Chapter 2

Ergodicity and Mixing on Locally Homogeneous Spaces

In this chapter we review some important notions and discuss ergodicity and mixing for actions of Lie groups.

2.1 Basic Notions in Ergodic Theory

Throughout, we will assume that an acting group G is σ-compact, locally compact, and metrizable. Moreover, we will assume that X, the space G acts on, is a σ-compact locally compact metric space, and that the action is jointly continuous (also see [53, Sec. 8] for more background). Such an action is said to be

- *measure-preserving* with respect to a probability measure μ on X if

$$\mu(g^{-1}\bullet B) = \mu(B)$$

  for any $g \in G$ and measurable set $B \subseteq X$, in which case we also say that μ is *invariant*;
- *ergodic* with respect to a probability measure μ if any measurable $B \subseteq X$ with the property $\mu(g^{-1}\bullet B \triangle B) = 0$ for all $g \in G$ has $\mu(B) \in \{0, 1\}$; and
- *mixing* with respect to a probability measure μ if

$$\mu(g^{-1}\bullet A \cap B) \longrightarrow \mu(A)\mu(B)$$

  as $g \to \infty$ in G for any measurable sets $A, B \subseteq X$.

Here the notation $g \to \infty$ is shorthand for elements g of G running through a sequence $(g_n)_{n\geqslant 1}$ with the property that for any compact set $K \subseteq G$ there is an $N = N(K)$ such that $n \geqslant N(K)$ implies $g_n \notin K$. Notice that the property of mixing (of non-compact groups) is much stronger than ergodicity in the following sense. Mixing for the action implies that each element $g \in G$ with $g^n \to \infty$ as $n \to \infty$ is itself a mixing (and ergodic) transformation in the usual sense (where the acting group is a copy of $\mathbb{Z}$), while ergodicity *a priori* does not tell us anything at all about properties of the action of individual elements of G (see Exercise 2.4.1).

We will now recall also that ergodicity and mixing are *spectral properties* in the sense that they can be phrased in terms of the associated Koopman representation or unitary action π of G defined by $\pi(g)f = f \circ g^{-1}$ for $f \in L^2(X,\mu)$ and $g \in G$. We note that this unitary representation has the following natural continuity property (which we will assume for all unitary representations discussed): given a function $f \in L^2(X,\mu)$ the map $g \in G \mapsto \pi(g)f \in L^2(X,\mu)$ is continuous (with respect to the given topology on G and the norm topology on $L^2(X,\mu)$), see [53, Def. 11.16 and Lem. 11.17] and [54, Lem. 3.74].

Assuming the action is measure-preserving, then:

- the G-action is *ergodic* if and only if the constant function $\mathbb{1}$ is the only eigenfunction for the representation (up to multiplication by scalars);
- the G-action is *mixing* if and only if

$$\langle \pi(g)f_1, f_2\rangle \longrightarrow \int f_1 \,\mathrm{d}\mu \int \overline{f_2}\,\mathrm{d}\mu = \langle f_1, \mathbb{1}\rangle \langle \mathbb{1}, f_2\rangle$$

  as $g \to \infty$ for any $f_1, f_2 \in L^2(X,\mu)$.

As a motivation for the study of ergodicity in this chapter we recall the pointwise ergodic theorem. The pointwise ergodic theorem holds quite generally for actions of amenable groups$^{(5)}$, but here we wish to only discuss the case of $\mathbb{R}^d$-flows (measure-preserving actions of $\mathbb{R}^d$).

Theorem 2.1. *Let $(t,x) \mapsto t{\bullet}x$ be a jointly continuous action of $\mathbb{R}^d$ on a σ-compact locally compact metric space X preserving a Borel probability measure μ. Then, for any $f \in L^1_\mu(X)$,*

$$\frac{1}{m_{\mathbb{R}^d}(B_r)} \int_{B_r} f(t{\bullet}x)\,\mathrm{d}t \longrightarrow E_\mu(f \big| \mathcal{E})(x) \tag{2.1}$$

as $r \to \infty$ for μ-almost every $x \in X$. Here

$$B_r = \{t = (t_1, \dots, t_d) \in \mathbb{R}^d \mid 0 \leqslant t_i \leqslant r \text{ for } i = 1, \dots, d\}.$$

denotes a cube of side length r with 0 at one of its corners,

$$\mathcal{E} = \{B \subseteq X \mid \mu(B \triangle g{\bullet}B) = 0 \text{ for all } g \in G\}$$

denotes the σ-algebra of invariant sets under the action, and $E_\mu(f \big| \mathcal{E})$ denotes the conditional expectation with respect to $\mathcal{E}$.

Remark 2.2. (1) This is a special case of [53, Th. 8.19], and the use of d-dimensional cubes as the averaging sequence is not necessary. As may be seen from conditions (P), (D), and (F) in [53, Sec. 8.6.2] any reasonable choice of metric balls containing the origin of $\mathbb{R}^d$ will suffice to achieve the almost everywhere convergence in (2.1).

(2) Notice that ergodicity for the action is equivalent to the invariant σ-algebra $\mathcal{E}$ being equivalent modulo μ-null sets to the trivial algebra $\{\varnothing, X\}$, so in this case the ergodic averages in (2.1) converge to $\int_X f \,\mathrm{d}\mu$.

(3) A consequence of Theorem 2.1 is that μ-almost every point in X has an orbit under the action that is not only dense in $\operatorname{supp}\mu$ but is equidistributed with respect to μ (see [53, Ch. 4.4.2] for the details in the case of a single transformation, and Section 6.3.1).

(4) The natural G-action on the quotient $X = \Gamma\backslash G$ by a lattice $\Gamma < G$ is ergodic with respect to the measure m_X inherited from Haar measure on G. However, as the group G is uncountable, it is not immediately obvious that the absence of nontrivial invariant sets (which is obvious for the G-action on X) implies the triviality of the measure of sets that are invariant modulo m_X (as is required for ergodicity). For the fact that this is indeed the case we refer to [53, Sec. 8.1].

(5) As mentioned above, mixing is of course a stronger property than ergodicity in many different ways. More significantly for our purposes, we will see in Chapter 5 situations in which mixing allows us to prove even stronger results on the behaviour of all orbits for certain subgroups, rather than just almost all orbits. This is significant, because knowledge of the behaviour of almost every point tells you nothing about the behaviour of any one specific point, and in some situations the easiest way to describe the behaviour of a specific point one is interested in is to describe the behaviour of all points.

(6) We note that for general groups and their measure-preserving actions ergodicity is a universal notion, as any invariant measure can be decomposed into 'ergodic components' (see, for example, [53, Sec. 8.7]). However, in general — and in particular for $G = \mathbb{Z}$ or $G = \mathbb{R}^d$ — mixing is a rather special property.

2.2 Real Lie Algebras and Lie Groups

†In this section we will set up the language concerning real Lie algebras and Lie groups that we need. For brevity we assume the basic definitions and properties of Lie groups are known. For proofs, background, and more details

† This section can be skipped if the reader is familiar with the theory. Also, most of the section can be skipped if the reader is only interested in some main examples of the theory, for example, the important cases of the simple Lie group $G = \mathrm{SL}_d(\mathbb{R})$ or the semi-simple Lie groups

$$G = \mathrm{SL}_d(\mathbb{R}) \times \cdots \times \mathrm{SL}_d(\mathbb{R}).$$

In the latter case, the reader will need to familiarize herself with the notions used in Section 2.2.1, the notion of simple Lie ideals and Lie groups, and should also do Exercise 2.2.1.

we refer to Knapp [103]. Not all of the theorems that we mention here will be used in an essential way, but for the most general theorem in this chapter we will use both the Levi decomposition and the Jacobson–Morozov theorem (Theorem 2.10).

2.2.1 Basic Notions

Recall that for any real Lie group G there is an associated real Lie algebra $\mathfrak{g}$ that *describes* G *near the identity.* There is a smooth map $\exp : \mathfrak{g} \to G$ with a local inverse $\log : B_\delta^G(I) \to \mathfrak{g}$ defined on some neighborhood $B_\delta^G(I)$ of the identity $I \in G$ with $\delta > 0$.

There is a linear representation of G on $\mathfrak{g}$, the *adjoint* representation

$$\mathrm{Ad}_g \colon \mathfrak{g} \longrightarrow \mathfrak{g}$$

for $g \in G$, satisfying

$$\exp(\mathrm{Ad}_g(v)) = g \exp(v) g^{-1}$$

for $g \in G$ and $v \in \mathfrak{g}$. Furthermore, there is a bilinear anti-symmetric Lie bracket

$$[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

and a related map $\mathrm{ad}_u : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\mathrm{ad}_u(v) = [u, v]$$

for $u, v \in \mathfrak{g}$, which satisfies

$$\mathrm{Ad}_g([u, v]) = [\mathrm{Ad}_g(u), \mathrm{Ad}_g(v)] \tag{2.2}$$

and

$$\exp(\mathrm{ad}_u) = \mathrm{Ad}_{\exp(u)} \tag{2.3}$$

for all $u, v \in \mathfrak{g}$ and all $g \in G$. Here $\mathrm{ad}_u \colon \mathfrak{g} \to \mathfrak{g}$ is an element of the algebra of linear maps $\mathrm{End}(\mathfrak{g})$,

$$\exp\colon\ \mathrm{End}(\mathfrak{g}) \longrightarrow \mathrm{GL}(\mathfrak{g})$$

is the exponential map from $\mathrm{End}(\mathfrak{g})$ to the group $\mathrm{GL}(\mathfrak{g})$ of linear automorphisms of the vector space $\mathfrak{g}$, and $\mathrm{Ad}_{\exp(u)}$ is the adjoint representation defined by the element $\exp(u) \in G$.

Finally, the Lie bracket satisfies the *Jacobi identity*

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$$

for all $u, v, w \in \mathfrak{g}$. In the special case where G is a closed linear subgroup of $\mathrm{SL}_d(\mathbb{R})$ for some $d \geqslant 2$ (which is more than sufficient for all of our applications) the claims above are easy to verify. Indeed, in these cases we have

$$\mathfrak{g} \subseteq \mathfrak{sl}_d(\mathbb{R}) = \{u \in \mathrm{Mat}_d(\mathbb{R}) \mid \mathrm{tr}(u) = 0\},$$

$$\mathrm{Ad}_g(u) = gug^{-1},$$

and

$$[u, v] = uv - vu$$

for all $g \in G$ and $u, v \in \mathfrak{g}$.

2.2.2 Classification and Complex Lie Algebras

The local relationship between a Lie group and its Lie algebra mentioned in Section 2.2.1 in fact goes much further. If G is connected and simply connected then its Lie algebra uniquely determines G. That is, any two connected and simply connected Lie groups with isomorphic Lie algebras are themselves isomorphic. Even without the assumption that the Lie groups G_1, G_2 are simply connected, one obtains a diffeomorphism ϕ between neighborhoods U_1 and U_2 of the identities in G_1 and G_2 if they have the same Lie algebra, such that products are mapped to products $\phi(gh) = \phi(g)\phi(h)$ as long as all the terms $g, h, gh \in U_1$ stay in the domain of the map ϕ. In this case we say that G_1 and G_2 are *locally isomorphic*. For this reason, one usually starts with a classification of Lie algebras, and this classification is easier in the case of complex Lie algebras, making this the conventional first case to consider.

2.2.3 The Structure of Lie Algebras

A Lie ideal $\mathfrak{f} \lhd \mathfrak{g}$ is a subspace of $\mathfrak{g}$ with $[\mathfrak{f}, \mathfrak{g}] \subseteq \mathfrak{f}$. Lie ideals of Lie algebras of real Lie groups correspond to normal subgroups in the following sense. If $F \lhd G$ is a closed normal subgroup, then its Lie algebra $\mathfrak{f} \subseteq \mathfrak{g}$ is a Lie ideal (see Exercise 2.2.3). On the other hand, if $\mathfrak{f} \lhd \mathfrak{g}$ is a Lie ideal, then there is an immersed normal subgroup $F \lhd G$ with Lie algebra $\mathfrak{f}$. Here the term *immersed* allows for the possibility that the subgroup $F = \langle \exp(\mathfrak{f}) \rangle$ generated by $\mathfrak{f}$ is not closed in G (this arises, for example, for the abelian Lie algebras $\mathfrak{f} = \mathbb{R}v$ and $\mathfrak{g} = \mathbb{R}^2$ for the group $G = \mathbb{R}^2/\mathbb{Z}^2$ for most choices of v). In the situation where $F \lhd G$ is not closed, we note that $\overline{F} \lhd G$ would then correspond to another Lie ideal $\overline{\mathfrak{f}} \lhd \mathfrak{g}$ (which is determined by $\mathfrak{f}$ and G, but in general not by $\mathfrak{f}$ and $\mathfrak{g}$ alone).

In group theory the notion of the commutator subgroup

$$[G,G] = \langle [g,h] \mid g,h \in G \rangle \lhd G$$

(where $[g,h] = g^{-1}h^{-1}gh$) is an important measure of the extent to which G fails to be abelian. Recall that a group G is said to be *nilpotent* if the *lower central series* (G_i) defined by

$$\begin{aligned} G_0 &= G, \\ G_{i+1} &= [G,G_i] = \langle [g,h] \mid g \in G, h \in G_i \rangle \lhd G \end{aligned}$$

for $i \geqslant 1$ reaches the trivial group $G_r = \{e\}$ for some $r \geqslant 1$ (the minimal such r is called the *nilpotency degree*). Similarly G is called *solvable* if the *commutator series* (G^i) defined by

$$\begin{aligned} G^0 &= G, \\ G^1 &= [G,G] \lhd G, \\ G^{i+1} &= [G^i,G^i] \lhd G \end{aligned}$$

for $i \geqslant 1$ reaches the trivial group $G_s = \{e\}$ for some $s \geqslant 1$. Every nilpotent group is solvable, while the group

$$G = B = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}$$

is solvable but not nilpotent.

These fundamental notions in group theory have natural translations into the theory of Lie algebras. A Lie algebra $\mathfrak{g}$ is *nilpotent* if the lower central series

$$\mathfrak{g}_0 = \mathfrak{g} \rhd \mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}_0] \rhd \cdots \rhd \mathfrak{g}_{i+1} = [\mathfrak{g},\mathfrak{g}_i] \rhd \cdots$$

ends with the trivial subalgebra $\mathfrak{g}_r = \{0\}$ for some $r \geqslant 1$, and $\mathfrak{g}$ is *solvable* if the commutator series

$$\mathfrak{g}^0 = \mathfrak{g} \rhd \mathfrak{g}^1 = [\mathfrak{g}^0,\mathfrak{g}^0] \rhd \cdots \rhd \mathfrak{g}^{i+1} = [\mathfrak{g}^i,\mathfrak{g}^i] \rhd \cdots$$

ends with the trivial subalgebra $\mathfrak{g}^s = \{0\}$ for some $s \geqslant 1$.

By Ado's theorem [103, Th. B.8], every real (or complex) Lie algebra $\mathfrak{g}$ can be realized as a linear Lie algebra, meaning that $\mathfrak{g}$ can be embedded into $\mathfrak{gl}_d(\mathbb{R}) = \mathrm{Mat}_d(\mathbb{R})$ (or into $\mathfrak{gl}_d(\mathbb{C}) = \mathrm{Mat}_d(\mathbb{C})$) for some $d \geqslant 1$. By Lie's theorem [103, Th. 1.25], a complex Lie algebra $\mathfrak{g}$ is solvable if and only if it can be embedded into

$$\mathfrak{b}(\mathbb{C}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1d} \\ & a_{22} & a_{23} & \cdots & a_{2d} \\ & & \ddots & & \\ & & & a_{d-1,d-1} & a_{d-1,d} \\ & & & & a_{dd} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{C} \text{ for } i \leqslant j \right\}.$$

Since every real Lie algebra $\mathfrak{g}$ has a complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \mathrm{i}\mathfrak{g}$ (see below) it also follows that every real Lie algebra can be embedded into $\mathfrak{b}(\mathbb{C})$ (but maybe not into the analogous real Lie algebra $\mathfrak{b}(\mathbb{R})$.)

By Engel's theorem [103, Th. 1.35], a real Lie algebra $\mathfrak{g}$ is nilpotent if and only if it can be embedded into

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & \cdots & a_{1d} \\ & 0 & a_{23} & \cdots & a_{2d} \\ & & \ddots & & \\ & & & 0 & a_{d-1,d} \\ & & & & 0 \end{pmatrix} \middle| \, a_{ij} \in \mathbb{R} \text{ for } i < j \right\}.$$

It is interesting to note that the commutator $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ of a solvable Lie algebra is nilpotent (since $[\mathfrak{b}(\mathbb{C}), \mathfrak{b}(\mathbb{C})] \subseteq \mathfrak{n}(\mathbb{C})$) — there is no analogue of this fact for abstract groups.

For a general Lie algebra $\mathfrak{g}$, the *radical* $\operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ is defined to be the subspace generated by all solvable Lie ideals $\mathfrak{f} \lhd \mathfrak{g}$, and this is a solvable Lie ideal of $\mathfrak{g}$.

A (real or complex) Lie algebra $\mathfrak{g}$ is said to be *semi-simple* if $\operatorname{rad}\mathfrak{g} = \{0\}$. A (real or complex) Lie algebra is called *simple* if $\mathfrak{g}$ is non-abelian (that is, if $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$) and $\mathfrak{g}$ has no Lie ideals other than $\mathfrak{g}$ and $\{0\}$. We note that a real simple Lie algebra always has a semi-simple complexification

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \mathrm{i}\mathfrak{g},$$

with the complexified Lie bracket defined by

$$[u + \mathrm{i}v, w + \mathrm{i}z] = [u, w] - [v, z] + \mathrm{i}\left([v, w] + [u, z]\right),$$

(but that this complexification might not be simple; see Exercise 2.2.2).

Every (real or complex) semi-simple Lie algebra $\mathfrak{g}$ is a direct sum of (real or complex) simple Lie subalgebras, each of which is a Lie ideal in $\mathfrak{g}$.

Finally, we note that solvable Lie algebras and semi-simple Lie algebras complement each other, and any Lie algebra can be described using Lie algebras of these two types in the following sense. The Levi decomposition

$$\mathfrak{g} = \mathfrak{g}_s + \operatorname{rad}\mathfrak{g}$$

of a (real or complex) Lie algebra consists of a semi-simple Lie subalgebra $\mathfrak{g}_s$ of $\mathfrak{g}$ and the radical $\operatorname{rad}\mathfrak{g} \lhd \mathfrak{g}$. In this decomposition $\operatorname{rad}\mathfrak{g}$ is unique, but in general $\mathfrak{g}_s$ is not.

2.2.4 Almost Direct Simple Factors

A connected real (or complex) Lie group G is called *simple* or *semi-simple* if its Lie algebra $\mathfrak{g}$ is simple or semi-simple respectively.

If $\mathfrak{g}$ is a real (or complex) semi-simple Lie algebra then, as mentioned above, we have a decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

with simple Lie ideals $\mathfrak{g}_i \lhd \mathfrak{g}$ for $i = 1, \ldots, r$. If G is a real (or complex) connected simply connected semi-simple Lie group then the stronger property

$$G \cong G_1 \times \cdots \times G_r, \tag{2.4}$$

holds, where each $G_i \lhd G$ is a connected simply connected Lie group with Lie algebra $\mathfrak{g}_i$.

The product decomposition in (2.4) does not hold for general semi-simple Lie groups without the assumption that the group is simply connected. However, the reason why the product decomposition fails is easy to understand.

Example 2.3. Let

$$G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\{(I, I), (-I, -I)\}$$

be the quotient by the normal subgroup N generated by $(-I, -I)$ in

$$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}).$$

Notice that the Lie algebra of G is isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ and that G is not simply connected. Furthermore,

$$G_1 = \mathrm{SL}_2(\mathbb{R}) \times \{I\}N/N$$

and

$$G_2 = \{I\} \times \mathrm{SL}_2(\mathbb{R})N/N$$

are both normal subgroups of G, are both isomorphic to $\mathrm{SL}_2(\mathbb{R})$, but

$$G \ncong G_1 \times G_2$$

unlike the simply connected case discussed above. Also note that $G_1 \cap G_2$ is generated by $(-I, I)N = (I, -I)N$ which is contained in the center of G.

Allowing for such phenomena along the center, one does get an almost direct product decomposition into almost direct factors of a real semi-simple Lie group as follows. Let G be a real semi-simple Lie group, and suppose that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

is the decomposition of its Lie algebra into real simple Lie subalgebras. Then for each $i = 1, \dots, r$ there is a normal closed connected simple Lie subgroup G_i, which we will refer to as an *almost direct factor*, with Lie algebra $\mathfrak{g}_i$. These almost direct factors have the following properties.

- G_i commutes with G_j for $i \neq j$;
- $G = G_1 \cdots G_r$; and
- the kernel of the homomorphism

$$\begin{aligned} G_1 \times \cdots \times G_r &\longrightarrow G_1 \cdots G_r = G \\ (g_1, \dots, g_r) &\longmapsto g_1 \cdots g_r \end{aligned}$$

  is contained in the center of $G_1 \times \cdots \times G_r$.

We define $G^+ \subseteq G$ to be the almost direct product of (i.e. the normal subgroup of G generated by) those almost direct factors G_i of G that are noncompact.

From now on, unless explicitly identified to be complex, we will always consider real Lie groups and Lie algebras.

Exercises for Section 2.2

Exercise 2.2.1. Show that $\mathfrak{sl}_d(\mathbb{R})$ (or $\mathfrak{sl}_d(\mathbb{C})$) is a real (resp. complex) simple Lie algebra for $d \geqslant 2$. Show that $\mathrm{SL}_d(\mathbb{R})$ and $\mathrm{SL}_d(\mathbb{C})$ are connected simple Lie groups.

Exercise 2.2.2. Show that $\mathfrak{sl}_d(\mathbb{C})$ for $d \geqslant 2$, when viewed as a real Lie algebra, is simple but its complexification is not.

Exercise 2.2.3. Show that if $F \lhd G$ is a closed normal subgroup of a Lie group G, then its Lie algebra $\mathfrak{f} \subseteq \mathfrak{g}$ is a Lie ideal.

Exercise 2.2.4. Let G be a real simple connected Lie group. Show that any proper normal subgroup of G for $d \geqslant 2$ is contained in the center of G.

Exercise 2.2.5. Show that the connected component of

$$\mathrm{SO}(2,2)(\mathbb{R}) = \{g \in \mathrm{SL}_4(\mathbb{R}) \mid g \text{ preserves the quadratic form } ad - bc\}$$

is isomorphic to the almost direct product discussed in Example 2.3.

2.3 Mautner Phenomenon, First Cases

The following key lemma(6) will be the main tool used for proving that ergodicity sometimes has an inheritance property from the acting group to some of its subgroups.

Lemma 2.4 (The key lemma). *Let $\mathcal{H}$ be a Hilbert space carrying a unitary representation of a topological group G. Suppose that $v_0 \in \mathcal{H}$ is fixed by some subgroup $L \leqslant G$. Then v_0 is also fixed under every other element $h \in G$ with the property that there exist sequences (g_n) in G and (ℓ_n), (ℓ'_n) in L with $\lim_{n\to\infty} g_n = e$ and $h = \lim_{n\to\infty} \ell_n g_n \ell'_n$.*

Proof. By assumption, there exist three sequences (g_n) in G, (ℓ_n) in L, and (ℓ'_n) in L with $g_n \to e$ and $\ell_n g_n \ell'_n \to h$ as $n \to \infty$. This implies that

$$\|\pi(\ell_n g_n \ell'_n)v_0 - v_0\| = \|\pi(\ell_n)(\pi(g_n\ell'_n)v_0 - \pi(\ell_n^{-1})v_0)\| = \|\pi(g_n)v_0 - v_0\|$$

by invariance of v_0 under all elements of L and unitarity of $\pi(\ell_n)$. However, the left-hand side converges to $\|\pi(h)v_0 - v_0\|$ by continuity of the representation and the right-hand side converges to 0. □

As we will see, this simple observation can be used to show that ergodicity of a measure-preserving action of G sometimes forces ergodicity of a subgroup L. Indeed, suppose G acts ergodically and preserving μ on a probability space (X,μ) as in Section 2.1, $L \leqslant G$ is a subgroup, and $f \in L^2_\mu(X)$ is invariant under L. Applying Lemma 2.4 with various choices of sequences, one may hope to prove that f is in fact invariant under other elements of G. In good situations one obtains in this way enough elements of G to generate G, which implies that f is invariant under G, hence f is constant, and so the action of L is ergodic.

2.3.1 The Case of $\mathrm{SL}_2(\mathbb{R})$

We now turn to the special (but important) case of $G = \mathrm{SL}_2(\mathbb{R})$. Any element $g \in \mathrm{SL}_2(\mathbb{R})$ is conjugate to one of the following three type of elements:

- an $\mathbb{R}$-diagonal matrix, that is one of the form $a = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ with $\lambda \in \mathbb{R}^\times$;
- a unipotent matrix $u = \begin{pmatrix} 1 & \pm 1 \\ & 1 \end{pmatrix}$ or $u = \begin{pmatrix} -1 & \pm 1 \\ & -1 \end{pmatrix}$; or
- a matrix in the compact subgroup $\mathrm{SO}(2,\mathbb{R})$, that is one of the form

  $$k = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

  for some $\phi \in \mathbb{R}$.

For the last case we can make no claim concerning ergodicity of the action of g. However, for the first two types we find the following phenomenon, where we write

$$C_G = \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

for the center of G. We note that $C_{\mathrm{SL}_2(\mathbb{R})} = \{\pm I\}$.

Proposition 2.5 (Mautner for $\mathrm{SL}_2(\mathbb{R})$). *Let $G = \mathrm{SL}_2(\mathbb{R})$ act unitarily on a Hilbert space $\mathcal{H}$, and suppose that $g \in G \smallsetminus C_G$ has the property that g is unipotent, $-g$ is unipotent, or g is $\mathbb{R}$-diagonalizable. Suppose g fixes a vector $v_0 \in \mathcal{H}$. Then all of G fixes v_0 also. The same holds for a connected Lie group G locally isomorphic*[†] *to $\mathrm{SL}_2(\mathbb{R})$ where $g \in G \smallsetminus C_G$ is such that Ad_g is unipotent or $\mathbb{R}$-diagonalizable with an eigenvalue λ with $|\lambda| \neq 1$.*

Suppose $g \in G$ satisfies the hypotheses of Proposition 2.5, and $h \in G$ has the property that hgh^{-1} fixes $v_0 \in \mathcal{H}$. Then g fixes $\pi^{-1}(h)v_0$ and so $v_0 = \pi^{-1}(h)v_0$ is fixed by G as needed. Thus it is sufficient to consider one representative of each conjugacy class for the proof of Proposition 2.5 and for the proof of similar statements that come later.

Proof of Proposition 2.5 for $\mathrm{SL}_2(\mathbb{R})$. For $a = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ with $\lambda \neq \pm 1$ a direct calculation shows that we can apply Lemma 2.4 with $L = a^{\mathbb{Z}}$ and any element of the unipotent subgroups $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$. For example,

$$a^n \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} a^{-n} = \begin{pmatrix} 1 & \lambda^{2n} s \\ & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

if $\lambda^{2n} \to 0$ as $n \to \infty$. It follows that if a fixes some $v_0 \in \mathcal{H}$, then so do these two unipotent subgroups, and as they together generate $\mathrm{SL}_2(\mathbb{R})$ (see Exercise 1.2.5 and Lemma 1.24), we obtain Proposition 2.5 for this case.

If $u = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ then

$$\begin{aligned} u^n \begin{pmatrix} 1+\delta & \\ & \frac{1}{1+\delta} \end{pmatrix} u^{-n} &= \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \begin{pmatrix} 1+\delta & \\ & \frac{1}{1+\delta} \end{pmatrix} \begin{pmatrix} 1 & -n \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+\delta & \left(\frac{1}{1+\delta} - 1 - \delta\right) n \\ & \frac{1}{1+\delta} \end{pmatrix} \end{aligned}$$

can be made (since n can be chosen arbitrary) to converge to $\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$ for $\delta \to 0$. It follows that if v_0 is fixed by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ then it is also fixed by $\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$ for any $s \in \mathbb{R}$ by Lemma 2.4 applied with

$$L = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

[†] This second case is not needed if one is only interested in closed linear subgroups G in $\mathrm{SL}_d(\mathbb{R})$. If G is a closed linear group linearly isomorphic to $\mathrm{SL}_2(\mathbb{R})$, then the theory of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$ implies that $G \cong \mathrm{SL}_2(\mathbb{R})$ or $G \cong \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$, and both of these cases are handled by the first part of the proposition.

Applying Lemma 2.4 once more with

$$L = \left\{ \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$$

to the matrix

$$\begin{pmatrix} 1 & s_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \delta & 1 \end{pmatrix} \begin{pmatrix} 1 & s_2 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 + \delta s_1 & s_2(1 + \delta s_1) + s_1 \\ \delta & 1 + \delta s_2 \end{pmatrix} = g_\delta \tag{2.5}$$

with s_1 chosen to have

$$1 + \delta s_1 = \mathrm{e}^\alpha$$

for some fixed $\alpha \in \mathbb{R}$, and with s_2 chosen to have

$$s_2(1 + \delta s_1) + s_1 = 0$$

shows that v_0 is also fixed by

$$\begin{pmatrix} \mathrm{e}^\alpha & \\ & \mathrm{e}^{-\alpha} \end{pmatrix} = \lim_{\delta \to 0} g_\delta.$$

Assuming $\alpha \neq 0$ and applying the previous (diagonal) case, we see once again that v_0 is fixed by all of $\mathrm{SL}_2(\mathbb{R})$. This finishes the proof of the proposition for $\mathrm{SL}_2(\mathbb{R})$, and also the proof of the Howe–Moore theorem (Theorem 2.15) for $\mathrm{SL}_2(\mathbb{R})$ and for products of several copies of $\mathrm{SL}_2(\mathbb{R})$. □

For the second case of Proposition 2.5 where G is only assumed to be locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ we are going to use the following more general lemma and also the calculations of the proof above. Here and in the following we will work more and more with elements $v \in \mathfrak{g}$ of the Lie algebra of G.

Definition 2.6 (Lie algebra fixing vectors). Let G be a Lie group, and let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. We say that $v \in \mathfrak{g}$ *fixes* $w \in \mathcal{H}$ if $\pi(\exp(tv))w = w$ for all $t \in \mathbb{R}$. We say that a Lie subalgebra $\mathfrak{f} \subseteq \mathfrak{g}$ *fixes* $w \in \mathcal{H}$ if every $v \in \mathfrak{f}$ fixes w.

Lemma 2.7 (Key lemma for unipotent conjugation). *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let π be a unitary representation on a Hilbert space $\mathcal{H}$. Suppose that $g \in G$ fixes v_0. Then v_0 is also fixed by all elements of the subspace*

$$\mathrm{Im}\,(\mathrm{Ad}_g - I) \cap \ker\,(\mathrm{Ad}_g - I) \subseteq \mathfrak{g},$$

and all of these elements are nilpotent.

In particular, this applies to $g = \exp(u)$ if $u \in \mathfrak{g}$ is nilpotent and the subspace $\mathrm{Im}\,\mathrm{ad}_u \cap \ker \mathrm{ad}_u$.

PROOF. Let $v \in \mathrm{Im}\,(\mathrm{Ad}_g - I) \cap \ker\,(\mathrm{Ad}_g - I)$. We wish to show that

$$\pi(\exp(v))\,v_0 = v_0.$$

By assumption, there exists some $w \in \mathfrak{g}$ with

$$(\mathrm{Ad}_g - I)(w) = v$$

and

$$(\mathrm{Ad}_g - I)(v) = 0,$$

so that

$$\mathrm{Ad}_g(w) = w + v$$

and

$$\mathrm{Ad}_g(v) = v.$$

For $n \geqslant 1$ this gives

$$\mathrm{Ad}_g^n\left(\tfrac{1}{n}w\right) = \tfrac{1}{n}w + v,$$

and so

$$g^n \exp\left(\tfrac{1}{n}w\right) g^{-n} = \exp\left(\tfrac{1}{n}w + v\right). \tag{2.6}$$

The exponential in the left-hand side of (2.6) converges to I, but the right-hand side converges to $\exp(v)$ as $n \to \infty$. It follows by Lemma 2.4 that $\exp(v)$ fixes v_0.

For the last claim of the first part of the lemma, we calculate

$$\mathrm{ad}_v = \lim_{n\to\infty} \mathrm{ad}_{\frac{1}{n}w+v} = \lim_{n\to\infty} \mathrm{ad}_{\mathrm{Ad}_g^n(\frac{1}{n}w)} = \lim_{n\to\infty} \mathrm{Ad}_g^n \circ \left(\tfrac{1}{n}\,\mathrm{ad}_w\right) \circ \mathrm{Ad}_g^{-n},$$

where we used (2.2). Since conjugation does not change the eigenvalues, it follows that ad_v is nilpotent.

Let now $u \in \mathfrak{g}$ be nilpotent as in the last part of the lemma, and let

$$g = \exp(u).$$

Then $\mathrm{Ad}_g = \exp(\mathrm{ad}_u) = I + \mathrm{ad}_u + \ldots + \frac{1}{n!}\,\mathrm{ad}_u^n$ for some n (see (2.3)). If now $v = \mathrm{ad}_u(w) \in \ker \mathrm{ad}_u$, then $\mathrm{Ad}_g(v) = v$ and $\mathrm{Ad}_g(w) = w + v$ and so the first part of the lemma applies. □

Proof of Proposition 2.5. Suppose now that G is only locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$. If Ad_g is $\mathbb{R}$-diagonalizable with an eigenvalue λ with $|\lambda| \neq 1$, then we may argue as before. Indeed suppose that $x \in \mathfrak{g}$ has $\mathrm{Ad}_g(x) = \lambda x$ with $|\lambda| < 1$. Then

$$g^n \exp(tx) g^{-n} = \exp\left(t\,\mathrm{Ad}_g^n(x)\right) \longrightarrow e$$

as $n \to \infty$, and Lemma 2.4 for $L = g^{\mathbb{Z}}$ shows that $\exp(\mathbb{R}x) \subseteq G$ fixes v_0. The same holds for $\exp(\mathbb{R}y)$ for any $y \in \mathfrak{g}$ with

$$\mathrm{Ad}_g(y) = \mu y$$

for some $|\mu| > 1$ by applying the same argument with $n \to -\infty$. Notice that the latter eigenvector must also exist, since otherwise $g \mapsto |\det \mathrm{Ad}_g|$ would be a non-trivial character from the simple group G to $\mathbb{R}^\times$. It follows that $[x, y]$ is an eigenvector for another eigenvalue since

$$\mathrm{Ad}_g([x, y]) = [\mathrm{Ad}_g(x), \mathrm{Ad}_g(y)] = \lambda\mu[x, y].$$

Hence $\exp(\mathbb{R}x)$ and $\exp(\mathbb{R}y)$ generate the 3-dimensional group G, which therefore fixes v_0.

So suppose now that we are in the second case where $\mathrm{Ad}_g \neq I$ is unipotent. Applying Lemma 2.7, we see that v_0 is fixed by all elements of

$$\mathrm{Im}\,(\mathrm{Ad}_g - I) \cap \ker\,(\mathrm{Ad}_g - I)\,.$$

By assumption ($g \notin C_G$ and Ad_g is unipotent) we know that this subspace is nontrivial. Therefore, there exists some $v \in \mathfrak{g} \smallsetminus \{0\}$ such that ad_v is nilpotent and v_0 is fixed by $\exp(\mathbb{R}v)$.

Now choose the isomorphism ϕ between $\mathfrak{g}$ and $\mathfrak{sl}_2(\mathbb{R})$ in such a way that v is mapped to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}).$$

This is possible by the following simple observations. If $w \in \mathfrak{sl}_2(\mathbb{C})$ has eigenvalues $\lambda \in \mathbb{C}$ (and so also $-\lambda \in \mathbb{C}$), then ad_w has eigenvalues $2\lambda, 0, -2\lambda$. Since

$$w = \phi(v) \in \mathfrak{sl}_2(\mathbb{R})$$

has the property that ad_w is nilpotent, it follows that w also has to be nilpotent. Now recall that the Jordan normal form for matrices in $\mathbb{R}^2$ shows that there is only one conjugacy class $[u]$ of non-zero elements of $\mathfrak{sl}_2(\mathbb{R})$ for which u (and also ad_u) is nilpotent. Hence composing ϕ with an appropriate conjugation gives a new ϕ with $\phi(v) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

For α and $\delta > 0$ we define

$$s_1(\alpha) = \frac{\mathrm{e}^\alpha - 1}{\delta},$$

and

$$s_2(\alpha) = \frac{-s_1(\alpha)}{1 + \delta s_1(\alpha)}$$

so that

$$\begin{pmatrix} 1 & s_1(\alpha) \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \delta & 1 \end{pmatrix} \begin{pmatrix} 1 & s_2(\alpha) \\ & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^\alpha & \\ \delta & \mathrm{e}^{-\alpha} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^\alpha & \\ & \mathrm{e}^{-\alpha} \end{pmatrix} \begin{pmatrix} 1 & \\ \mathrm{e}^\alpha\delta & 1 \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbb{R})$ by a simple calculation using (2.5). Clearly, if $\delta > 0$ and $\alpha > 0$ are chosen small enough and in that order, then the local isomorphism is defined on the matrices above. So let $g_\delta \in G$ be the element corresponding to

$$\begin{pmatrix} 1 & \\ \delta & 1 \end{pmatrix},$$

and let $h \in G$ be the element corresponding to

$$\begin{pmatrix} \mathrm{e}^{\alpha} & \\ & \mathrm{e}^{-\alpha} \end{pmatrix}.$$

We then have

$$\exp(s_1(\alpha)v) g_\delta \exp(s_2(\alpha)v) = h g_{\mathrm{e}^{\alpha}\delta} \tag{2.7}$$

as an identity in G. We wish to conjugate both sides of this expression by h^n. Note that

$$h^n \exp(sv) h^{-n} = \exp(\mathrm{e}^{2n\alpha} sv)$$

is already known to fix v_0 and that

$$h^n g_\delta h^{-n} = g_{\mathrm{e}^{-2n\alpha}\delta}$$

converges to the identity as $n \to \infty$. Therefore, conjugating (2.7) by h^n gives

$$\exp(\mathrm{e}^{2n\alpha} s_1(\alpha)v) g_{\mathrm{e}^{-2n\alpha}\delta} \exp(\mathrm{e}^{2n\alpha} s_2(\alpha)v) = h g_{\mathrm{e}^{\alpha-2n\alpha}\delta}$$

It follows that h satisfies the assumptions of Lemma 2.4, and so fixes $v_0 \in \mathcal{H}$. We are therefore reduced to the first case of the proof. □

2.3.2 Big and Small Eigenvalues

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. In this section we will show an inheritance claim, which uses the notion of horospherical algebras. The unstable and stable horospherical Lie subalgebras ($\mathfrak{g}^+$ and $\mathfrak{g}^-$ respectively) for $g \in G$ are defined as follows:

- $\mathfrak{g}^+$ is the sum of all generalized† subspaces corresponding to eigenvalues of Ad_g with absolute value bigger than one, so

$$\mathfrak{g}^+ = \{v \in \mathfrak{g} \mid \mathrm{Ad}_g^n(v) \longrightarrow 0 \text{ as } n \longrightarrow -\infty\},$$

  and
- $\mathfrak{g}^-$ is the sum of all generalized subspaces with eigenvalues of Ad_g with absolute value smaller than one, so

† Here we allow for Jordan blocks corresponding to eigenvalues of absolute value bigger than one as well as for (generalized) eigenspaces corresponding to pairs of complex eigenvalues of absolute value bigger than one.

$$\mathfrak{g}^- = \{v \in \mathfrak{g} \mid \mathrm{Ad}_g^n(v) \longrightarrow 0 \text{ as } n \longrightarrow \infty\}.$$

To see that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are subalgebras, the characterization in terms of the adjoint action is most useful. If $v_1, v_2 \in \mathfrak{g}^-$, then

$$\mathrm{Ad}_g^n(v_1 + v_2) = \mathrm{Ad}_g^n(v_1) + \mathrm{Ad}_g^n(v_2) \longrightarrow 0$$

and

$$\mathrm{Ad}_g^n\left([v_1, v_2]\right) = [\mathrm{Ad}_g^n(v_1), \mathrm{Ad}_g^n(v_2)] \longrightarrow 0$$

as $n \to \infty$, showing that $v_1 + v_2, [v_1, v_2] \in \mathfrak{g}^-$ also; the same argument (but using $n \to -\infty$) shows that $\mathfrak{g}^+$ is also a subalgebra.

Lemma 2.8 (Auslander ideal). *Let G and $\mathfrak{g}$ be as in Theorem* 2.16*, and let g be an element of G. Then the Lie algebra $\mathfrak{f} = \langle \mathfrak{g}^+, \mathfrak{g}^- \rangle$ generated by the unstable and stable horospherical Lie subalgebras of $\mathfrak{g}$ is a Lie ideal of $\mathfrak{g}$, called the* Auslander ideal *of $\mathfrak{g}$.*

Proof. The proof relies on the Jacobi identity. Let $\mathfrak{g}^0$ be the sum of the generalized eigenspaces for all eigenvalues of absolute value one, so that

$$\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^0 + \mathfrak{g}^-,$$

and we need to show that $[\mathfrak{g}, \mathfrak{f}] \subseteq \mathfrak{f}$. Since $\mathfrak{f}$ is a subalgebra by definition, it is sufficient to show that $[\mathfrak{g}^0, \mathfrak{f}] \subseteq \mathfrak{f}$. Notice first that $[\mathfrak{g}^0, \mathfrak{g}^-] \subseteq \mathfrak{g}^-$ (and similarly $[\mathfrak{g}^0, \mathfrak{g}^+] \subseteq \mathfrak{g}^+$). Indeed, if $u \in \mathfrak{g}^0$ and $v \in \mathfrak{g}^-$, then $\|\mathrm{Ad}_g^n(u)\|$ is either bounded or goes to infinity at most at a polynomial rate as $n \to \infty$, while $\|\mathrm{Ad}_g^n(v)\|$ decays to 0 at exponential speed. It follows that

$$\mathrm{Ad}_g^n\left([u, v]\right) = [\mathrm{Ad}_g^n(u), \mathrm{Ad}_g^n(v)] \longrightarrow 0$$

as $n \to \infty$, as required.

If now $u \in \mathfrak{g}^+, v \in \mathfrak{g}^-$, so that $[u, v] \in \mathfrak{f}$, then for any $w_0 \in \mathfrak{g}^0$ we have

$$[w_0, [u, v]] + \underbrace{[u, \underbrace{[v, w_0]}_{\in \mathfrak{f}}]}_{\in \mathfrak{f}} + \underbrace{[v, \underbrace{[w_0, u]}_{\in \mathfrak{f}}]}_{\in \mathfrak{f}} = 0$$

by the Jacobi identity, the case above, and the fact that $\mathfrak{f}$ is a subalgebra. It follows that $[\mathfrak{g}^0, [\mathfrak{g}^+, \mathfrak{g}^-]] \subseteq \mathfrak{f}$. Repeating the argument under the assumptions $w \in \mathfrak{g}^0$, $u, v \in \mathfrak{f}$ with $[w_0, u], [w_0, v] \in \mathfrak{f}$ we obtain $[w_0, [u, v]] \in \mathfrak{f}$. Hence $\{u \in \mathfrak{f} : [w_0, u] \in \mathfrak{f}\}$ is a subalgebra and so equals $\mathfrak{f}$. As $w_0 \in \mathfrak{g}^0$ was arbitrary, it follows that $\mathfrak{f}$ is a Lie ideal as claimed. □

Proposition 2.9 (Mautner phenomenon for the Auslander ideal). *Let G and $\mathfrak{g}$ be as in Theorem* 2.16*, and suppose that G acts unitarily on a Hilbert space $\mathcal{H}$ and that $g \in G$ fixes $v_0 \in \mathcal{H}$. Then v_0 is fixed by* $\exp \mathfrak{f}$*, where $\mathfrak{f}$ is the Auslander ideal from Lemma* 2.8*.*

Proof. Lemma 2.4 applied to $h = \exp(v)$ with $v \in \mathfrak{g}^{\pm}$ shows that $v_0 \in \mathcal{H}$ is fixed by $\exp(v)$ for $v \in \mathfrak{g}^{\pm}$. It follows that v_0 is fixed by the closed subgroup F generated by the sets $\exp(\mathfrak{g}^{+})$ and $\exp(\mathfrak{g}^{-})$. In particular, there exists a Lie subalgebra (the Lie algebra of F) containing $\mathfrak{g}^{+}$ and $\mathfrak{g}^{-}$ that fixes v_0. Since $\mathfrak{f}$ is the Lie subalgebra generated by $\mathfrak{g}^{+}$ and $\mathfrak{g}^{-}$, we deduce that every element of $\mathfrak{f}$ fixes v_0. □

2.3.3 The case of Semi-simple Lie Algebras

In this subsection we will assume that G is a connected semi-simple Lie group. To study actions of such a group, we will combine the arguments from Section 2.3.2, the Jacobson–Morozov theorem(7), and the case of $\mathrm{SL}_2(\mathbb{R})$ from Section 2.3.1. The Jacobson–Morozov theorem (we refer to Knapp [103, Sec. X.2] for the proof) is the reason that the special case $G = \mathrm{SL}_2(\mathbb{R})$ is so useful.

Theorem 2.10 (Jacobson–Morozov). *Suppose that $\mathfrak{g}$ is a real semi-simple Lie algebra, and let $x \in \mathfrak{g}$ be a nilpotent element. Then there exist elements $y, h \in \mathfrak{g}$ so that (x, y, h) form an* $\mathfrak{sl}_2$-triple, *meaning that they span a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, and in fact*

$$\begin{aligned}
[h, x] &= 2x, \\
[h, y] &= -2y, \ \textit{and} \\
[x, y] &= h.
\end{aligned}$$

It may be useful to be more explicit about Theorem 2.10 in two low-dimensional examples. In $\mathfrak{sl}_2(\mathbb{R})$ we have

$$x_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In $\mathrm{SL}_3(\mathbb{R})$ there are two (fundamentally different) choices, the first via the most obvious embedding $\mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{sl}_3(\mathbb{R})$ giving

$$x_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The second choice for $\mathrm{SL}_3(\mathbb{R})$ (which is not conjugate to the first) comes from the embedding $\mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{sl}_3(\mathbb{R})$ defined by

$$x_3 = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, y_3 = \begin{pmatrix} 0 & & \\ 2 & 0 & \\ & 2 & 0 \end{pmatrix}, h_3 = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix} = [x_3, y_3].$$

One can easily check the fundamental relations from Theorem 2.10:

$$[h_3, x_3] = 2x_3,\ [h_3, y_3] = -2y_3,\ \text{and } [x_3, y_3] = h_3.$$

Proposition 2.11 (Mautner phenomenon for semi-simple groups). *Let G be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$ which acts unitarily on a Hilbert space $\mathcal{H}$. If $g \in G$ is* Ad*-diagonalizable with positive eigenvalues or $g = \exp(x)$ for some nilpotent $x \in \mathfrak{g}$, and g fixes some vector $v_0 \in \mathcal{H}$, then there is a normal subgroup of G containing g which also fixes v_0.*

Proof. If $g = a \in G$ has the property that Ad_a is diagonalizable with positive eigenvalues, then we can split $\mathfrak{g}$ as before into three spaces

$$\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^0 + \mathfrak{g}^-,$$

where $\mathfrak{g}^0$ is the eigenspace of Ad_a with eigenvalue one. Since the Lie algebra generated by $\mathfrak{g}^+$ and $\mathfrak{g}^-$ is a Lie ideal $\mathfrak{f}$ by Lemma 2.8, $\mathfrak{f}$ is a direct sum of some of the direct simple factors of $\mathfrak{g}$. Hence it has to contain any simple factor of $\mathfrak{g}$ that intersects either of the spaces $\mathfrak{g}^+$ or $\mathfrak{g}^-$ nontrivially. Let $F_1 = \langle \exp(\mathfrak{f}) \rangle$ be the normal subgroup containing these simple factors. Since the eigenvalues of Ad_a are by assumption positive, it follows that (the linear map induced by) Ad_a acts trivially on the Lie algebra of G/F_1 (which may be identified with a sub-algebra of $\mathfrak{g}^0$). Therefore, aF_1 belongs to the center of G/F_1, and so generates a normal subgroup of G/F_1. Let $F = \langle a, F_1 \rangle$ be the pre-image in G of this normal subgroup. Then $a \in F$, F is a normal subgroup in G, and F fixes $v_0 \in \mathcal{H}$ as required.

Suppose now that $g = u = \exp(x)$ is unipotent. Then by the Jacobson–Morozov theorem there exists a connected subgroup $H < G$ locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ containing u such that x corresponds under the isomorphism to an upper nilpotent element of $\mathfrak{sl}_2(\mathbb{R})$. By the case of $\mathfrak{sl}_2(\mathbb{R})$ considered in Section 2.3.1, we see that H fixes v_0. Since H also contains the image of

$$a = \begin{pmatrix} \mathrm{e}^{\alpha} & \\ & \mathrm{e}^{-\alpha} \end{pmatrix}$$

for small $\alpha > 0$ (under the local isomorphism), we have produced the situation of the first case, which was considered above. Let F be again the normal subgroup corresponding to a. Now recall that

$$u = \exp(x) \in \exp(\mathfrak{g}^+)$$

if $\mathfrak{g}^+$ is defined using the element in H corresponding to $a \in \mathrm{SL}_2(\mathbb{R})$. Therefore, it follows from the above that $g \in F$ once again. □

Exercises for Section 2.3

Exercise 2.3.1. Let $a \in G = \mathrm{SL}_d(\mathbb{R})$ be a diagonal matrix such that

$$G_a^{\pm} = \{u \in G \mid a^n u a^{-n} \to I \text{ as } n \to \mp\infty\}$$

are nontrivial subgroups. Show directly that $\langle G_a^+, G_a^- \rangle = G$.

Exercise 2.3.2. Show that $\mathfrak{g}^0$ from the proof of Lemma 2.8 is a Lie subalgebra.

Exercise 2.3.3. Let G be a simple Lie group and let $\Gamma < G$ be a lattice. Let $a \in G$ and recall that the Lie algebra of G splits as a direct sum $\mathfrak{g}^+ + \mathfrak{g}^0 + \mathfrak{g}^-$ as in the proof of Lemma 2.8. Assume that Ad_a is diagonalizable when restricted to $\mathfrak{g}^0$ and that 1 is the only eigenvalue of this restriction (so that $\mathfrak{g}^0$ is the Lie algebra of $C_G(a) = \{g \in G \mid ag = ga\}$). Using the pointwise ergodic theorem (Theorem 2.1) show that for any $x \in X = \Gamma\backslash G$ and $m_{G_a^+}$-almost every $u \in G_a^+$ the forward orbit $\{a^n \cdot (u \cdot x) : n \geqslant 0\}$ of $u \cdot x$ equidistributes[†] in X with respect to the Haar measure m_X.

Exercise 2.3.4. Prove Proposition 2.5 (and hence Theorem 2.14) for the case of $\mathrm{SL}_d(\mathbb{R})$ for $d = 3$ or more generally for $d \geqslant 3$, either directly by a similar argument or using the case $\mathrm{SL}_2(\mathbb{R})$ considered above.

Exercise 2.3.5. Prove the analogue of Proposition 2.5 for the case $\mathrm{SL}_2(\mathbb{Q}_p)$ (or for the case $\mathrm{SL}_d(\mathbb{Q}_p)$ for $d \geqslant 2$), where $\mathbb{Q}_p$ is the field of p-adic rational numbers. More precisely show that $\mathrm{SL}_2(\mathbb{Q}_p)$ fixes $v_0 \in \mathcal{H}$ if $\mathrm{SL}_2(\mathbb{Q}_p)$ acts unitarily on $\mathcal{H}$ and either
(a) v_0 is fixed by some diagonal element with at least one eigenvalue of absolute value not equal to one, or
(b) v_0 is fixed by a one-parameter[‡] unipotent subgroup $\{I + sw \mid s \in \mathbb{Q}_p\}$ defined by some nilpotent $w \in \mathrm{Mat}_{2,2}(\mathbb{Q}_p)$.

Exercise 2.3.6. Prove Theorem 2.15 for $\mathrm{SL}_2(\mathbb{Q}_p)$ (or for $\mathrm{SL}_d(\mathbb{Q}_p)$ for $d \geqslant 2$), where $\mathbb{Q}_p$ is the field of p-adic rational numbers, using Exercise 2.3.5 in place of Theorem 2.14. For the analogous KAK-decomposition of $\mathrm{SL}_d(\mathbb{Q}_p)$ set $K = \mathrm{SL}_d(\mathbb{Z}_p)$ and let A consist of all diagonal matrices whose diagonal entries are integer powers of p.

Exercise 2.3.7. Prove (direclty or using Exercise 2.3.6) that an unbounded open subgroup $H < \mathrm{SL}_d(\mathbb{Q}_p)$ necessarily equals $\mathrm{SL}_d(\mathbb{Q}_p)$.

2.4 The Howe–Moore Theorem

The main goal of this chapter is to relate the algebraic properties of G to properties of its measure-preserving actions, by showing that for certain Lie groups ergodicity forces mixing (in contrast to the abelian case, where, for example, an ergodic action of $\mathbb{Z}^2$ could have no ergodic elements).

[†] We note that the results of this section and Remark 2.2 (3) immediately show that the forward orbit is equidistributed for m_X-almost every $x \in X$, but the desired statement is stronger as it involves a Haar measure on a subgroup.

[‡] We note that in this p-adic case a single element of this subgroup generates a compact subgroup and so could not satisfy the Mautner phenomenon.

Theorem 2.12 (Howe–Moore, automatic mixing). *An ergodic and measure-preserving action on a probability space by a simple connected Lie group G with finite center is mixing.*

The assumption that the center be finite is necessary. If $G = \widetilde{\mathrm{SL}_2(\mathbb{R})}$ is the universal cover of $\mathrm{SL}_2(\mathbb{R})$, then there are ergodic actions of G on non-trivial probability spaces in which the infinite center (which is isomorphic to $\mathbb{Z}$) acts trivially (as for example the action of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ induced by the natural ergodic action of $\mathrm{SL}_2(\mathbb{R})$ on X_2).

A more general formulation expresses this result in terms of *vanishing of matrix coefficients at infinity* in the associated unitary representations. Here a *unitary representation* is an action $\pi : G \times \mathcal{H} \to \mathcal{H}$ by unitary maps $\pi(g)$ for $g \in G$ such that for any given $v \in \mathcal{H}$ the map $G \ni g \mapsto \pi(g)v$ is continuous (with respect to the given topology on G and the norm topology on $\mathcal{H}$). Given a continuous action of a metric locally compact group G on a locally compact metric space X and a locally finite measure μ on X that is preserved by the action, the associated unitary representation

$$\pi(g)(f) = f \circ g^{-1}$$

for $f \in \mathcal{H} = L^2(X, \mu)$ indeed satisfies this continuity property (this may be seen, for example, in [53, Lem. 8.7] or [54, Lem. 3.74]).

Theorem 2.13 (Howe–Moore, vanishing of matrix coefficients). *If a simple connected Lie group G with finite center acts unitarily on a Hilbert space $\mathcal{H}$, and the action has no non-trivial fixed vectors, then the associated matrix coefficients vanish at infinity in the sense that*

$$\langle \pi(g)v, w \rangle \longrightarrow 0$$

as $g \to \infty$ in G for any $v, w \in \mathcal{H}$.

One of the most important ingredients in the proof of the Howe–Moore theorem is the following weaker statement, which says that ergodicity of a G-action is inherited by unbounded subgroups of simple groups(8). As mentioned earlier, this is far from true in the setting of abelian groups (see Exercise 2.4.1).

Theorem 2.14 (Mautner phenomenon for simple groups). *Let G be a simple connected Lie group with finite center acting unitarily on a Hilbert space $\mathcal{H}$. If $g \in G$ does not belong† to a compact subgroup of G, and $v \in \mathcal{H}$ is fixed under the action of g, then v is fixed under the action of G.*

† Equivalently, if $g^n \to \infty$ in G as $n \to \infty$.

2.4.1 A More General Howe–Moore Theorem and its Proof

Assuming the Mautner phenomenon in Theorem 2.14 for simple groups with finite center, we will deduce the following general version of the Howe–Moore theorem on vanishing of matrix coefficients. In order to state the theorem, we will use† the terminology and results from Section 2.2.4.

Theorem 2.15 (Howe–Moore for semi-simple groups). *Let G be a semi-simple Lie group with finite center, and let $\pi : G \times \mathcal{H} \to \mathcal{H}$ be a unitary representation on a Hilbert space $\mathcal{H}$. For v_1, v_2 in $\mathcal{H}$ we have*

$$\langle \pi(g_n)v_1, v_2\rangle \longrightarrow 0 \tag{2.8}$$

as $n \to \infty$ in either of the following two situations:

(1) *For any of the simple non-compact factors G_i of G, there are no non-trivial G_i-fixed vectors in $\mathcal{H}$ and $g_n \to \infty$ as $n \to \infty$.*
(2) *$\mathcal{H}$ has no non-trivial G^+-fixed vectors, $g_n = g_n^{(1)} \cdots g_n^{(r)}$ with $g_n^{(i)} \in G_i$, and $g_n^{(i)} \to \infty$ as $n \to \infty$ for each simple non-compact factor‡ $G_i \subseteq G^+$ of G.*

In the proof of Theorem 2.15 we will make use of the general Cartan decomposition for semi-simple Lie groups with finite center, also known as the KAK decomposition (the existence of this decomposition with K compact is where the essential hypothesis that G has finite center enters the argument). Here $K < G$ is a maximal compact subgroup and $A < G$ is a Cartan subgroup§. For the case $G = \mathrm{SL}_d(\mathbb{R})$ this decomposition is easy to exhibit, as in this case $K = \mathrm{SO}(d)$, A is the subgroup of diagonal matrices with positive entries down the diagonal, and every matrix $g \in \mathrm{SL}_d(\mathbb{R})$ can be written in the form $g = ka\ell$ with $k, \ell \in K$ and $a \in A$ (see Exercise 2.4.2). We refer to Knapp [103, Sec. VII.3] or [102] for the proof in the general case.

Proof of Theorem 2.15 (Theorems 2.12–2.13) assuming Theorem 2.14. Assume that $g_n \to \infty$ in G as $n \to \infty$. We will show (2.8) by showing that there always exists a subsequence for which (2.8) holds.

This suffices by a simple indirect argument. Assume (2.8) does not hold. Then there exists some $\varepsilon > 0$ and some subsequence (n_k) with

$$|\langle \pi(g_{n_k})v_1, v_2\rangle| \geqslant \varepsilon$$

† This is only needed because we state the theorem in greater generality. At its core the argument only needs basic functional analysis, see Exercise 2.4.3.

‡ Even though the decomposition of g_n into $g_n^{(1)} \cdots g_n^{(r)}$ with $g_n^{(i)} \in G_i$ is not unique, the requirement that $g_n^{(i)} \to \infty$ as $n \to \infty$ does make sense as the ambiguity in the decomposition is only up to the finite center of G.

§ Recall that a Cartan subgroup A is a maximal abelian connected subgroup of G for which Ad_a is $\mathbb{R}$-diagonalizable for all $a \in A$.

for all $k \geqslant 1$. However, applying the above claim to this subsequence we find a subsequence of (n_k) for which (2.8) holds — a contradiction to the choice of (n_k).

By passing to a subsequence if necessary (and dropping the resulting subscript for convenience), we may also assume that

$$v^* = \lim_{n\to\infty} \pi(a_n)v_1 \in \mathcal{H}$$

exists in the weak*-topology by the Banach–Alaoglu theorem, since

$$\|\pi(a_n)v_1\| = \|v_1\|$$

by unitarity. The claim in (2.8) (for this subsequence and any $v_2 \in \mathcal{H}$) is the statement $v^* = 0$.

Let us explain the main step first in the case of $G = \mathrm{SL}_d(\mathbb{R})$ and a sequence of diagonal elements $(g_n = a_n)$. Since $a_n \to \infty$ as $n \to \infty$, we can choose a subsequence and find some nontrivial element u of the elementary unipotent subgroups appearing in Lemma 1.24 such that

$$a_n^{-1} u a_n \to e \tag{2.9}$$

as $n \to \infty$. We claim that this implies that

$$\pi(u)v^* = v^*. \tag{2.10}$$

To prove the claim, let $w \in \mathcal{H}$ be any element. Then

$$\begin{aligned}\langle \pi(u)v^*, w\rangle &= \left\langle v^*, \pi(u)^{-1}w\right\rangle \\ &= \lim_{n\to\infty} \left\langle \pi(a_n)v_1, \pi(u^{-1})w\right\rangle \\ &= \lim_{n\to\infty} \left\langle \pi(a_n^{-1}ua_n)v_1, \pi(a_n^{-1})w\right\rangle.\end{aligned}$$

However,

$$\lim_{n\to\infty} \|\pi(a_n^{-1}ua_n)v_1 - v_1\| = 0,$$

so

$$\begin{aligned}\langle \pi(u)v^*, w\rangle &= \lim_{n\to\infty} \left\langle \pi(a_n^{-1}ua_n)v_1, \pi(a_n^{-1})w\right\rangle \\ &= \lim_{n\to\infty} \left\langle v_1, \pi(a_n^{-1})w\right\rangle \\ &= \lim_{n\to\infty} \langle \pi(a_n)v, w\rangle = \langle v^*, w\rangle.\end{aligned}$$

However, this implies that $\pi(u)v^* = v^*$, i.e. (2.9) implies (2.10) as claimed.

The theorem now follows in the case of $G = \mathrm{SL}_d(\mathbb{R})$ from the claim. Indeed by the Mautner phenomenon (Theorem 2.14) and the assumption that there are no nontrivial fixed vectors we see that $v^* = 0$, which as explained above implies (2.8).

We return to the general case and claim that it sufficies to consider the case of a sequence $g_n = a_n$ belonging to the Cartan subgroup of G. (With the above this will conclude the case of $G = \mathrm{SL}_d(\mathbb{R})$.) In fact using the Cartan decomposition of G we may write the terms of any sequence $g_n \to \infty$ as $n \to \infty$ in the form

$$g_n = k_n a_n \ell_n$$

with $k_n, \ell_n \in K$ for all $n \geqslant 1$ with $a_n \to \infty$ as $n \to \infty$ in $A < G$. Since K is compact and the representation is continuous the study of $\langle \pi(k_n a_n \ell_n) v_1, v_2 \rangle$ can be reduced to the study of $\langle \pi(k a_n \ell) v_1, v_2 \rangle$ for some $k, \ell \in K$. Indeed by choosing a subsequence we may assume $k_n \to k$ and $\ell_n \to \ell$ as $n \to \infty$. Now we apply continuity of the representation and the Cauchy-Schwartz inequality to see that

$$\begin{aligned}
\big|\langle \pi(k_n a_n \ell_n)\, v_1, v_2\rangle - \langle \pi(k a_n \ell)\, v_1, v_2\rangle\big| \\
&\leqslant \big|\langle \pi(a_n \ell_n)\, v_1, \pi(k_n)^* v_2\rangle - \langle \pi(a_n \ell)\, v_1, \pi(k_n)^* v_2\rangle\big| \\
&\quad + \big|\langle \pi(a_n \ell)\, v_1, \pi(k_n)^* v_2\rangle - \langle \pi(a_n \ell)\, v_1, \pi(k)^* v_2\rangle\big| \\
&\leqslant \big\|\pi(\ell_n)\, v_1 - \pi(\ell)\, v_1\big\| \big\|v_2\big\| + \big\|v_1\big\| \big\|\pi(k_n)^* v_2 - \pi(k) v_2\big\|
\end{aligned}$$

which goes to 0 as $n \to \infty$.

Recall that $a_n \in A < G$ is the product $a_n = a_n^{(1)} \cdots a_n^{(r)}$ with $a_n^{(i)} \in G_i$ for $i = 1, \ldots, r$. We claim that v^* is fixed under a non-trivial unipotent† element of every factor G_i of G (with respect to the action of π on $\mathcal{H}$) for which $a_n^{(i)} \to \infty$ as $n \to \infty$, for each $i = 1, \ldots, r$. This claim implies the theorem via the Mautner phenomenon (Theorem 2.14): the vector v^* is fixed under all almost direct factors G_i of G for which $a_n^{(i)} \to \infty$. In both case (1) and case (2), this implies that $v^* = 0$, and hence the theorem.

Then since $a_n^{(i)} \to \infty$ as $n \to \infty$ by assumption on G_i, we may choose a subsequence so that there exists a non-trivial unipotent element $u \in G_i$ with

$$(a_n^{(i)})^{-1} u (a_n^{(i)}) \to e$$

as $n \to \infty$. In fact by the structure of simple groups, u can be chosen as an element of one of the restricted root subgroups. However, we have already shown that (2.9) implies (2.10) and so the theorem follows. □

† If $G \leqslant \mathrm{SL}_d(\mathbb{R})$ is a linear group, then $u \in G$ is unipotent if 1 is the only eigenvalue of u. In general we say that $u \in G$ is unipotent if $\mathrm{Ad}_u \in \mathrm{SL}(\mathfrak{g})$ is unipotent — this is often referred to as being Ad-unipotent.

Exercises for Section 2.4

Exercise 2.4.1. (a) Let $G = \mathbb{Z}^d$ with $d \geqslant 2$. Find an ergodic action of G with the property that no subgroup of G with lower rank acts ergodically.
(b) Let $G = \mathbb{R}^d$ with $d \geqslant 1$. Prove that in any ergodic action of G almost every element of $\mathbb{R}^d$ acts ergodically. (This relies on the standing assumptions regarding X, which imply in particular that $L^2(X)$ is separable.)

Exercise 2.4.2. Prove that every element of $\mathrm{SL}_d(\mathbb{R})$ can be written in the form $ka\ell$ as claimed on page 65.

Exercise 2.4.3. Extract from the general proof of Theorem 2.15 above the special case of $\mathrm{SL}_2(\mathbb{R})$ (or $\mathrm{SL}_d(\mathbb{R})$ for $d \geqslant 2$), still assuming Theorem 2.14 for this (or these) groups.

Exercise 2.4.4. Let G be a simple Lie group, let $L < G$ be an unbounded subgroup, let π be a unitary representation of G on a Hilbert space $\mathcal{H}$, and let $v_0 \in \mathcal{H}$ be a vector fixed by all elements of L. Show that v_0 is then also fixed by all elements of G.

2.5 The General Mautner Phenomenon

[†]We will now consider the general Mautner phenomena, which was proven by Moore [137] in 1980.

Theorem 2.16 (Mautner phenomenon). *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $L < G$ be a closed subgroup, and suppose that G acts unitarily on a Hilbert space $\mathcal{H}$ with a non-zero vector v_0 fixed by every element of L. Then there exists a Lie ideal $\mathfrak{f} \lhd \mathfrak{g}$ (the* Mautner ideal*) such that*

- *v_0 is fixed by $\exp(\mathfrak{f}) \leqslant G$ and*
- *the map $A_g \colon \mathfrak{g}/\mathfrak{f} \to \mathfrak{g}/\mathfrak{f}$ induced by Ad_g for $g \in L$ is diagonalizable with all eigenvalues of absolute value one.*

The proof of Theorem 2.16 will combine the key lemma (Lemma 2.4), the special case of $\mathrm{SL}_2(\mathbb{R})$ from Section 2.3.1, and techniques from the theories of Lie groups and Lie algebras. It subsumes the ergodicity of many natural actions.

2.5.1 The Structure of the Inductive Steps

[‡]For the solvable and then the general case below, we would like to use an induction process to be outlined in this section. For this, notice first that in

[†] The material of this section will not be used later.

[‡] As the following proof will show, semi-simple groups are easier to work with and are, fortunately, sufficient for many purposes. For this reason the reader may initially skip the remainder of Chapter 2 and return to it when she needs it.

proving Theorem 2.16 we may assume that v_0 is a *cyclic vector* in the sense that

$$\mathcal{H} = \overline{\langle \pi(G)v_0 \rangle}$$

is the smallest closed subspace containing the orbit of v_0 under the action of G, since if this is not the case we may simply restrict the unitary representation to this subspace.

This remark allow us to use induction on the dimension of G. In the inductive steps we will show that there is a non-trivial Lie ideal $\mathfrak{f} \lhd \mathfrak{g}$ that fixes v_0. Taking exponentials gives a normal subgroup $F \lhd G$ generated by $\exp(\mathfrak{f})$. Let $\bar{F}$ be the closure of F (*a priori* there is no reason for F to be closed), so that $\bar{F} \lhd G$ is a closed normal subgroup that fixes v_0. We claim that $\bar{F}$ acts trivially on $\mathcal{H}$ since $\mathcal{H}$ is the closure of the orbit of v_0. Indeed, if $g \in G$ and $h \in F$ then $hg = gh'$ for some $h' \in F$, and

$$\pi(h)\pi(g)v_0 = \pi(g)\pi(h')v_0 = \pi(g)v_0$$

and since $\mathcal{H} = \overline{\langle \pi(G)v_0 \rangle}$ we see that both F and $\bar{F}$ act trivially. Therefore we may consider the unitary representation of $G/\bar{F}$ on $\mathcal{H}$ induced by the unitary representation of G that we started with. If $\mathfrak{f} \lhd \mathfrak{g}$ was a non-trivial Lie ideal, then the dimension of $\widetilde{G} = G/F$ is smaller.

By induction we may assume that Theorem 2.16 already holds for $\widetilde{G}$ (with the subgroup $\widetilde{L} = LF/F < \widetilde{G} = G/F$) acting on $\mathcal{H}$. This in turn then implies the theorem also for G.

2.5.2 The Inductive Step for Elements in the Radical

Recall from Section 2.2.3 that a real Lie algebra $\mathfrak{g}$ has a Levi decomposition[(9)]

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$$

where $\mathfrak{l}$ is a semi-simple real Lie algebra, and $\mathfrak{r} \lhd \mathfrak{g}$ is the radical (the maximal solvable Lie ideal of $\mathfrak{g}$). Also recall from Knapp [103, Prop. 1.40] that

$$\mathfrak{n} = [\mathfrak{r}, \mathfrak{g}] \lhd \mathfrak{g}$$

is a nilpotent Lie ideal. Using this we can prove the following part of the inductive step.

Proposition 2.17 (Mautner phenomenon for nilpotent elements of the radical). *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let π be a unitary representation on the Hilbert space $\mathcal{H}$. Suppose that $v_0 \in \mathcal{H}$ is a cyclic vector as in Section* 2.5.1. *Suppose there is a nilpotent element $u \in \mathfrak{r} \smallsetminus \{0\}$ (with $\mathrm{Ad}_{\exp(u)}$ unipotent) in the radical of the Lie algebra that fixes v_0. Then there is a non-trivial Lie ideal $\mathfrak{f} \lhd \mathfrak{g}$ that fixes v_0.*

This proposition shows that in the situation above we can always apply the inductive step outlined in Section 2.5.1, so that in particular we can also conclude from the inductive argument that there exists an $\mathfrak{f}$ as in the proposition containing u which fixes v_0.

PROOF OF PROPOSITION 2.17. Suppose as in the statement of the proposition that the nilpotent element $u \in \mathfrak{r} \smallsetminus \{0\}$ fixes v_0. If u lies in the center of $\mathfrak{g}$ (that is, if $[u, \mathfrak{g}] = 0$), then we can take $\mathfrak{f} = \mathbb{R}u$. Otherwise we claim that we may use Lemma 2.7 finitely many times to find vectors $v_1, \dots, v_\ell$ that all fix v_0 and such that $v_2, \dots, v_\ell \in \mathfrak{n} = [\mathfrak{r}, \mathfrak{g}]$ and such that the last vector $v_\ell \neq 0$ lies in the center of $\mathfrak{n}$. Initially set $v_1 = u$. Whenever $[v_j, \mathfrak{n}] \neq 0$ for $j \geqslant 1$ then we may take some $w_j \in \mathfrak{n}$ with

$$v_{j+1} = [v_j, w_j] \neq 0$$

and

$$[v_j, v_{j+1}] = 0.$$

This is possible because v_j (that is, ad_{v_j}) is nilpotent, by assumption for $j = 1$ and also for $j \geqslant 2$ since in that case $v_j \in \mathfrak{n}$. Hence by Lemma 2.7 and induction, v_{j+1} fixes v_0. Clearly by construction

$$v_1 = u \in \mathfrak{r}, v_2 \in [\mathfrak{r}, \mathfrak{n}] \subseteq \mathfrak{n}, v_3 \in [\mathfrak{n}, \mathfrak{n}], \dots.$$

Since $\mathfrak{n}$ is a nilpotent Lie algebra, this sequence stops with $v_\ell \in \mathfrak{n}$ and

$$[v_\ell, \mathfrak{n}] = 0$$

for some ℓ.

Let

$$\mathfrak{c} = \{w \in \mathfrak{n} \mid [w, \mathfrak{n}] = 0\}$$

be the center of $\mathfrak{n}$. Since $\mathfrak{n} \lhd \mathfrak{g}$, this is an abelian Lie ideal of $\mathfrak{g}$. We will define $\mathfrak{f}$ as a subspace of $\mathfrak{c}$ containing v_ℓ; indeed we define $\mathfrak{f}$ to be the Lie ideal of $\mathfrak{g}$ generated by v_ℓ. It remains to show that $\mathfrak{f}$ fixes v_0, and this follows as before: If $w \in \mathfrak{g}$ and we have some $v \in \mathfrak{f}$ that fixes v_0, then $[v, w] \in \mathfrak{f}$ also fixes v_0, because $[v, w] \in \mathfrak{c}$, $[v, [v, w]] = 0$ and we may apply Lemma 2.7 as before. As the Lie ideal $\mathfrak{f}$ generated by v_ℓ is obtained by taking the sum of $\mathbb{R}v_\ell$, $[v_\ell, \mathfrak{g}]$, $[[v_\ell, \mathfrak{g}], \mathfrak{g}], \dots$, the proposition follows. □

2.5.3 The General Case of Theorem 2.16

Let $G, \pi, \mathcal{H}, L, v_0$ be as in Theorem 2.16, and suppose that the allowed assumptions of Section 2.5.1 are satisfied.

Let $g \in L$. If Ad_g has an eigenvalue of absolute value greater than or smaller than 1, then we may apply Section 2.3.2 to find the non-trivial Auslander ideal that fixes v_0, and use induction. Suppose therefore that all the eigenvalues of Ad_g have absolute value equal to 1, but that Ad_g is not diagonalizable over $\mathbb{C}$ (since in that case the theorem already holds trivially for g). Then there exist two vectors $v, w \in \mathfrak{g}$ with

$$\mathrm{Ad}_g(v) = \lambda v,$$
$$\mathrm{Ad}_g(w) = \lambda(w + v),$$

and so for $n \in \mathbb{N}$,

$$\mathrm{Ad}_g^n\,(w) = \lambda^n\,(w + nv)\,. \tag{2.11}$$

These expressions have the obvious meaning if $\lambda \in \{\pm 1\}$, but if $\lambda \in \mathbb{S}^1 \smallsetminus \{\pm 1\}$ then we are using the symbol λ as a convenient shorthand for a rotation of the real linear space corresponding to a complex eigenvalue. In any case, there is a sequence (n_k) with $n_k \to \infty$ as $k \to \infty$ along which λ^{n_k} converges to the identity. Using this sequence we can divide (2.11) by n_k and find

$$\lim_{k\to\infty} \mathrm{Ad}_g^{n_k}\left(\tfrac{1}{n_k} w\right) = v. \tag{2.12}$$

We apply the arguments from Lemma 2.7 again to conclude that $v \in \mathfrak{g}$ fixes v_0 and that ad_v is unipotent.

Now let $\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$ be the Levi decomposition. If $v \in \mathfrak{r}$, then we can apply the argument from Proposition 2.17. Thus we may assume that

$$v = x + v_{\mathfrak{r}}$$

with $x \in \mathfrak{l} \smallsetminus \{0\}$ and $v_{\mathfrak{r}} \in \mathfrak{r}$. We note that $x \in \mathfrak{l} \smallsetminus \{0\}$ is a nilpotent element of the semi-simple Lie algebra $\mathfrak{l}$ (because, for example, the adjoint of x on $\mathfrak{l} \cong \mathfrak{g}/\mathfrak{r}$ coincides with the adjoint of v on $\mathfrak{g}/\mathfrak{r}$). Furthermore, we claim that $v_{\mathfrak{r}}$ lies in $\mathfrak{n}$ and so is also nilpotent. This follows from the construction of v. Indeed, since $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{n} < \mathfrak{r}$ it follows that $\mathfrak{g}/\mathfrak{n} \cong \mathfrak{l} + \mathfrak{r}/\mathfrak{n}$ is a direct sum of Lie algebras and that $\mathfrak{r}/\mathfrak{n}$ is in the center of $\mathfrak{g}/\mathfrak{n}$. Therefore, the action of Ad_g is trivial on $\mathfrak{r}/\mathfrak{n}$ for any $g \in G$. Splitting $w + \mathfrak{n}$ into its components in $\mathfrak{l}$ and in $\mathfrak{r}/\mathfrak{n}$, the definition of v in (2.12) shows that $v_{\mathfrak{r}} \in \mathfrak{n}$.

Knowing that $x \in \mathfrak{l}$ is nilpotent and nontrivial, we may apply the Jacobson–Morozov theorem (Theorem 2.10) and choose an $\mathfrak{sl}_2$-triple (x, y, h) in $\mathfrak{l}^3$.

Note that if we would have $v = x$ then we could apply the already established semi-simple case (see below). Our aim is therefore to always reduce the proof via induction to this case.

If $[v, \mathfrak{r}] \neq 0$ then we can apply Lemma 2.7 once again to find a non-trivial element of $\mathfrak{n}$ fixing v_0 after which we may apply Proposition 2.17.

So assume that $[v, \mathfrak{r}] = 0$. Then we have

$$\begin{aligned}[v,h] &= [x,h] + [v_{\mathfrak{r}},h] && (\text{since } v = x + v_{\mathfrak{r}})\\ &= -2x + [v_{\mathfrak{r}},h] && (\text{since } [h,x] = 2x)\end{aligned}$$

and so

$$\begin{aligned}[v,[v,h]] &= [v,-2x+[v_{\mathfrak{r}},h]]\\ &= [v,-2x]+0 && (\text{since } [v_{\mathfrak{r}},h]\in\mathfrak{r})\\ &= [x+v_{\mathfrak{r}},-2x]\\ &= -2[v_{\mathfrak{r}},x]\in\mathfrak{r}.\end{aligned}$$

Furthermore,

$$[v,[v,[v,h]]] = [v,\underbrace{-2[v_{\mathfrak{r}},x]}_{\in\mathfrak{r}}] = 0.$$

Hence we may apply Lemma 2.7 if $[v,[v,h]] \neq 0$, and then use Proposition 2.17 and induction. So assume that $[v,[v,h]] = 0$. By Lemma 2.7, $[v,h]$ fixes v_0. If $[v,h] \neq -2v$, then (recall that $v = x + v_{\mathfrak{r}}$)

$$[v,h] + 2v = -2x + [v_{\mathfrak{r}},h] + 2x + 2v_{\mathfrak{r}} \neq 0$$

belongs to $\mathfrak{n}$ and fixes[†] v_0, so we may apply Proposition 2.17 and induction.

So assume now that $[v,h] = -2v$. We claim that this implies (only using structure theory of Lie groups) that $v_{\mathfrak{r}} = 0$, so that $v = x$ is a member of an $\mathfrak{sl}_2$-triple inside $\mathfrak{l}$.

So assume (for the purposes of obtaining a contradiction) that $v_{\mathfrak{r}} \neq 0$. Also recall that $v_{\mathfrak{r}} \in \mathfrak{n}$. There exists a Lie ideal $\mathfrak{f} = \mathfrak{n}_i \lhd \mathfrak{g}$ from the lower central series as in Section 2.2.3 with $v_{\mathfrak{r}} \notin \mathfrak{f}$ but $v_{\mathfrak{r}} \in \mathfrak{n}_{i-1}$ so that $[v_{\mathfrak{r}},\mathfrak{n}] \subseteq \mathfrak{f} = \mathfrak{n}_i$. Thus $\mathfrak{g}$ acts on $\mathfrak{n}/\mathfrak{f}$, both v (since $[v,\mathfrak{r}] = 0$ by one of our allowed assumptions from above) and $v_{\mathfrak{r}}$ (by construction of $\mathfrak{f}$) act trivially on $\mathfrak{n}/\mathfrak{f}$, and so x also acts trivially on $\mathfrak{n}/\mathfrak{f}$. However, if x acts trivially on the whole space there must be a Lie ideal in $\mathfrak{g}$ (the kernel of the representation) which acts trivially. Therefore, we see that h acts trivially on $\mathfrak{n}/\mathfrak{f}$ and so

$$[h,v] = 2v = 2x + 2v_{\mathfrak{r}} = [h,x] + [h,v_{\mathfrak{r}}] \in 2x + \mathfrak{f}$$

gives the contradiction $v_{\mathfrak{r}} \in \mathfrak{f}$. Therefore, $v_{\mathfrak{r}} = 0$ as claimed.

To finish the proof we wish to apply Proposition 2.11. Since $v = x$ fixes v_0, there exists a Lie ideal $\mathfrak{h} \lhd \mathfrak{l}$ containing v that fixes v_0. As the proof of Proposition 2.11 shows, $\mathfrak{h}$ is the Auslander ideal of $a = \exp(h) \in G$ inside $\mathfrak{l}$ and contains h. By Proposition 2.9 the non-trivial Auslander ideal $\mathfrak{f} \lhd \mathfrak{g}$

[†] Since addition in the Lie algebra and taking products in the Lie group are not quite the same we should add some explanations for this step. Actually in this proof we even know that all elements of $\mathbb{R}v$ and $\mathbb{R}[v,h]$ fix v_0. Hence both $[v,h]$ and v belong to the Lie algebra of the subgroup that fixes v_0, which implies that also $[v,h]+2v$ fixes v_0.

defined by a within $\mathfrak{g}$ also fixes v_0, contains $\mathfrak{h}$, and so also v. This concludes the induction and hence also the proof of Theorem 2.16.

Notes to Chapter 2

(5)(Page 46) The main result here is due to Lindenstrauss [121], who showed that any locally compact amenable group has a Følner sequence along which the pointwise ergodic theorem holds. We refer to a survey of Nevo [141] for an overview of both the amenable case and the case of certain non-amenable groups, and to [53, Ch. 8] for an accessible discussion of the case of groups with polynomial growth.

(6)(Page 53) This argument comes from Margulis [127], and the argument is also presented in [53, Prop. 11.18].

(7)(Page 61) Theorem 2.10 was stated by Morozov [138] and a complete proof was provided by Jacobson [86].

(8)(Page 64) The Mautner phenomenon was developed for the study of geodesic flows on symmetric spaces by Mautner [130] and has been significantly extended since then, notably by Moore [137].

(9)(Page 69) This decomposition, conjectured by Killing and Cartan, was shown by Levi [118], and Malcev [123] later showed that any two Levi factors (the semi-simple Lie algebra viewed as a factor-algebra of $\mathfrak{g}$) are conjugate by a specific form of inner automorphism; we refer to Knapp [103, Th. B.2] for the proof.