Chapter 2
Ergodicity and Mixing on Locally Homogeneous Spaces

In this chapter we review some important notions and discuss ergodicity and mixing for actions of Lie groups.

2.1 Basic Notions in Ergodic Theory

Throughout, we will assume that an acting group G is σ-compact, locally compact, and metrizable. Moreover, we will assume that X, the space G acts on, is a σ-compact locally compact metric space, and that the action is jointly continuous (also see [52, Sec. 8] for more background). Such an action is said to be

- *measure-preserving* with respect to a measure μ on X if
$$\mu(g^{-1}\bullet B) = \mu(B)$$
for any $g \in G$ and measurable set $B \subseteq X$, in which case we also say that μ is *invariant*;
- *ergodic* with respect to a probability measure μ if any measurable $B \subseteq X$ with the property $\mu(g^{-1}\bullet B \triangle B) = 0$ for all $g \in G$ has $\mu(B) \in \{0, 1\}$; and
- *mixing* with respect to a probability measure μ if
$$\mu(g^{-1}\bullet A \cap B) \longrightarrow \mu(A)\mu(B)$$
as $g \to \infty$ in G for any measurable sets $A, B \subseteq X$.

Here the notation $g \to \infty$ is shorthand for elements g of G running through a sequence $(g_n)_{n\geqslant 1}$ with the property that for any compact set $K \subseteq G$ there is an $N = N(K)$ such that $n \geqslant N(K)$ implies $g_n \notin K$. Notice that the property of mixing (of non-compact groups) is much stronger than ergodicity in the following sense. Mixing for the action implies that each element $g \in G$ with $g^n \to \infty$ as $n \to \infty$ is itself a mixing (and ergodic) transformation in the usual sense (where the acting group is a copy of $\mathbb{Z}$), while ergodicity *a priori* does not tell us anything at all about properties of the action of individual elements of G (see Exercise 2.14).

We note that the transitive action of G on a quotient $X = G/\Gamma$ by a lattice is always ergodic. In fact this follows from the definition and the following more general result from [52, Prop. 8.3].

Proposition 2.1. *Let G be a σ-compact metric group acting continuously on a compact metric space X preserving a measure $\mu \in \mathcal{M}(X)$. Then for $B \in \mathcal{B}$ the following properties are equivalent:*

(1) *B is invariant in the sense that $\mu(g{\bullet}B \triangle B) = 0$ for all $g \in G$;*
(2) *B is invariant in the sense that there is a set $B' \in \mathcal{B}$ with $\mu(B \triangle B') = 0$ and with $g{\bullet}B' = B'$ for all $g \in G$.*

We will now recall also that ergodicity and mixing are *spectral properties* in the sense that they can be phrased in terms of the associated Koopman representation. The latter is the unitary representation π of G defined by

$$\pi_g f = f \circ g^{-1}$$

for $f \in L^2(X, \mu)$ and $g \in G$. In particular, π has the following natural continuity property: Given a function $f \in L^2(X, \mu)$ the map $G \ni g \mapsto \pi_g f \in L^2(X, \mu)$ is continuous (with respect to the given topology on G and the norm topology on $L^2(X, \mu)$); see Exercise 2.3, [52, Def. 11.16 and Lem. 11.17], or [53, Lem. 3.74].

Assuming the action is measure-preserving for a probability measure μ on X, then:

- The G-action is *ergodic* if and only if the constant functions are the only eigenfunction for the representation with eigenvalue 1.
- The G-action is *mixing* if and only if

$$\langle \pi_g f_1, f_2 \rangle = \int_X (f_1 \circ g^{-1}) \overline{f_2} \, \mathrm{d}\mu \longrightarrow \int f_1 \, \mathrm{d}\mu \int \overline{f_2} \, \mathrm{d}\mu = \langle f_1, \mathbb{1} \rangle \langle \mathbb{1}, f_2 \rangle$$

  as $g \to \infty$ for any $f_1, f_2 \in L^2(X, \mu)$.

As a motivation for the study of ergodicity in this chapter we recall the pointwise ergodic theorem. The pointwise ergodic theorem holds quite generally for actions of amenable groups,[(5)] but here we wish to only discuss the case of $\mathbb{R}^d$-flows (measure-preserving actions of $\mathbb{R}^d$).

Theorem 2.2. *Let $\mathbb{R}^d \times X \ni (t, x) \mapsto t{\bullet}x \in X$ be a jointly continuous action of $\mathbb{R}^d$ on a σ-compact locally compact metric space X preserving a Borel probability measure μ. Then, for any $f \in L^1_\mu(X)$,*

$$\frac{1}{m_{\mathbb{R}^d}(B_r)} \int_{B_r} f(t{\bullet}x) \, \mathrm{d}t \longrightarrow E_\mu(f \big| \mathcal{E})(x) \tag{2.1}$$

as $r \to \infty$ for μ-almost every $x \in X$. Here

$$B_r = \{t = (t_1, \dots, t_d) \in \mathbb{R}^d \mid 0 \leqslant t_i \leqslant r \text{ for } i = 1, \dots, d\}.$$

denotes a cube of side length r with 0 at one of its corners,

$$\mathcal{E} = \{B \subseteq X \mid \mu(B \triangle g\bullet B) = 0 \text{ for all } g \in G\}$$

denotes the σ-algebra of invariant sets under the action, and $E_\mu(f \big| \mathcal{E})$ denotes the conditional expectation with respect to $\mathcal{E}$.

We close with some remarks on the notions of ergodicity and mixing for group actions. Theorem 2.2 is a special case of [52, Th. 8.19], and the use of d-dimensional cubes as the averaging sequence is not necessary. As may be seen from conditions (P), (D), and (F) in [52, Sec. 8.6.2] any reasonable choice of metric balls containing the origin of $\mathbb{R}^d$ will suffice to achieve the almost everywhere convergence in (2.1).

Notice that ergodicity for the action is equivalent to the invariant σ-algebra $\mathcal{E}$ being equivalent modulo μ-null sets to the trivial algebra $\{\varnothing, X\}$, so in this case the ergodic averages in (2.1) converge to $\int_X f \,\mathrm{d}\mu$. A consequence of Theorem 2.2 and ergodicity is that μ-almost every point in X has an orbit under the action that is not only dense in $\operatorname{supp}\mu$ but is equidistributed with respect to μ. We say $x \in X$ is *generic* (for μ and the action considered) or that the orbit of x is *equidistributed* (with respect to μ) if

$$\frac{1}{m_{\mathbb{R}^d}(B_r)} \int_{B_r} f(t\bullet x) \,\mathrm{d}t \longrightarrow \int_X f \,\mathrm{d}\mu$$

as $r \to \infty$ for all $f \in C_c(X)$ (see [52, Ch. 4.4.2] for the details in the case of a single transformation, Exercise 2.4, and Section 6.3.1).

The natural G-action on the quotient $X = \Gamma\backslash G$ by a lattice $\Gamma < G$ is ergodic with respect to the measure m_X inherited from Haar measure on G. However, as the group G is uncountable, it is not immediately obvious that the absence of nontrivial invariant sets (which is obvious for the transitive G-action on X) implies the triviality of the measure of sets that are invariant modulo m_X (as is required for ergodicity). For the fact that this is indeed the case we refer to [52, Sec. 8.1].

As mentioned above, mixing is of course a stronger property than ergodicity in many different ways. It amounts to an asymptotic independence of measurable sets of the form A and $g^{-1}\bullet B$ as $g \to \infty$. More significantly for our purposes, we will see in Chapter 5 situations in which mixing allows us to prove even stronger results on the behaviour of all orbits for certain subgroups, rather than just almost all orbits. This is significant, because knowledge of the behaviour of almost every point tells you nothing about the behaviour of any one specific point, and in some situations the easiest way to describe the behaviour of a specific point one is interested in is to describe the behaviour of all points.

For general groups and their measure-preserving actions ergodicity is a universal notion, as any invariant measure can be decomposed into 'ergodic components' (see, for example, [52, Sec. 8.7]). However, in general—and in particular for $G = \mathbb{Z}$ or $G = \mathbb{R}^d$—mixing is a rather special property.

Exercise 2.3. Let G act on X preserving a locally finite measure μ. Show that the unitary representation $G \ni g \mapsto \pi_g$ satisfies the continuity property discussed on page 54.

Exercise 2.4. Assuming Theorem 2.2 and ergodicity show that μ-almost every $x \in X$ is generic.

The following general result is the reason why ergodic theory is only interesting if the acting group is assumed to be non-compact.

Exercise 2.5. Let K be a compact metric group acting continuously on a σ-compact locally compact metric space X. Characterize all ergodic probability measures on X.

2.2 Real Lie Algebras and Lie Groups

†In this section we will set up the language concerning real Lie algebras and Lie groups that we need. For brevity we assume the basic definitions and properties of Lie groups are known. For proofs, background, and more details we refer to Knapp [102]. Not all of the theorems that we mention here will be used in an essential way, but for the most general theorem in this chapter we will use both the Levi decomposition and the Jacobson–Morozov theorem (Theorem 2.23).

2.2.1 Basic Notions

Recall that for any real Lie group G there is an associated real Lie algebra $\mathfrak{g}$ that *describes G near the identity*. There is a smooth map $\exp\colon \mathfrak{g} \to G$ with a local inverse $\log\colon B_\delta^G(I) \to \mathfrak{g}$ defined on some neighborhood $B_\delta^G(I)$ of the identity $I \in G$ with $\delta > 0$.

There is a linear representation of G on $\mathfrak{g}$, the *adjoint* representation

$$\mathrm{Ad}_g\colon \mathfrak{g} \longrightarrow \mathfrak{g}$$

for $g \in G$, satisfying

$$\exp(\mathrm{Ad}_g(v)) = g \exp(v) g^{-1}$$

for $g \in G$ and $v \in \mathfrak{g}$. Furthermore, there is a bilinear anti-symmetric Lie bracket

† This section can be skipped if the reader is familiar with the theory. Also, most of the section can be skipped if the reader is only interested in some main examples of the theory, for example, the important cases of the simple Lie group $G = \mathrm{SL}_d(\mathbb{R})$ or the semi-simple Lie groups

$$G = \mathrm{SL}_d(\mathbb{R}) \times \cdots \times \mathrm{SL}_d(\mathbb{R}).$$

In the latter case, the reader will need to familiarize herself with the notions used in Section 2.2.1, the notion of simple Lie ideals and Lie groups, and should also do Exercise 2.7.

$$[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$$

and a related map $\mathrm{ad}_u\colon \mathfrak{g}\to\mathfrak{g}$ defined by

$$\mathrm{ad}_u(v)=[u,v]$$

for $u,v\in\mathfrak{g}$, which satisfies

$$\mathrm{Ad}_g([u,v])=[\mathrm{Ad}_g(u),\mathrm{Ad}_g(v)] \tag{2.2}$$

and

$$\exp(\mathrm{ad}_u)=\mathrm{Ad}_{\exp(u)} \tag{2.3}$$

for all $u,v\in\mathfrak{g}$ and all $g\in G$. Here $\mathrm{ad}_u\colon\mathfrak{g}\to\mathfrak{g}$ is an element of the algebra of linear maps $\mathrm{End}(\mathfrak{g})$,

$$\exp\colon\ \mathrm{End}(\mathfrak{g})\longrightarrow\mathrm{GL}(\mathfrak{g})$$

is the exponential map from $\mathrm{End}(\mathfrak{g})$ to the group $\mathrm{GL}(\mathfrak{g})$ of linear automorphisms of the vector space $\mathfrak{g}$, and $\mathrm{Ad}_{\exp(u)}$ is the adjoint representation defined by the element $\exp(u)\in G$. Finally, the Lie bracket satisfies the *Jacobi identity*

$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$$

for all $u,v,w\in\mathfrak{g}$.

In the special case where G is a closed linear subgroup of $\mathrm{SL}_d(\mathbb{R})$ with $d\geqslant 2$ (which is more than sufficient for all of our applications) the claims above are easy to verify. Indeed, in these cases we have

$$\mathfrak{g}\subseteq\mathfrak{sl}_d(\mathbb{R})=\{u\in\mathrm{Mat}_d(\mathbb{R})\mid \mathrm{tr}(u)=0\},$$

$$\mathrm{Ad}_g(u)=gug^{-1},$$

and

$$[u,v]=uv-vu$$

for all $g\in G$ and $u,v\in\mathfrak{g}$.

2.2.2 Classification and Complex Lie Algebras

The local relationship between a Lie group and its Lie algebra mentioned in Section 2.2.1 in fact goes much further. If G is connected and simply connected then its Lie algebra uniquely determines G. That is, any two connected and simply connected Lie groups with isomorphic Lie algebras are themselves isomorphic. Even without the assumption that the Lie groups G_1,G_2 are simply connected, one obtains a diffeomorphism ϕ between neighborhoods U_1 and U_2 of the identities in G_1 and G_2 if they have the same Lie algebra, such that products are mapped to products $\phi(gh)=\phi(g)\phi(h)$ as long as all the terms $g,h,gh\in U_1$

stay in the domain of the map ϕ. In this case we say that G_1 and G_2 are *locally isomorphic*. For this reason, one usually starts with a classification of Lie algebras, and this classification is easier in the case of complex Lie algebras, making this the conventional first case to consider.

2.2.3 The Structure of Lie Algebras

A Lie ideal $\mathfrak{f} \lhd \mathfrak{g}$ is a subspace of $\mathfrak{g}$ with $[\mathfrak{f}, \mathfrak{g}] \subseteq \mathfrak{f}$. Lie ideals of Lie algebras of real Lie groups correspond to normal subgroups in the following sense. If $F \lhd G$ is a closed normal subgroup, then its Lie algebra $\mathfrak{f} \subseteq \mathfrak{g}$ is a Lie ideal (see Exercise 2.6). On the other hand, if $\mathfrak{f} \lhd \mathfrak{g}$ is a Lie ideal and G is connected, then there is an immersed normal subgroup $F \lhd G$ with Lie algebra $\mathfrak{f}$. Here the term *immersed* allows for the possibility that the subgroup $F = \langle \exp(\mathfrak{f}) \rangle$ generated by $\mathfrak{f}$ is not closed in G (this arises, for example, for the abelian Lie algebras $\mathfrak{f} = \mathbb{R}v$ and $\mathfrak{g} = \mathbb{R}^2$ for the group $G = \mathbb{R}^2/\mathbb{Z}^2$ for most choices of v). In the situation where $F \lhd G$ is not closed, we note that $\overline{F} \lhd G$ would then correspond to another Lie ideal $\overline{\mathfrak{f}} \lhd \mathfrak{g}$ (which is determined by $\mathfrak{f}$ and G, but in general not by $\mathfrak{f}$ and $\mathfrak{g}$ alone).

In group theory the notion of the commutator subgroup

$$[G, G] = \langle [g, h] \mid g, h \in G \rangle \lhd G$$

(where $[g, h] = g^{-1}h^{-1}gh$) is an important measure of the extent to which G fails to be abelian. Recall that a group G is said to be *nilpotent* if the *lower central series* (G_i) defined by

$$\begin{aligned} G_0 &= G, \\ G_{i+1} &= [G, G_i] = \langle [g, h] \mid g \in G, h \in G_i \rangle \lhd G \end{aligned}$$

for $i \geqslant 1$ reaches the trivial group $G_r = \{e\}$ for some $r \geqslant 1$ (the minimal such r is called the *nilpotency degree*). Similarly G is called *solvable* if the *commutator series* (G^i) defined by

$$\begin{aligned} G^0 &= G, \\ G^1 &= [G, G] \lhd G, \\ G^{i+1} &= [G^i, G^i] \lhd G \end{aligned}$$

for $i \geqslant 1$ reaches the trivial group $G_s = \{e\}$ for some $s \geqslant 1$. Every nilpotent group is solvable, while the group

$$G = B = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} \,\middle|\, a > 0, b \in \mathbb{R} \right\}$$

is solvable but not nilpotent.

These fundamental notions in group theory have natural translations into the theory of Lie algebras. A Lie algebra $\mathfrak{g}$ is *nilpotent* if the lower central series

$$\mathfrak{g}_0 = \mathfrak{g} \rhd \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}_0] \rhd \cdots \rhd \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i] \rhd \cdots$$

ends with the trivial subalgebra $\mathfrak{g}_r = \{0\}$ for some $r \geqslant 1$, and $\mathfrak{g}$ is *solvable* if the commutator series

$$\mathfrak{g}^0 = \mathfrak{g} \rhd \mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0] \rhd \cdots \rhd \mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i] \rhd \cdots$$

ends with the trivial subalgebra $\mathfrak{g}^s = \{0\}$ for some $s \geqslant 1$.

By Ado's theorem [102, Th. B.8], every real (or complex) Lie algebra $\mathfrak{g}$ can be realized as a linear Lie algebra, meaning that $\mathfrak{g}$ can be embedded into $\mathfrak{gl}_d(\mathbb{R}) = \mathrm{Mat}_d(\mathbb{R})$ (or into $\mathfrak{gl}_d(\mathbb{C}) = \mathrm{Mat}_d(\mathbb{C})$) for some $d \geqslant 1$. By Lie's theorem [102, Th. 1.25], a complex Lie algebra $\mathfrak{g}$ is solvable if and only if it can be embedded into

$$\mathfrak{b}(\mathbb{C}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1d} \\ & a_{22} & a_{23} & \cdots & a_{2d} \\ & & \ddots & & \\ & & & a_{d-1,d-1} & a_{d-1,d} \\ & & & & a_{dd} \end{pmatrix} \middle|\ a_{ij} \in \mathbb{C} \text{ for } i \leqslant j \right\}.$$

Since every real Lie algebra $\mathfrak{g}$ has a complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \mathrm{i}\mathfrak{g}$ (see below) it also follows that every real Lie algebra can be embedded into $\mathfrak{b}(\mathbb{C})$ (but maybe not into the analogous real Lie algebra $\mathfrak{b}(\mathbb{R})$.)

By Engel's theorem [102, Th. 1.35], a real Lie algebra $\mathfrak{g}$ is nilpotent if and only if it can be embedded into

$$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & \cdots & a_{1d} \\ & 0 & a_{23} & \cdots & a_{2d} \\ & & \ddots & & \\ & & & 0 & a_{d-1,d} \\ & & & & 0 \end{pmatrix} \middle|\ a_{ij} \in \mathbb{R} \text{ for } i < j \right\}.$$

It is interesting to note that the commutator $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ of a solvable Lie algebra is nilpotent (since $[\mathfrak{b}(\mathbb{C}), \mathfrak{b}(\mathbb{C})] \subseteq \mathfrak{n}(\mathbb{C})$)—there is no analogue of this fact for abstract groups.

For a general Lie algebra $\mathfrak{g}$, the *radical* $\operatorname{rad}\mathfrak{g}$ of $\mathfrak{g}$ is defined to be the subspace generated by all solvable Lie ideals $\mathfrak{f} \lhd \mathfrak{g}$, and this is a solvable Lie ideal of $\mathfrak{g}$.

A (real or complex) Lie algebra $\mathfrak{g}$ is said to be *semi-simple* if $\operatorname{rad}\mathfrak{g} = \{0\}$. A (real or complex) Lie algebra is called *simple* if $\mathfrak{g}$ is non-abelian (that is, if $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$) and $\mathfrak{g}$ has no Lie ideals other than $\mathfrak{g}$ and $\{0\}$. We note that a real simple Lie algebra always has a semi-simple complexification

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \mathrm{i}\mathfrak{g},$$

with the complexified Lie bracket defined by

$$[u+\mathrm{i}v, w+\mathrm{i}z] = [u,w] - [v,z] + \mathrm{i}\left([v,w]+[u,z]\right),$$

(but that this complexification might not be simple; see Exercise 2.8).

Every (real or complex) semi-simple Lie algebra $\mathfrak{g}$ is a direct sum of (real or complex) simple Lie subalgebras, each of which is a Lie ideal in $\mathfrak{g}$.

Finally, we note that solvable Lie algebras and semi-simple Lie algebras complement each other, and any Lie algebra can be described using Lie algebras of these two types in the following sense. The Levi decomposition

$$\mathfrak{g} = \mathfrak{g}_s + \operatorname{rad}\mathfrak{g}$$

of a (real or complex) Lie algebra consists of a semi-simple Lie subalgebra $\mathfrak{g}_s$ of $\mathfrak{g}$ and the radical $\operatorname{rad}\mathfrak{g} \lhd \mathfrak{g}$. In this decomposition $\operatorname{rad}\mathfrak{g}$ is unique, but in general $\mathfrak{g}_s$ is only unique up to an automorphism.

Exercise 2.6. Show that if $F \lhd G$ is a closed normal subgroup of a Lie group G, then its Lie algebra $\mathfrak{f} \subseteq \mathfrak{g}$ is a Lie ideal.

Exercise 2.7. Show that $\mathfrak{sl}_d(\mathbb{R})$ (or $\mathfrak{sl}_d(\mathbb{C})$) is a real (resp. complex) simple Lie algebra for $d \geqslant 2$. Show that $\mathrm{SL}_d(\mathbb{R})$ and $\mathrm{SL}_d(\mathbb{C})$ are connected simple Lie groups.

Exercise 2.8. Show that $\mathfrak{sl}_d(\mathbb{C})$ for $d \geqslant 2$, when viewed as a real Lie algebra, is simple but its complexification is not.

2.2.4 Almost Direct Simple Factors

A connected real (or complex) Lie group G is called† *simple* or *semi-simple* if its Lie algebra $\mathfrak{g}$ is simple or semi-simple respectively.

If $\mathfrak{g}$ is a real (or complex) semi-simple Lie algebra then, as mentioned above, we have a decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

with simple Lie ideals $\mathfrak{g}_i \lhd \mathfrak{g}$ for $i = 1, \ldots, r$. If G is a real (or complex) connected simply connected semi-simple Lie group then the stronger property

$$G \cong G_1 \times \cdots \times G_r, \tag{2.4}$$

holds, where each $G_i \lhd G$ is a connected simply connected Lie group with Lie algebra $\mathfrak{g}_i$.

A real (or complex) semi-simple Lie group G is called *adjoint* if its centre is trivial. If G is a real (or complex) connected adjoint semi-simple Lie group, then (2.4) holds similarly.

However, the product decomposition in (2.4) does not hold for general semi-simple Lie groups without the assumption that the group is simply connected or adjoint. The reason why the product decomposition fails is easy to understand.

† Some authors prefer the term *almost simple* as G will often have a non-trivial centre.

Example 2.9. Let

$$G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\{(I,I),(-I,-I)\}$$

be the quotient by the normal subgroup N generated by $(-I,-I)$ in

$$\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}).$$

Notice that the Lie algebra of G is isomorphic to $\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ and that G is not simply connected. Furthermore,

$$G_1 = \mathrm{SL}_2(\mathbb{R}) \times \{I\}N/N$$

and

$$G_2 = \{I\} \times \mathrm{SL}_2(\mathbb{R})N/N$$

are both normal subgroups of G, are both isomorphic to $\mathrm{SL}_2(\mathbb{R})$, but G is not isomorphic to $G_1 \times G_2$ unlike the simply connected case discussed above. Also note that $G_1 \cap G_2$ is generated by $(-I,I)N = (I,-I)N$ which is contained in the centre of G.

Allowing for such phenomena along the centre, one does get an almost direct product decomposition into almost direct factors of a real semi-simple Lie group as follows. Let G be a real semi-simple Lie group, and suppose that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

is the decomposition of its Lie algebra into real simple Lie subalgebras. Then for each $i = 1, \ldots, r$ there is a normal closed connected simple Lie subgroup G_i, which we will refer to as an *almost direct factor*, with Lie algebra $\mathfrak{g}_i$. These almost direct factors have the following properties.

- G_i commutes with G_j for $i \neq j$;
- $G = G_1 \cdots G_r$; and
- the kernel of the homomorphism

$$\begin{aligned} G_1 \times \cdots \times G_r &\longrightarrow G_1 \cdots G_r = G \\ (g_1, \ldots, g_r) &\longmapsto g_1 \cdots g_r \end{aligned}$$

  is contained in the centre of $G_1 \times \cdots \times G_r$.

We define $G^+ \subseteq G$ to be the almost direct product of (i.e. the normal subgroup of G generated by) those almost direct factors G_i of G that are non-compact.

From now on, unless explicitly identified to be complex, we will always consider real Lie groups and Lie algebras.

Exercise 2.10. Let G be a real simple connected Lie group ($G = \mathrm{SL}_d(\mathbb{R})$ for $d \geqslant 2$, for example). Show that any proper normal subgroup of G is contained in the centre of G.

Exercise 2.11. Show that the connected component of

$$\mathrm{SO}_{2,2}(\mathbb{R}) = \{g \in \mathrm{SL}_4(\mathbb{R}) \mid g \text{ preserves the quadratic form } ad - bc\}$$

is isomorphic to the almost direct product discussed in Example 2.9.

2.2.5 Two Decompositions for Simple Groups

Let $G \leqslant \mathrm{SL}_d(\mathbb{R})$ be a simple closed linear group. Then one can find three subgroups allowing two important decompositions as follows.

(1) The *Cartan* subgroup $A \leqslant G$ is abelian, connected, and consists of diagonalizable matrices. For $G = \mathrm{SL}_d(\mathbb{R})$ the subgroup A consists of the full connected diagonal subgroup.
(2) The subgroup $N \leqslant G$ is a particular nilpotent subgroup normalized by A. For $G = \mathrm{SL}_d(\mathbb{R})$ the subgroup N is the upper triangular unipotent subgroup.
(3) Finally, the subgroup $K \leqslant G$ is a maximal compact subgroup containing the finite centre of G. For $G = \mathrm{SL}_d(\mathbb{R})$ we have $K = \mathrm{SO}_d(\mathbb{R})$.

We note that A acts diagonally on the Lie algebra $\mathfrak{g}$ of G (as it acts diagonally on $\mathfrak{sl}_d(\mathbb{R})$). The eigenspaces for the action of A are called the *root subspaces* and consist of nilpotent matrices. Moreover we note that G is compact if and only if $G = K$, or if and only if $A = \{I\}$ is trivial.

The first decomposition takes the form $G = KAN$ and is called the *Iwasawa decomposition*. In fact every $g \in G$ can be written in a unique way as $g = kan$ with $k \in K$, $a \in A$, and $n \in N$. In the case of $G = \mathrm{SL}_d(\mathbb{R})$ this corresponds to the Gram–Schmidt procedure (see Proposition 1.52).

The second decomposition takes the form $G = KAK$ and is called the *Cartan decomposition*. In the case of $G = \mathrm{SL}_d(\mathbb{R})$ this corresponds to the polar or singular value decomposition of matrices. This decomposition fails to be unique: For instance, uniqueness fails for the identity element (but more can be said).

Exercise 2.12. Prove that every element of $\mathrm{SL}_d(\mathbb{R})$ can be written in the form kak' with $k, k' \in \mathrm{SO}_d(\mathbb{R})$ and a diagonal matrix a as claimed above.

2.3 Mautner Phenomenon

We recall that a *unitary representation* is an action $\pi\colon G \times \mathcal{H} \to \mathcal{H}$ by unitary maps π_g for $g \in G$ such that for any given $v \in \mathcal{H}$ the map $G \ni g \mapsto \pi_g v$ is continuous (with respect to the given topology on G and the norm topology on $\mathcal{H}$). We say that $v \in \mathcal{H}$ is *fixed by* or *invariant under* $g \in G$ if $\pi_g v = v$.

Let G act on X as in Section 2.1 and let μ be a locally finite measure on X invariant under this action. Then by Exercise 2.3 (also see its hint on page 434) the associated unitary representation

$$\pi_g(f) = f \circ g^{-1}$$

for $f \in \mathcal{H} = L^2(X, \mu)$ satisfies this continuity property and so defines a unitary representation (see also [52, Lem. 8.7] or [53, Lem. 3.74]).

The following lemma(6) will be the main tool used for proving that ergodicity sometimes has an inheritance property from the acting group to some of its subgroups.

Lemma 2.13 (The key lemma). *Let $\mathcal{H}$ be a Hilbert space carrying a unitary representation of a topological group G. Suppose that $v_0 \in \mathcal{H}$ is fixed by some subgroup $L \leqslant G$. Then v_0 is also fixed under every other element $h \in G$ with the property that there exist sequences (g_n) in G and (ℓ_n), (ℓ'_n) in L with $\lim_{n\to\infty} g_n = e$ and $h = \lim_{n\to\infty} \ell_n g_n \ell'_n$.*

Proof. By assumption, there exist three sequences (g_n) in G, (ℓ_n) in L, and (ℓ'_n) in L with $g_n \to e$ and $\ell_n g_n \ell'_n \to h$ as $n \to \infty$. This implies that

$$\|\pi_{\ell_n g_n \ell'_n} v_0 - v_0\| = \|\pi_{\ell_n}(\pi_{g_n \ell'_n} v_0 - \pi_{\ell_n^{-1}} v_0)\| = \|\pi_{g_n} v_0 - v_0\|$$

by unitarity of π_{ℓ_n} and invariance of v_0 under all elements of L. However, the left-hand side converges to $\|\pi_h v_0 - v_0\|$ by continuity of the representation and the right-hand side converges to 0. □

As we will see, this simple observation can be used to show that ergodicity of a measure-preserving action of G sometimes forces ergodicity of a subgroup L. Indeed, suppose G acts ergodically and preserving μ on a probability space (X, μ) as in Section 2.1, $L \leqslant G$ is a subgroup, and $f \in L^2_\mu(X)$ is invariant under L. Applying Lemma 2.13 with various choices of sequences, one may hope to prove that f is in fact invariant under other elements of G. In good situations one obtains in this way enough elements of G to generate G, which implies that f is invariant under G, hence f is constant, and so the action of L is ergodic.

Exercise 2.14. (a) Let $G = \mathbb{Z}^d$ with $d \geqslant 2$. Find an ergodic action of G with the property that no subgroup of G with lower rank acts ergodically.
(b) Let $G = \mathbb{R}^d$ with $d \geqslant 1$. Prove that in any ergodic action of G almost every element of $\mathbb{R}^d$ acts ergodically. (This relies on the standing assumptions regarding X, which imply in particular that $L^2(X)$ is separable.)

2.3.1 The Case of $\mathrm{SL}_2(\mathbb{R})$

We now turn to the special (but important) case of $G = \mathrm{SL}_2(\mathbb{R})$. Any element $g \in \mathrm{SL}_2(\mathbb{R})$ is conjugate to one of the following three type of elements:

- an $\mathbb{R}$-diagonal matrix, that is one of the form $a = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ with $\lambda \in \mathbb{R}^\times$;
- a unipotent matrix $u = \begin{pmatrix} 1 & \pm 1 \\ & 1 \end{pmatrix}$ or $u = \begin{pmatrix} -1 & \pm 1 \\ & -1 \end{pmatrix}$; or
- a matrix in the compact subgroup $\mathrm{SO}(2, \mathbb{R})$, that is one of the form

$$k = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

for some $\phi \in \mathbb{R}$.

For the last case we can make no claim concerning ergodicity of the action of g (see Exercise 2.5 concerning compact groups and its hint on page 434). However, for the first two types we find the following phenomenon, where we write

$$C_G = \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

for the centre of a group G. We note that $C_{\mathrm{SL}_2(\mathbb{R})} = \{\pm I\}$.

Proposition 2.15 (Mautner for $\mathrm{SL}_2(\mathbb{R})$). *Let $G = \mathrm{SL}_2(\mathbb{R})$ act unitarily on a Hilbert space $\mathcal{H}$. Suppose that $g \in G \smallsetminus C_G$ has the property that g is unipotent, $-g$ is unipotent, or g is $\mathbb{R}$-diagonalizable. If g fixes a vector $v_0 \in \mathcal{H}$, then all of G fixes v_0 also.*

Suppose $g \in G$ exhibits the Mautner phenomenon of Proposition 2.15, and $h \in G$ has the property that hgh^{-1} fixes $v_0 \in \mathcal{H}$. Then g fixes $\pi_h^{-1} v_0$ and so all of G fixes $\pi_h^{-1} v_0 = v_0$. Thus it is sufficient to consider one representative of each conjugacy class for the proof of Proposition 2.15 and for the proof of similar statements that come later.

Proof of Proposition 2.15. For $a = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ with $\lambda \neq \pm 1$ a direct calculation shows that we can apply Lemma 2.13 with $L = a^{\mathbb{Z}}$ and any element of the unipotent subgroups $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{R})$. For example,

$$a^n \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} a^{-n} = \begin{pmatrix} 1 & \lambda^{2n} s \\ & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

if $\lambda^{2n} \to 0$ as $n \to \infty$. It follows that if a fixes some $v_0 \in \mathcal{H}$, then so do these two unipotent subgroups. As they together generate $\mathrm{SL}_2(\mathbb{R})$ (see Exercise 1.4 and Lemma 1.57), we obtain Proposition 2.15 for this case.

If $u = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ then

$$\begin{aligned} u^n \begin{pmatrix} 1+\delta & \\ & \frac{1}{1+\delta} \end{pmatrix} u^{-n} &= \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \begin{pmatrix} 1+\delta & \\ & \frac{1}{1+\delta} \end{pmatrix} \begin{pmatrix} 1 & -n \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1+\delta & \kappa_\delta n \\ & \frac{1}{1+\delta} \end{pmatrix} \end{aligned}$$

for $\kappa_\delta = \frac{1}{1+\delta} - 1 - \delta \neq 0$ converging to 0 for $\delta \to 0$. Hence for any $s \in \mathbb{R}$ the above matrix can be made (since $n \in \mathbb{Z}$ can be chosen arbitrary) to converge to $\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$ for $\delta \to 0$. It follows that if v_0 is fixed by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ then it is also fixed by $\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$ for any $s \in \mathbb{R}$ by Lemma 2.13 applied with

$$L = \left\{ \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}.$$

Applying Lemma 2.13 once more with

$$L = \left\{ \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\}$$

to the matrix

$$\begin{pmatrix} 1 & s_1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ \delta & 1 \end{pmatrix} \begin{pmatrix} 1 & s_2 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1+\delta s_1 & s_2(1+\delta s_1)+s_1 \\ \delta & 1+\delta s_2 \end{pmatrix} = g_\delta \tag{2.5}$$

with s_1 chosen to have

$$1 + \delta s_1 = \mathrm{e}$$

and with s_2 chosen to have

$$s_2(1+\delta s_1) + s_1 = 0$$

shows that v_0 is also fixed by

$$\begin{pmatrix} \mathrm{e} & \\ & \mathrm{e}^{-1} \end{pmatrix} = \lim_{\delta \to 0} g_\delta.$$

Applying the previous (diagonal) case, we see once again that v_0 is fixed by all of $\mathrm{SL}_2(\mathbb{R})$. □

Exercise 2.16. Let $a \in G = \mathrm{SL}_d(\mathbb{R})$ be a diagonal matrix such that

$$G_a^{\pm} = \{u \in G \mid a^n u a^{-n} \to I \text{ as } n \to \mp\infty\}$$

are nontrivial subgroups. Show directly that $\langle G_a^+, G_a^- \rangle = G$.

Exercise 2.17. Prove Proposition 2.15 for the case of $\mathrm{SL}_d(\mathbb{R})$ for $d = 3$ or more generally for $d \geqslant 3$, either directly by a similar argument or using the case $\mathrm{SL}_2(\mathbb{R})$ considered above.

2.3.2 Big and Small Eigenvalues

Let G be a Lie group with Lie algebra $\mathfrak{g}$. In this section we will show an inheritance claim, which uses the notion of horospherical algebras. The unstable and stable horospherical Lie subalgebras ($\mathfrak{g}^+$ and $\mathfrak{g}^-$ respectively) for $g \in G$ are defined as follows:

- $\mathfrak{g}^+$ is the sum of all generalized[†] subspaces corresponding to eigenvalues of Ad_g with absolute value bigger than one; equivalently we have

$$\mathfrak{g}^+ = \{v \in \mathfrak{g} \mid \mathrm{Ad}_g^n(v) \longrightarrow 0 \text{ as } n \longrightarrow -\infty\}.$$

- $\mathfrak{g}^-$ is the sum of all generalized subspaces with eigenvalues of Ad_g with absolute value smaller than one; equivalently we have

$$\mathfrak{g}^- = \{v \in \mathfrak{g} \mid \mathrm{Ad}_g^n(v) \longrightarrow 0 \text{ as } n \longrightarrow \infty\}.$$

To see that $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are subalgebras, the characterization in terms of the adjoint action is most useful. By linearity of Ad_g it is clear that $\mathfrak{g}^-$ is a linear subspace. Moreover, if $v_1, v_2 \in \mathfrak{g}^-$, then

$$\mathrm{Ad}_g^n([v_1, v_2]) = [\mathrm{Ad}_g^n(v_1), \mathrm{Ad}_g^n(v_2)] \longrightarrow 0$$

as $n \to \infty$, showing that $[v_1, v_2] \in \mathfrak{g}^-$ also; the same argument using $n \to -\infty$ shows that $\mathfrak{g}^+$ is also a subalgebra.

Lemma 2.18 (Auslander ideal). *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $g \in G$. Then the Lie algebra $\mathfrak{f} = \langle \mathfrak{g}^+, \mathfrak{g}^- \rangle$ generated by the unstable and stable horospherical Lie subalgebras of $\mathfrak{g}$ is a Lie ideal of $\mathfrak{g}$, called the* Auslander ideal *of g.*

PROOF. The proof relies on the Jacobi identity. Let $\mathfrak{g}^0$ be the sum of the generalized eigenspaces for all eigenvalues of absolute value one, so that

$$\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^0 + \mathfrak{g}^-,$$

and we need to show that $[\mathfrak{g}, \mathfrak{f}] \subseteq \mathfrak{f}$. Since $\mathfrak{f}$ is a subalgebra by definition, it is sufficient to show that $[\mathfrak{g}^0, \mathfrak{f}] \subseteq \mathfrak{f}$.

Notice first that $[\mathfrak{g}^0, \mathfrak{g}^-] \subseteq \mathfrak{g}^-$ (and similarly $[\mathfrak{g}^0, \mathfrak{g}^+] \subseteq \mathfrak{g}^+$). Indeed, suppose $u \in \mathfrak{g}^0$ and $v \in \mathfrak{g}^-$. By the theory of Jordan normal forms $\| \mathrm{Ad}_g^n(u)\|$ is either bounded or goes to infinity at most at a polynomial rate as $n \to \infty$, while $\| \mathrm{Ad}_g^n(v)\|$ decays to 0 at exponential speed. It follows by bi-linearity of $[\cdot, \cdot]$ that

$$\mathrm{Ad}_g^n([u, v]) = [\mathrm{Ad}_g^n(u), \mathrm{Ad}_g^n(v)] \longrightarrow 0$$

as $n \to \infty$, as required.

If now $u \in \mathfrak{g}^+, v \in \mathfrak{g}^-$, so that $[u, v] \in \mathfrak{f}$, then for any $w_0 \in \mathfrak{g}^0$ we have

$$[w_0, [u, v]] + \underbrace{[u, \underbrace{[v, w_0]}_{\in \mathfrak{f}}]}_{\in \mathfrak{f}} + \underbrace{[v, \underbrace{[w_0, u]}_{\in \mathfrak{f}}]}_{\in \mathfrak{f}} = 0$$

[†] Here we allow for Jordan blocks corresponding to eigenvalues of absolute value bigger than one as well as for (generalized) eigenspaces corresponding to pairs of complex eigenvalues of absolute value bigger than one.

by the Jacobi identity, the case above, and the fact that $\mathfrak{f}$ is a subalgebra. It follows that $[\mathfrak{g}^0, [\mathfrak{g}^+, \mathfrak{g}^-]] \subseteq \mathfrak{f}$. Repeating the argument under the assumptions $w \in \mathfrak{g}^0$, $u, v \in \mathfrak{f}$ with $[w_0, u], [w_0, v] \in \mathfrak{f}$ we obtain $[w_0, [u, v]] \in \mathfrak{f}$. Hence $\{u \in \mathfrak{f} \mid [w_0, u] \in \mathfrak{f}\}$ is a subalgebra and so equals $\mathfrak{f}$. As $w_0 \in \mathfrak{g}^0$ was arbitrary, it follows that $\mathfrak{f}$ is a Lie ideal as claimed. □

Definition 2.19 (Lie algebra fixing vectors). Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. We say that $v \in \mathfrak{g}$ *fixes* $w \in \mathcal{H}$ if $\pi_{\exp(tv)} w = w$ for all $t \in \mathbb{R}$. We say that a Lie subalgebra $\mathfrak{f} \subseteq \mathfrak{g}$ *fixes* $w \in \mathcal{H}$ if every $v \in \mathfrak{f}$ fixes w.

Proposition 2.20 (Mautner phenomenon for the Auslander ideal). *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and suppose that G acts unitarily on a Hilbert space $\mathcal{H}$ and that $g \in G$ fixes $v_0 \in \mathcal{H}$. Then v_0 is fixed by $\mathfrak{f}$, where $\mathfrak{f}$ is the Auslander ideal from Lemma* 2.18.

Proof. Lemma 2.13 applied to $h = \exp(v)$ with $v \in \mathfrak{g}^{\pm}$ shows that $v_0 \in \mathcal{H}$ is fixed by $\exp(v)$ for $v \in \mathfrak{g}^{\pm}$. It follows that v_0 is fixed by the closed subgroup F generated by the sets $\exp(\mathfrak{g}^+)$ and $\exp(\mathfrak{g}^-)$. In particular, there exists a Lie subalgebra (the Lie algebra of F) containing $\mathfrak{g}^+$ and $\mathfrak{g}^-$ that fixes v_0. Since $\mathfrak{f}$ is the Lie subalgebra generated by $\mathfrak{g}^+$ and $\mathfrak{g}^-$, we deduce that $\mathfrak{f}$ fixes v_0. □

Exercise 2.21. Show that $\mathfrak{g}^0$ from the proof of Lemma 2.18 is a Lie subalgebra.

Exercise 2.22. Let G be a simple Lie group and let $\Gamma < G$ be a lattice. Let $a \in G$ and recall that the Lie algebra of G splits as a direct sum $\mathfrak{g}^+ + \mathfrak{g}^0 + \mathfrak{g}^-$ as in the proof of Lemma 2.18. Assume that Ad_a is diagonalizable when restricted to $\mathfrak{g}^0$ and that 1 is the only eigenvalue of this restriction (so that $\mathfrak{g}^0$ is the Lie algebra of $C_G(a) = \{g \in G \mid ag = ga\}$). Using the pointwise ergodic theorem (Theorem 2.2) show that for any $x \in X = \Gamma\backslash G$ and $m_{G_a^+}$-almost every $u \in G_a^+$ the forward orbit $\{a^n \bullet (u \bullet x) \mid n \geqslant 0\}$ of $u \bullet x$ equidistributes† in X with respect to the Haar measure m_X.

2.3.3 The case of Semi-simple Lie Groups

In this subsection we will assume that $G \subseteq \mathrm{SL}_d(\mathbb{R})$ is a connected semi-simple closed linear group. To study actions of such a group, we will combine the arguments from Section 2.3.2, the Jacobson–Morozov theorem,(7) and the case of $\mathrm{SL}_2(\mathbb{R})$ from Section 2.3.1. The Jacobson–Morozov theorem (we refer to Knapp [102, Sec. X.2] for the proof) is the reason that the special case $G = \mathrm{SL}_2(\mathbb{R})$ is so useful.

† We note that the results of this section and Exercise 2.4 immediately show that the forward orbit is equidistributed for m_X-almost every $x \in X$, but the desired statement is stronger as it involves a Haar measure on a subgroup.

Theorem 2.23 (Jacobson–Morozov). *Suppose that $\mathfrak{g}$ is a real semi-simple Lie algebra, and let $x \in \mathfrak{g}$ be a nilpotent element. Then there exist elements y and h in $\mathfrak{g}$ so that (h, x, y) form an* $\mathfrak{sl}_2$-triple*, meaning that they span a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, and in fact*

$$\begin{aligned}[h, x] &= 2x,\\ [h, y] &= -2y, \text{ and}\\ [x, y] &= h.\end{aligned}$$

It may be useful to be more explicit about Theorem 2.23 in two low-dimensional examples. In $\mathfrak{sl}_2(\mathbb{R})$ we have

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.6}$$

In $\mathrm{SL}_3(\mathbb{R})$ there are two (fundamentally different) choices. The first is via the most obvious embedding $\mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{sl}_3(\mathbb{R})$ using the upper-left block giving

$$h_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, x_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The second choice for $\mathrm{SL}_3(\mathbb{R})$ (which is not conjugate to the first) comes from the embedding $\mathfrak{sl}_2(\mathbb{R}) \hookrightarrow \mathfrak{sl}_3(\mathbb{R})$ defined by

$$h_3 = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix}, x_3 = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, y_3 = \begin{pmatrix} 0 & & \\ 2 & 0 & \\ & 2 & 0 \end{pmatrix}.$$

One can easily check the fundamental relations from Theorem 2.23:

$$[h_3, x_3] = 2x_3,\ [h_3, y_3] = -2y_3, \text{ and } [x_3, y_3] = h_3.$$

Proposition 2.24 (Mautner phenomenon for semi-simple groups). *Let G be a connected semi-simple closed linear group in $\mathrm{SL}_d(\mathbb{R})$ with Lie algebra $\mathfrak{g}$ which acts unitarily on a Hilbert space $\mathcal{H}$. Suppose $g \in G$ is diagonalizable with positive eigenvalues or $g = \exp(x)$ for some nilpotent $x \in \mathfrak{g}$, and g fixes some vector $v_0 \in \mathcal{H}$. Then there is a normal subgroup of G containing g which also fixes v_0.*

PROOF. Suppose that $g = a \in G$ is diagonalizable with positive eigenvalues. Then[†] Ad_a is also diagonalizable with positive eigenvectors. Hence we can split $\mathfrak{g}$ as before into three spaces

$$\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^0 + \mathfrak{g}^-,$$

[†] The eigenvalues of Ad_a on $\mathfrak{sl}_d(\mathbb{R})$ are quotients of eigenvalues of a, which implies the same for Ad_a restricted to $\mathfrak{g} \subseteq \mathfrak{sl}_d(\mathbb{R})$.

where $\mathfrak{g}^0$ is the eigenspace of Ad_a with eigenvalue one. Since the Lie algebra generated by $\mathfrak{g}^+$ and $\mathfrak{g}^-$ is a Lie ideal $\mathfrak{f}$ by Lemma 2.18, $\mathfrak{f}$ is a direct sum of some of the direct simple factors of $\mathfrak{g}$. Hence it has to contain any simple factor of $\mathfrak{g}$ that intersects either of the spaces $\mathfrak{g}^+$ or $\mathfrak{g}^-$ nontrivially. Let $F_1 = \langle \exp(\mathfrak{f}) \rangle$ be the normal subgroup containing these simple factors. By Proposition 2.20 we know that F_1 fixes v_0. Since the eigenvalues of Ad_a are positive, it follows that (the linear map induced by) Ad_a acts trivially on the Lie algebra of G/F_1 (which may be identified with a sub-algebra of $\mathfrak{g}^0$). Therefore, aF_1 belongs to the centre of G/F_1, and so generates a normal subgroup of G/F_1. Let $F = \langle a, F_1 \rangle$ be the pre-image in G of this normal subgroup. Then $a \in F$, F is a normal subgroup in G, and F fixes $v_0 \in \mathcal{H}$ as required.

Suppose now that $g = u = \exp(x)$ is unipotent. Then by the Jacobson–Morozov theorem there exists a connected subgroup $H < G$ locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ containing u such that x corresponds under the isomorphism to an upper nilpotent element of $\mathfrak{sl}_2(\mathbb{R})$. By the classification of finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$ (see Knapp [102], Fulton and Harris [64] or [41, Sec. 4.1]) it follows that H is isomorphic to $\mathrm{SL}_2(\mathbb{R})$ or to $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$. In either case we may apply Proposition 2.15 to see that H fixes v_0. Since H also contains the image of

$$a = \begin{pmatrix} \mathrm{e} & \\ & \mathrm{e}^{-1} \end{pmatrix},$$

we have produced the situation of the first case, which was considered above. Let F be the normal subgroup corresponding to the Auslander ideal of a and recall that

$$g = u = \exp(x) \in \exp(\mathfrak{g}^+) \subseteq F.$$

The theorem follows once more from Proposition 2.20. □

2.4 The Howe–Moore Theorem

The main goal of this chapter is to relate the algebraic properties of G to properties of its measure-preserving actions, by showing that for certain Lie groups ergodicity forces mixing (in contrast to the abelian case, where, for example, an ergodic action of $\mathbb{Z}^2$ could have no ergodic elements).

Theorem 2.25 (Howe–Moore, automatic mixing). *Let $G \subseteq \mathrm{SL}_d(\mathbb{R})$ be a connected simple closed linear group. An ergodic and measure-preserving action on a probability space by G is mixing.*

A more general formulation expresses this result in terms of *vanishing of matrix coefficients at infinity* in the associated unitary representations (restricted to the orthogonal complement of the constants).

Theorem 2.26 (Howe–Moore, vanishing of matrix coefficients). *Let G in $\mathrm{SL}_d(\mathbb{R})$ be a connected simple closed linear group acting unitarily on a Hilbert* *space $\mathcal{H}$, and suppose that the action has no non-trivial fixed vectors. Then the associated matrix coefficients vanish at infinity in the sense that*

$$\langle \pi_g v, w \rangle \longrightarrow 0$$

as $g \to \infty$ in G for any $v, w \in \mathcal{H}$.

One of the most important ingredients in the proof of the Howe–Moore theorem is the inheritance property in Proposition 2.24.

Exercise 2.27. Deduce Theorem 2.25 from Theorem 2.26.

2.4.1 A More General Howe–Moore Theorem and its Proof

In order to state the general version of the Howe–Moore theorem, we will use[†] the terminology and results from Section 2.2.4.

Theorem 2.28 (Howe–Moore for semi-simple groups). *Let $G \subseteq \mathrm{SL}_d(\mathbb{R})$ be a connected semi-simple closed linear group and let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. For v_1, v_2 in $\mathcal{H}$ we have*

$$\langle \pi_{g_n} v_1, v_2 \rangle \longrightarrow 0 \tag{2.7}$$

as $n \to \infty$ in either of the following two situations:

(1) *For any of the simple non-compact factors G_i of G, there are no non-trivial G_i-fixed vectors in $\mathcal{H}$ and $g_n \to \infty$ as $n \to \infty$.*
(2) *$\mathcal{H}$ has no non-trivial G^+-fixed vectors, $g_n = g_n^{(1)} \cdots g_n^{(r)}$ with $g_n^{(i)} \in G_i$, and $g_n^{(i)} \to \infty$ as $n \to \infty$ for each simple non-compact factor[‡] $G_i \subseteq G^+$ of G.*

Proof (of Theorems 2.25, 2.26 and 2.28). Assume that $g_n \to \infty$ in G as $n \to \infty$. We will show (2.7) by showing that there always exists a subsequence for which (2.7) holds.

Moreover, it suffices to consider the case of a sequence $(g_n = a_n)$ belonging to the Cartan subgroup of G. In fact using the Cartan decomposition of G we may write the terms of any sequence $g_n \to \infty$ as $n \to \infty$ in the form

$$g_n = k_n a_n k_n'$$

with $k_n, k_n' \in K$ for all $n \geqslant 1$ with $a_n \to \infty$ as $n \to \infty$ in $A < G$. Since K is compact and the representation is continuous the study of $\langle \pi_{k_n a_n k_n'} v_1, v_2 \rangle$

[†] This is only needed because we state the theorem in greater generality. At its core the argument only needs basic functional analysis.

[‡] Even though the decomposition of g_n into $g_n^{(1)} \cdots g_n^{(r)}$ with $g_n^{(i)} \in G_i$ is not unique, the requirement that $g_n^{(i)} \to \infty$ as $n \to \infty$ does make sense as the ambiguity in the decomposition is only up to the finite centre of G.

can be reduced to the study of $\langle \pi_{ka_nk'} v_1, v_2 \rangle$ for some $k, k' \in K$. Indeed using compactness of K we may choose a subsequence and assume $k_n \to k$ and $k'_n \to k'$ as $n \to \infty$. Now we apply continuity of the representation and the Cauchy-Schwartz inequality to see that

$$\begin{aligned}
&\left|\langle \pi_{k_na_nk'_n} v_1, v_2 \rangle - \langle \pi_{ka_nk'} v_1, v_2 \rangle\right| \\
&\qquad \leqslant \left|\langle \pi_{a_nk'_n} v_1, \pi^*_{k_n} v_2 \rangle - \langle \pi_{a_nk'} v_1, \pi^*_{k_n} v_2 \rangle\right| \\
&\qquad\qquad + \left|\langle \pi_{a_nk'} v_1, \pi^*_{k_n} v_2 \rangle - \langle \pi_{a_nk'} v_1, \pi^*_{k} v_2 \rangle\right| \\
&\qquad \leqslant \left\|\pi_{k'_n} v_1 - \pi_{k'} v_1\right\| \left\|v_2\right\| + \left\|v_1\right\| \left\|\pi^*_{k_n} v_2 - \pi^*_k v_2\right\|
\end{aligned}$$

which goes to 0 as $n \to \infty$. Hence it suffices to study $\langle \pi_{a_n} \pi_k v_1, \pi_k v_2 \rangle$. We assume from now on that $g_n = a_n$ belongs to the Cartan subgroup and to simplify the notation consider $\langle \pi_{a_n} v_1, v_2 \rangle$.

By passing to a subsequence we may also assume that

$$v^* = \lim_{n\to\infty} \pi_{a_n} v_1 \in \mathcal{H}$$

exists in the weak*-topology by the Banach–Alaoglu theorem, since

$$\|\pi_{a_n} v_1\| = \|v_1\|$$

by unitarity. The claim in (2.7) (for this subsequence and any $v_2 \in \mathcal{H}$) is the statement $v^* = 0$.

Let us explain the main step first in the case of $G = \mathrm{SL}_d(\mathbb{R})$. Since we know that $a_n \to \infty$ as $n \to \infty$, we can choose a subsequence and assume that at least one eigenvalue of a_n goes to ∞ while another goes to 0 as $n \to \infty$. Hence we can find some nontrivial element u of the elementary unipotent subgroups appearing in Lemma 1.57 such that

$$a_n^{-1} u a_n \longrightarrow I \tag{2.8}$$

as $n \to \infty$. We claim that this implies that

$$\pi_u v^* = v^*. \tag{2.9}$$

To prove the claim, let $w \in \mathcal{H}$ be any element. Then

$$\begin{aligned}
\langle \pi_u v^*, w \rangle &= \langle v^*, \pi_u^{-1} w \rangle \\
&= \lim_{n\to\infty} \langle \pi_{a_n} v_1, \pi_{u^{-1}} w \rangle \\
&= \lim_{n\to\infty} \left\langle \pi_{a_n^{-1}ua_n} v_1, \pi_{a_n^{-1}} w \right\rangle.
\end{aligned}$$

However we have

$$\lim_{n\to\infty} \|\pi_{a_n^{-1}ua_n} v_1 - v_1\| = 0.$$

Applying Cauchy–Schwarz to the difference we obtain

$$\begin{aligned}\langle \pi_u v^*, w\rangle &= \lim_{n\to\infty} \left\langle \pi_{a_n^{-1} u a_n} v_1, \pi_{a_n^{-1}} w \right\rangle \\ &= \lim_{n\to\infty} \left\langle v_1, \pi_{a_n^{-1}} w \right\rangle \\ &= \lim_{n\to\infty} \left\langle \pi_{a_n} v, w \right\rangle = \langle v^*, w\rangle .\end{aligned}$$

However, this implies that $\pi_u v^* = v^*$, i.e. (2.8) implies (2.9) as claimed.

The theorem now follows in the case of $G = \mathrm{SL}_d(\mathbb{R})$ from the claim. Indeed, by the Mautner phenomenon (Proposition 2.24) and the assumption that there are no nontrivial fixed vectors, we see that $v^* = 0$, which, as explained above, implies (2.7).

In the general case we apply a little more structure theory for simple Lie groups. Let $a_n \in A < G$ be the product $a_n = a_n^{(1)} \cdots a_n^{(r)}$ with $a_n^{(i)} \in G_i$ for $i = 1, \ldots, r$. Let i be chosen so that $a_n^{(i)} \to \infty$ as $n \to \infty$. Recall that the simple roots form a basis of the dual of the Lie algebra of A (see Knapp [102, Sec. II.5]). Hence there exists a subsequence and a non-trivial nilpotent $x \in \mathfrak{g}_i$ from one of the root spaces (corresponding to a simple or the negative of a simple root) so that $\mathrm{Ad}_{a_n}^{-1}(x) \to 0$ as $n \to \infty$. In other words, $u = \exp(x)$ satisfies (2.8). By the argument above this in turn implies (2.9). We now conclude using the Mautner phenomenon (Proposition 2.24): The vector v^* is fixed under all almost direct factors G_i of G for which $a_n^{(i)} \to \infty$. In both case (1) and case (2), this implies that $v^* = 0$, and hence the theorems. □

Exercise 2.29 (Mautner for simple groups). Let G be a simple closed linear group, let $L < G$ be an unbounded subgroup, let π be a unitary representation of G on a Hilbert space $\mathcal{H}$, and let $v_0 \in \mathcal{H}$ be a vector fixed by all elements of L. Show that v_0 is then also fixed by all elements of G.

Exercise 2.30 (Lie groups locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$). Suppose G is a connected Lie group locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ acting unitarily on a Hilbert space $\mathcal{H}$. Suppose a nilpotent element $u \neq 0$ of the Lie algebra of G fixes a vector $v_0 \in \mathcal{H}$. Then all of G fixes v_0.

Exercise 2.31 (Howe–Moore). Generalize the Howe–Moore theorem (Theorem 2.28) to a connected semi-simple Lie group G with finite centre. For this you may use the fact that the finite centre allows a Cartan decomposition $G = KAK$ with a compact subgroup $K \leqslant G$.

The assumption that the centre be finite is necessary. If $G = \widetilde{\mathrm{SL}_2(\mathbb{R})}$ is the universal cover of $\mathrm{SL}_2(\mathbb{R})$, then there are ergodic actions of G on non-trivial probability spaces in which the infinite centre (which is isomorphic to $\mathbb{Z}$) acts trivially (as for example the action of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ induced by the natural ergodic action of $\mathrm{SL}_2(\mathbb{R})$ on X_2).

2.5 *p*-Adic Groups

For number-theoretic applications it is often important to consider locally homogeneous spaces defined by closed linear p-adic groups $G \leqslant \mathrm{SL}_d(\mathbb{Q}_p)$ for a prime $p \in \mathbb{N}$. The results of Section 1.2 were phrased abstractly and so apply equally well to p-adic groups and to products of real Lie groups and p-adic groups (see Exercise 2.32).

Moreover, to some extent the discussions in Section 2.2 concerning Lie algebras and semi-simple Lie groups generalize to closed linear p-adic groups. In fact for $v \in \mathfrak{gl}_d(\mathbb{Q}_p) = \mathrm{Mat}_d(\mathbb{Q}_p)$ with its norm

$$\|v\| = \max_{i,j=1,\dots,d} |v_{i,j}|_p$$

sufficiently small (see Exercise 2.33) the exponential series

$$\exp(v) = \sum_{n=0}^{\infty} \frac{1}{n!} v^n$$

converges and defines an element of $\mathrm{GL}_d(\mathbb{Q}_p)$ close to I. Just as in the real case the local inverse exists and is given by

$$\log(g) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (g - I)$$

for $g \in \mathrm{GL}_d(\mathbb{Q}_p)$ sufficiently close to I. In this sense $\mathfrak{gl}_d(\mathbb{Q}_p)$ is again the Lie algebra of $\mathrm{GL}_d(\mathbb{Q}_p)$ and $\mathfrak{sl}_d(\mathbb{Q}_p) = \{v \in \mathfrak{gl}_d(\mathbb{Q}_p) \mid \operatorname{tr} v = 0\}$ is the Lie algebra of $\mathrm{SL}_d(\mathbb{Q}_p)$.

There are however fundamental differences between the real and the p-adic cases that require some care. To begin with, every element $x \in \mathbb{Q}_p$ and every element $g \in \mathrm{GL}_d(\mathbb{Q}_p)$ close enough to I generate compact subgroups (see Exercise 2.33). Moreover, as $\mathbb{Q}_p$ is totally disconnected no non-trivial closed linear p-adic group can be connected or simply connected as a topological space. This makes it unclear how to phrase, for example, the hypotheses of a p-adic version of Theorem 2.25 even though the argument generalizes to a large extent. For now we only claim that Theorems 2.25 and 2.26 still hold for $G = \mathrm{SL}_d(\mathbb{Q}_p)$ and $d \geqslant 2$ (see Exercise 2.37).

Exercise 2.32 (Left-invariant metric). Find a left-invariant on $G = \mathrm{SL}_d(\mathbb{Q}_p)$ that induces the topology on G inherited from $\mathrm{Mat}_d(\mathbb{Q}_p) \supseteq G$.

Exercise 2.33. (a) For a sequence (a_n) in $\mathbb{Q}_p$ define the *radius of convergence* R of its associated power series $\sum_{n=0}^{\infty} a_n x^n$ by the Hadamard formula

$$\frac{1}{R} = \limsup_{n \to \infty} \sqrt[n]{|a_n|_p}$$

and show that the power series converges in $\mathbb{Q}_p$ if $|x|_p < R$ and diverges if $|x|_p > R$.

(b) Calculate the radius of convergence of the power series $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ and $\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ over $\mathbb{Q}_p$.

(c) Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be power series and let R be the minimum of the two radii of convergence. Show the Cauchy product formula

$$\Big(\sum_{n=0}^{\infty} a_n x^n\Big)\Big(\sum_{n=0}^{\infty} b_n x^n\Big) = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty} a_k b_{n-k}\Big)x^n$$

for $x \in \mathbb{Q}_p$ with $|x|_p < R$.

(d) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R. Show that the power series $\sum_{n=0}^{\infty} a_n v^n$ converges for $v \in \mathrm{Mat}_d(\mathbb{Q}_p)$ with $\|v\| < R$.

(e) Show that exp is defined on a neighbourhood of $0 \in \mathfrak{gl}_d(\mathbb{Q}_p)$, that log is defined on a neighbourhood of $I \in \mathrm{GL}_d(\mathbb{Q}_p)$, and that locally they are inverses of each other.

(f) Show that every $g \in \mathrm{GL}_d(\mathbb{Q}_p)$ close enough to I generates a compact subgroup.

For $G = \mathrm{SL}_d(\mathbb{Q}_p)$ we define $K = \mathrm{SL}_d(\mathbb{Z}_p)$,

$$A = \left\{ \begin{pmatrix} p^{\alpha_1} & & \\ & \ddots & \\ & & p^{\alpha_n} \end{pmatrix} \mid \alpha_1, \dots, \alpha_n \in \mathbb{Z}, \alpha_1 + \dots + \alpha_n = 0 \right\},$$

and N to be the upper triangular subgroup with 1s along the diagonal.

Exercise 2.34 (Iwasawa decomposition). Prove that any $g \in \mathrm{SL}_d(\mathbb{Q}_p)$ can be written as $g = kan$ with $k \in K$, $a \in A$, and $n \in N$.

Exercise 2.35 (Cartan decomposition). Prove that any $g \in \mathrm{SL}_d(\mathbb{Q}_p)$ can be written as $g = kak'$ with $k, k' \in K$ and $a \in A$.

Exercise 2.36 (Mautner phenomenon for $\mathrm{SL}_d(\mathbb{Q}_p)$). Prove the analogue of Proposition 2.15 for the case $\mathrm{SL}_d(\mathbb{Q}_p)$ for $d \geqslant 2$. More precisely show that $\mathrm{SL}_d(\mathbb{Q}_p)$ fixes $v_0 \in \mathcal{H}$ if $\mathrm{SL}_d(\mathbb{Q}_p)$ acts unitarily on $\mathcal{H}$ and either

(a) v_0 is fixed by some diagonal element with at least one eigenvalue of absolute value not equal to one, or

(b) v_0 is fixed by a one-parameter† unipotent subgroup $\{\exp(sw) \mid s \in \mathbb{Q}_p\}$ defined by some non-trivial nilpotent $w \in \mathrm{Mat}_{2,2}(\mathbb{Q}_p)$.

Exercise 2.37 (Howe–Moore). Formulate and prove analogues of Theorems 2.25 and 2.26 for $G = \mathrm{SL}_d(\mathbb{Q}_p)$.

Exercise 2.38. Suppose that $L \leqslant \mathrm{SL}_d(\mathbb{Q}_p)$ is an unbounded and open subgroup. Show that this implies $L = \mathrm{SL}_d(\mathbb{Q}_p)$.

† We note that in this p-adic case a single element of this subgroup (isomorphic to $\mathbb{Q}_p$) generates a compact subgroup and so could not exhibit the Mautner phenomenon.

2.6 The General Mautner Phenomenon*

We will now present the general Mautner phenomena for Lie groups, which was proven by Moore [135] in 1980. We will only discuss the proof through a series of guided exercises.

Theorem 2.39 (Mautner phenomenon). *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $L < G$ be a closed subgroup, and let G act unitarily on a Hilbert space $\mathcal{H}$. We suppose $v_0 \in \mathcal{H}$ is fixed by every element of L. Then there exists a Lie ideal $\mathfrak{f} \lhd \mathfrak{g}$ (the* Mautner ideal*) such that*

- *v_0 is fixed by $\exp(\mathfrak{f}) \leqslant G$,*
- *$\mathfrak{f}$ is normalized by Ad_g for $g \in L$, and*
- *the map on $\mathfrak{g}/\mathfrak{f}$ induced by Ad_g for $g \in L$ is diagonalizable with all eigenvalues of absolute value one.*

The proof of Theorem 2.39 will combine the key lemma (Lemma 2.13) with techniques from the theories of Lie groups and Lie algebras. It subsumes the ergodicity of many natural actions.

2.6.1 The Structure of the Inductive Step

We notice first that in proving Theorem 2.39 we may assume $G = \langle L, G^o \rangle$ is generated by L and its connected component of the identity G^o. Moreover, we may assume that v_0 is a *cyclic vector* in the sense that $\mathcal{H} = \overline{\langle \pi_G v_0 \rangle}$ is the smallest closed subspace containing the orbit of v_0 under the action of G. Otherwise we may simply restrict to the open subgroup $\langle L, G^o \rangle$ and restrict the unitary representation to the subspace $\overline{\langle \pi_G v_0 \rangle}$.

This remark allow us to use induction on the dimension of G. In the inductive step we will show that there is a non-trivial Lie ideal $\mathfrak{f} \lhd \mathfrak{g}$ that fixes v_0. This implies for $g \in L$ that $\mathrm{Ad}_g \mathfrak{f} \lhd \mathfrak{g}$ is another Lie ideal that fixes v_0. Taking their linear hulls and exponentials gives a normal subgroup $F \lhd G$ generated by $g \exp(\mathfrak{f}) g^{-1}$ for $g \in L$. Let $\overline{F}$ be the closure of F (*a priori* there is no reason for F to be closed), so that $\overline{F} \lhd G$ is a closed normal subgroup that fixes v_0. We claim that $\overline{F}$ acts trivially on $\mathcal{H}$. Indeed, if $g \in G$ and $h \in \overline{F}$ then $hg = gh'$ for some $h' \in \overline{F}$, and

$$\pi_h \pi_g v_0 = \pi_g \pi_{h'} v_0 = \pi_g v_0.$$

Since $\mathcal{H} = \overline{\langle \pi_G v_0 \rangle}$ we see that $\overline{F}$ acts trivially. Therefore we may consider the unitary representation of $G/\overline{F}$ on $\mathcal{H}$ induced by the unitary representation of G that we started with. If $\mathfrak{f} \lhd \mathfrak{g}$ was a non-trivial Lie ideal, then the dimension of $\widetilde{G} = G/F$ is smaller than the dimension of G.

By induction we may assume that Theorem 2.39 already holds for $\widetilde{G}$ (with the subgroup $\widetilde{L} = LF/F < \widetilde{G} = G/F$) acting on $\mathcal{H}$. This in turn then implies the theorem also for G.

2.6.2 The Inductive Step

In the remainder of the section we will always assume that G, L, and $v_0 \in \mathcal{H}$ are as in Theorem 2.39 and Section 2.6.1. We will use the following reformulation of the key lemma to create ideals in $\mathfrak{g}$.

Exercise 2.40 (Key lemma for the Lie algebra). (a) Show that v_0 is also fixed for $g \in L$ by all elements of the subspace

$$\operatorname{Im}\left(\operatorname{Ad}_g - I\right) \cap \ker\left(\operatorname{Ad}_g - I\right) \subseteq \mathfrak{g},$$

and that all of these elements are nilpotent.
(b) Show that if $u \in \mathfrak{g}$ is nilpotent and fixes v_0, then the subspace $\operatorname{Im}\operatorname{ad}_u \cap \ker\operatorname{ad}_u$ also consists of nilpotent elements fixing v_0.

Recall from Section 2.2.3 that a real Lie algebra $\mathfrak{g}$ has a Levi decomposition(8)

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$$

where $\mathfrak{l}$ is a semi-simple real Lie algebra, and $\mathfrak{r} \lhd \mathfrak{g}$ is the radical (the maximal solvable Lie ideal of $\mathfrak{g}$). Also recall from Knapp [102, Prop. 1.40] that

$$\mathfrak{n} = [\mathfrak{r}, \mathfrak{g}] \lhd \mathfrak{g}$$

is a nilpotent Lie ideal.

Exercise 2.41 (Nilpotent elements of the radical). Suppose there is a nilpotent element $u \in \mathfrak{r} \smallsetminus \{0\}$ (with $\operatorname{Ad}_{\exp(u)}$ unipotent) in the radical of the Lie algebra that fixes v_0. Then there is a non-trivial Lie ideal $\mathfrak{f} \lhd \mathfrak{g}$ that fixes v_0.

This exercise shows that in the situation above we can always apply the inductive step outlined in Section 2.6.1. In particular we can also conclude from the inductive argument that there exists a Lie ideal $\mathfrak{f}$ containing u which fixes v_0.

Let $g \in L$. If Ad_g has an eigenvalue of absolute value greater than or smaller than 1, then we may apply Section 2.3.2 to find the non-trivial Auslander ideal of g that fixes v_0, and use induction.

Suppose therefore that all the eigenvalues of Ad_g have absolute value equal to 1, but that Ad_g is not diagonalizable over $\mathbb{C}$ (since in that case the theorem already holds trivially for g). Then there exist two vectors $u, w \in \mathfrak{g} \smallsetminus \{0\}$ with

$$\begin{aligned} \operatorname{Ad}_g(u) &= \lambda u, \\ \operatorname{Ad}_g(w) &= \lambda(w+u), \end{aligned}$$

and so for $n \in \mathbb{N}$,

$$\operatorname{Ad}_g^n(w) = \lambda^n (w + nu)\,. \tag{2.10}$$

These expressions have the obvious meaning if $\lambda \in \{\pm 1\}$, but if $\lambda \in \mathbb{S}^1 \smallsetminus \{\pm 1\}$ then we are using the symbol λ as a convenient shorthand for a rotation of the real linear space corresponding to a complex eigenvalue. In any case, there is a sequence (n_k) with $n_k \to \infty$ as $k \to \infty$ along which λ^{n_k} converges to the identity. Using this sequence we can divide (2.10) by n_k and find

$$\lim_{k\to\infty} \mathrm{Ad}_g^{n_k}\left(\tfrac{1}{n_k}w\right) = u. \tag{2.11}$$

Exercise 2.42. Show that u is nilpotent and fixes v_0.

If $u \in \mathfrak{r}$ belongs to the radical then Exercise 2.41 applies and gives a non-trivial Lie ideal fixing v_0. Thus we may assume that

$$u = x + u_{\mathfrak{r}}$$

with $x \in \mathfrak{l}\smallsetminus\{0\}$ and $u_{\mathfrak{r}} \in \mathfrak{r}$. We note that $x \in \mathfrak{l}\smallsetminus\{0\}$ is a nilpotent element of the semi-simple Lie algebra $\mathfrak{l}$ (because, for example, the adjoint of x on $\mathfrak{l} \cong \mathfrak{g}/\mathfrak{r}$ coincides with the adjoint of u on $\mathfrak{g}/\mathfrak{r}$). Hence we may apply the Jacobson–Morozov theorem (Theorem 2.23) and choose an $\mathfrak{sl}_2$-triple (h, x, y) in $\mathfrak{l}^3$.

Note that if we would have $u = x$ then we could apply the Mautner phenomenon for groups locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ in Exercise 2.30. Moreover, Section 2.3.2 allows us to obtain a non-trivial Lie ideal fixing v_0. Our aim is therefore to reduce the proof via induction to this case.

If $[u, \mathfrak{r}] \neq 0$ then we can apply Exercises 2.40 and 2.41 once again to find a non-trivial Lie ideal fixing v_0.

So assume that $[u, \mathfrak{r}] = 0$. Then we have $[u, h] = -2x + [u_{\mathfrak{r}}, h]$ and so

$$[u, [u, h]] = [u, -2x + [u_{\mathfrak{r}}, h]] = [u, -2x] = [x + u_{\mathfrak{r}}, -2x] = -2[u_{\mathfrak{r}}, x] \in \mathfrak{r}$$

since $[u_{\mathfrak{r}}, h] \in \mathfrak{r}$. This implies that $[u, [u, [u, h]]] = 0$. Hence if $[u, [u, h]] \neq 0$ we may apply Exercises 2.40 and 2.41 to find a non-trivial Lie ideal fixing v_0.

So assume that $[u, \mathfrak{r}] = 0$ and $[u, [u, h]] = 0$. By Exercise 2.40 $[u, h]$ fixes v_0.

Exercise 2.43. Show that if $[u, h] \neq -2u$, then there exists a non-trivial Lie ideal fixing v_0.

Exercise 2.44. Suppose $[u, \mathfrak{r}] = 0$ and $[u, h] = -2u$. Use Lie theory to show that $u_{\mathfrak{r}} = 0$ and hence that $u = x$ is a member of an $\mathfrak{sl}_2$-triple inside $\mathfrak{l}$.

To summarize, if for some $g \in L$ the map Ad_g has eigenvalues of absolute value not equal to 1 we find a non-trivial Auslander ideal fixing v_0. If all eigenvalues have absolute value 1 but Ad_g has non-trivial Jordan blocks we find (by using (2.11)) nilpotent elements fixing v_0. Using the Levy decomposition and general Lie theory this again leads via case distinction to the existence of a non-trivial Lie ideal fixing v_0.

Notes to Chapter 2

(5) (Page 54) The main result here is due to Lindenstrauss [120], who showed that any locally compact amenable group has a Følner sequence along which the pointwise ergodic theorem

holds. We refer to a survey of Nevo [139] for an overview of both the amenable case and the case of certain non-amenable groups, and to [52, Ch. 8] for an accessible discussion of the case of groups with polynomial growth.

(6)(Page 62) This argument comes from Margulis [126], and the argument is also presented in [52, Prop. 11.18].

(7)(Page 67) Theorem 2.23 was stated by Morozov [136] and a complete proof was provided by Jacobson [86].

(8)(Page 76) This decomposition, conjectured by Killing and Cartan, was shown by Levi [117], and Malcev [122] later showed that any two Levi factors (the semi-simple Lie algebra viewed as a factor-algebra of $\mathfrak{g}$) are conjugate by a specific form of inner automorphism; we refer to Knapp [102, Th. B.2] for the proof.