})$ on

$$\mathcal{H}_\pi = \left\{v\colon A \to \mathcal{H}_\rho \mid v(aa_0) = \rho(a_0)^{-1} v(a) \text{ for all } a \in A, a_0 \in A_0 \text{ with } \|v\|_{\mathcal{H}_\pi} < \infty\right\}$$

equipped with the Hilbert space norm

$$\|v\|_{\mathcal{H}_\pi}^2 = \sum_{aA_0 \in A/A_0} \|v(a)\|_{\mathcal{H}_\rho}^2$$

by $(\pi_b v)(a) = v(b^{-1}a)$ and $(\pi_h v)(a) = \langle h, at\rangle v(a)$ for all $a, b \in A$, $h \in H$, and $v \in \mathcal{H}_\pi$.
(a) Show that π is a unitary representation of G on $\mathcal{H}_\pi$.
(b) Show that the spectral multiplicity of $\pi|_H$ is $\dim \mathcal{H}_\rho$.
(c) Show that π is irreducible.
(d) Assume now that the A-action on $\widehat{H}$ has countable separation of orbits. Show that the construction above gives rise to all irreducible representations of G (up to isomorphism).
(e) Show that the A-action on $\widehat{H}$ has countable separation of orbits if A is finite.

3.6 Two More Groups With Unreasonable Unitary Duals*

3.6.1 The Discrete Heisenberg Group

The (3-dimensional) discrete Heisenberg group is defined by

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z} \right\}.$$

We will show that despite G being nilpotent, and hence as close as possible to being abelian without being abelian, its unitary dual is quite wild and has no simple description.

As in Section 3.3.4, we will again use the notation $(a, b, c) \in \mathbb{Z}^3$ for the matrices in G and the subgroups

$$\begin{aligned} C(G) &= \{(0,0,c) \mid c \in \mathbb{Z}\}, \\ H &= \left\{h_{b,c} = (0,b,c) \;\middle|\; b, c \in \mathbb{Z}\right\}, \end{aligned}$$

and

$$A = \{g_a = (a, 0, 0) \mid a \in \mathbb{Z}\}.$$

Let us begin with the by now obvious candidates for irreducible unitary representations:

- old representations defined by characters on $G/C(G) \cong \mathbb{Z}^2$,
- new representations arising from A-orbits on $\widehat{H} \cong \mathbb{T}^2$, and
- new representations arising from A-invariant and ergodic probability measures on $\mathbb{T}^2$.

We now explain the sense in which the second type is itself 'unreasonable' and how to modify the third type to also give rise to an 'unreasonably large collection' of irreducible unitary representations.

By the same calculation as in Section 3.3.4, the dual automorphisms corresponding to elements of $A \cong \mathbb{Z}$ on $\mathbb{T}^2$ are defined by powers of the matrix

$$M = \widehat{\theta}_{(1,0,0)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \mathbb{T}^2 \ni \begin{pmatrix} x \\ \xi \end{pmatrix} \longmapsto \begin{pmatrix} x + \xi \\ \xi \end{pmatrix} \in \mathbb{T}^2.$$

Fixing some $\xi \in \mathbb{T} \smallsetminus \{0\}$, noting that $X = \mathbb{T} \times \{\xi\} \cong \mathbb{T}$ is invariant under M, and recalling that the A-action defined in (3.1) uses $\widehat{\theta}_{(1,0,0)}^{-1}$, we therefore need to study the rotation map

$$R_\xi \colon \mathbb{T} \ni x \longmapsto x - \xi \in \mathbb{T}.$$

For $\xi \in \mathbb{Q}/\mathbb{Z}$ every orbit under this map is finite. However, for an irrational ξ the orbit of every $x_0 \in \mathbb{T}$ is free, and given by $x_0 + \mathbb{Z}\xi \subseteq \mathbb{T}$. Hence each such orbit gives rise to an irreducible unitary representation. What is unreasonable about this? After taking into account possible isomorphisms of irreducible representations, the irreducible unitary representations arising from ξ and orbits of A on $\widehat{H}$ are in one-to-one correspondence with elements of the quotient $\mathbb{T}/\mathbb{Z}\xi$. Since $\mathbb{Z}\xi$ is a dense subgroup of $\mathbb{T}$ for every irrational ξ, the quotient $\mathbb{T}/\mathbb{Z}\xi$ is 'unreasonable' because (for example) any cross-section S (that is, a unique choice of representative x_0 for each coset $x_0 + \mathbb{Z}\xi \subseteq \mathbb{T}$, and hence for its associated irreducible unitary representation of G) is necessarily non-measurable.

We now move on to the third type in the list above, namely irreducible unitary representations arising from A-invariant and ergodic probability measures. Unlike the discussion of the solvable discrete group in Section 3.2.1, the A-invariant and ergodic probability measures on $\widehat{H} = \mathbb{T}^2$ are not at all complicated.† If ξ is rational, every orbit on $\mathbb{T} \times \{\xi\}$ is finite and the normalized Dirac measure on the orbit is an A-invariant and ergodic probability

† The set of quasi-invariant measures (see Lemma 3.4), on the other hand, is more complicated. Nonetheless, even by restricting to invariant probability measures, we can construct many irreducible representations of G.

measure on $\mathbb{T} \times \{\xi\} \subseteq \mathbb{T}^2$. If, on the other hand, ξ is irrational, then the Lebesgue measure on $\mathbb{T} \times \{\xi\}$ is the only A-invariant and ergodic probability measure on $\mathbb{T} \times \{\xi\} \subseteq \mathbb{T}^2$. However, we have only scratched the surface of the possible constructions of irreducible unitary representations. In particular, the following exercises will show that a generalization of Theorem 3.20 from measures on orbits to ergodic probability measures is not at all possible.

In fact, we can use an irrational $\xi \in \mathbb{T}$, Lebesgue measure μ on $X = \mathbb{T} \times \{\xi\}$, and a measurable map (called a *cocycle*) $c\colon A \times X \longrightarrow \mathbb{S}^1$ satisfying the *cocycle equation*

$$c(a_1 a_2, x) = c(a_1, x) c(a_2, a^{-1} \bullet x) \tag{3.15}$$

for all $a_1, a_2 \in A$ and almost all $x \in X$ to define a unitary representation $\pi^{\mu,c}$ by

$$\left(\pi^{\mu,c}_{h_{b,c} g_a} f\right)(x) = \mathrm{e}^{2\pi\mathrm{i}(xb+\xi c)} c(a,x) f(a^{-1} \bullet x)$$

for $a, b, c \in \mathbb{Z}$, $f \in L^2_\mu(X)$, and $x \in X$. We note that in the case that the cocycle $c\colon A \times X \to \mathbb{S}^1$ does not depend on the point in X, the property in (3.15) becomes the defining equation for a homomorphism from A to $\mathbb{S}^1$. We will call a cocycle arising from a homomorphism a *constant cocycle*.

Exercise 3.35 (Unitarity and irreducibility).
(a) Generalize Proposition 1.5 to allow for measurable cocycles, while assuming that the group is discrete.
(b) Formulate and prove a generalization of Lemma 3.5 to allow for measurable cocycles (assuming that A is discrete).
(c) Generalize Lemma 3.6 to allow measurable cocycles.

What makes these methods for producing irreducible unitary representations more difficult to understand are the questions that arise regarding equivalence of the resulting representations.

Exercise 3.36 (Coboundaries). Let G be the discrete Heisenberg group as above. Let μ be an A-invariant σ-finite measure on $\widehat{H}$, let $d \geqslant 1$, and let $c_1, c_2\colon A \times \widehat{H} \to \mathbb{S}^1$ be two cocycles (that is, measurable maps satisfying (3.15)). Show that the unitary representations π^{μ,c_1} and π^{μ,c_2} are unitarily equivalent if and only if there is a measurable map

$$b\colon \widehat{H} \longrightarrow \mathbb{S}^1$$

(a *coboundary*) satisfying the *coboundary equation*

$$c_2(a,x) = b(x) c_1(a,x) b(a^{-1} \bullet x)^{-1} \tag{3.16}$$

for all $a \in A$ and μ-almost every $x \in \widehat{H}$.

In a way, b as in the exercise above corresponds to a point-dependent coordinate change in $\mathbb{C}$. Cocycles related by (3.16) are said to be *cohomologous*.

The following exercise now shows that cocycles can be used for the discrete Heisenberg group to define additional irreducible unitary representations.

Exercise 3.37 (Constant cocycles and irreducible representations). Let G be the discrete Heisenberg group as above, let $\xi \in \mathbb{T}$ be irrational, and let $\alpha \in \mathbb{T} \smallsetminus \mathbb{Z}\xi$. Then the

Lebesgue measure μ on $X = \mathbb{T} \times \{\xi\} \subseteq \mathbb{T}^2 \cong \widehat{H}$ and the constant cocycle c_α defined by $c_\alpha((a,0,0),x) = \mathrm{e}^{2\pi\mathrm{i}a\alpha}$ for $x \in X$ define an irreducible unitary representation $\pi^{\xi,\alpha}$ that is not unitarily equivalent to the irreducible unitary representation π^μ defined by the Lebesgue measure μ on $\mathbb{T} \times \{\xi\}$ alone. Indeed, two representations π^{ξ,α_1} and π^{ξ,α_2} defined as above by homomorphisms and $\alpha_1, \alpha_2 \in \mathbb{T}$ are unitarily equivalent if and only if $\alpha_2 - \alpha_1$ is an integer multiple of ξ.

Notice that the exercise above shows that $\widehat{G}$ contains ‘another copy of the unreasonable quotient’ $\mathbb{T}/\mathbb{Z}\xi$ for every irrational $\xi \in \mathbb{T}$ that arise not from orbits but from twisting the representation π^μ by constant cocycles.

3.6.2 The Mautner Group

The examples considered so far, and in particular those in Sections 3.2.1 and 3.6.1, may give the impression that only discrete groups have unreasonably complicated unitary duals, while connected groups have better behaviour. To see that this is not the case, we consider here the connected group $G = G_{\mathrm{Mautner}}$ known as the *Mautner group*.

To define the Mautner group, we first recall the isometry group

$$G_2 = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$$

of the plane considered in Section 3.3.1, and define

$$G_4 = G_2 \times G_2 \cong \mathbb{T}^2 \ltimes (\mathbb{R}^2)^2.$$

We now let $A < \mathbb{T}^2$ be an immersed subgroup isomorphic to $\mathbb{R}$ and define the Mautner group by $G_{\mathrm{Mautner}} = A \ltimes (\mathbb{R}^2)^2$.

By this stage we expect that the description of $\widehat{G}$ (or our inability to give a description of it) has to do with the action of A on $\widehat{H}$ for $H = (\mathbb{R}^2)^2$. For the vector

$$t_0 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \in (\mathbb{R}^2)^2 \cong \widehat{H}$$

the $\mathbb{T}^2$-orbit is free and is given by $X = \mathbb{T}^2 \centerdot t_0 = \mathbb{S}^1 \times \mathbb{S}^1$. The restriction of the A-action to X is isomorphic to the translation action of the dense subgroup A on $\mathbb{T}^2$. In particular, we can:

- use every coset in $\mathbb{T}^2/A$ (equivalently, every A-orbit in X) to obtain an irreducible unitary representation of G,
- use the Haar measure on $\mathbb{T}^2$ pushed down to an A-invariant and ergodic probability measure on X to obtain another irreducible unitary representation π^μ of G, or
- make use of (continuous or, with a bit more effort, measurable) unitary cocycles $c\colon A \times X \to \mathbb{S}^1$ for the A-action on X, together with the ergodic

probability measure μ, to obtain irreducible unitary representations $\pi^{\mu,c}$ of G.

In particular, the first and (due to a generalization of Exercise 3.37, also the) third type again give rise to an unreasonable set of irreducible unitary representations.

3.7 Summary and Outlook

Our discussions regarding the interplay between $\pi|_H$ and the G-action on $\widehat{H}$ for a normal abelian subgroup $H \lhd G$ in Section 3.1 will be important in, for example, Chapter 7.

The complete description of $\widehat{G}$ for some concrete metabelian groups will provide interesting examples for the discussion of the Fell topology in the next chapter. The reader interested in pursuing the so-called Mackey machine in full strength may consult Folland [27].