Chapter 6
Lie Algebras and Unitary Representations of SU(2)

To continue the discussion of unitary representations, we need to introduce Lie algebras and their (for now) finite-dimensional representations. In particular this will allow us to describe the representation theory of the compact group $\mathrm{SU}_2(\mathbb{R})$, which represents an important example. For this we will sometimes use the following notational conventions in addition to the standing assumptions and notations of Section 1.1.

- If G is a real (or complex) Lie group (which is not assumed to be connected), then we write $\mathfrak{g} = \mathrm{Lie}\, G$ for its real (or complex) Lie algebra and write $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f} \in \mathfrak{g}$ for its elements. Recall that there is a smooth map $\exp\colon \mathfrak{g} \to G$ with a local inverse $\log\colon B_\delta^G(I) \to \mathfrak{g}$ defined on some neighbourhood $B_\delta^G(I)$ of the identity $I \in G$ with $\delta > 0$.
- We will use the letters s, t to denote real numbers.

6.1 Finite-Dimensional Representation Theory of SL(2)

For the classification of simple (and semi-simple) Lie groups and their finite-dimensional representations the most important Lie group to understand is $\mathrm{SL}_2(\mathbb{R})$. As we will see later (in Chapters 8 and 9), this remains true for the theory of unitary representations. As a warm-up for this discussion as well as because of its independent interest, we study in this chapter the twin sibling $\mathrm{SU}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$. In fact, as we will explain shortly these two groups are strongly related: $\mathrm{SU}_2(\mathbb{R})$ is the *compact real form*, and $\mathrm{SL}_2(\mathbb{R})$ is the *split real form*, of the complex Lie group $\mathrm{SL}_2(\mathbb{C})$, and they have identical descriptions of their finite-dimensional representations (which can be made unitary for $\mathrm{SU}_2(\mathbb{R})$ but are not unitary except in trivial cases for $\mathrm{SL}_2(\mathbb{R})$; see Exercise 1.87 and 1.89). As before, we will always study representations on complex vector spaces.

6.1.1 A Quick Reminder on Closed Linear Groups

We wish to recall the most basic concepts for Lie group, with a focus on closed linear groups and refer to Knapp [44, Sec. 0] for more details. In fact, many facts concerning the connection between Lie groups and their Lie algebras are easy to verify for closed linear Lie groups $G < \mathrm{GL}_n(\mathbb{R})$ using only the properties of the exponential map

$$\begin{aligned}\exp\colon \mathrm{Mat}_{n,n}(\mathbb{R}) &\longrightarrow \mathrm{GL}_n(\mathbb{R})\\ m &\longmapsto \exp(m) = \sum_{k=0}^{\infty} \tfrac{1}{k!} m^k.\end{aligned}$$

One such basic property is that the derivative of exp is the identity map on $\mathrm{End}(\mathbb{R}^n) = \mathrm{Mat}_{n,n}(\mathbb{R})$, which is also referred to as the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of $\mathrm{GL}_n(\mathbb{R})$, and thought of as the tangent space of $\mathrm{GL}_n(\mathbb{R})$ at the identity I. Using this, we can calculate, for example, the *adjoint representation*, denoted Ad, of $\mathrm{GL}_n(\mathbb{R})$ on its Lie algebra $\mathfrak{gl}_n(\mathbb{R})$, which is defined by the derivative of conjugation at the identity for any $g \in \mathrm{GL}_n(\mathbb{R})$. This gives

$$\mathrm{Ad}_g(\mathbf{m}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big(g \exp(t\mathbf{m}) g^{-1}\big) = g \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big(\exp(t\mathbf{m})\big) g^{-1} = g\mathbf{m}g^{-1}$$

for all $\mathbf{m} \in \mathfrak{gl}_n(\mathbb{R})$, where the last three expressions involve the usual product of matrices. Moreover, the *adjoint representation*, denoted ad of $\mathfrak{gl}_n(\mathbb{R})$ on $\mathfrak{gl}_n(\mathbb{R})$ is defined as the derivative of the map

$$\mathrm{Ad}\colon \mathrm{GL}_n(\mathbb{R}) \ni g \longmapsto \mathrm{Ad}_g \in \mathrm{GL}\big(\mathfrak{gl}_n(\mathbb{R})\big)$$

at I, which is a map from $\mathfrak{gl}_n(\mathbb{R})$ into the Lie algebra $\mathrm{End}\big(\mathfrak{gl}_n(\mathbb{R})\big)$ of the group $\mathrm{GL}\big(\mathfrak{gl}_n(\mathbb{R})\big)$. By the chain and product rules, this gives

$$\mathrm{ad}_{\mathbf{a}}(\mathbf{m}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \big(\exp(t\mathbf{a})\mathbf{m}\exp(-t\mathbf{a})\big) = \mathbf{am} - \mathbf{ma} \tag{6.1}$$

for all $\mathbf{a}, \mathbf{m} \in \mathfrak{gl}_n(\mathbb{R})$.

We also call the map

$$\begin{aligned}[\cdot,\cdot]\colon\ \mathfrak{gl}_n(\mathbb{R}) \times \mathfrak{gl}_n(\mathbb{R}) &\longrightarrow \mathfrak{gl}_n(\mathbb{R})\\ (\mathbf{a}, \mathbf{m}) &\longmapsto [\mathbf{a}, \mathbf{m}] = \mathbf{am} - \mathbf{ma}\end{aligned}$$

the *Lie bracket*. This is a bilinear map satisfying

$$[\mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}]$$

for all $\mathbf{a}, \mathbf{b} \in \mathfrak{gl}_n(\mathbb{R})$. Moreover, the *Jacobi identity*

$$[\mathbf{a},[\mathbf{b},\mathbf{c}]]+[\mathbf{b},[\mathbf{c},\mathbf{a}]]+[\mathbf{c},[\mathbf{a},\mathbf{b}]]=0$$

can be checked by a direct calculation. Equivalently, we have

$$[[\mathbf{a},\mathbf{b}],\mathbf{c}]=[\mathbf{a},[\mathbf{b},\mathbf{c}]]-[\mathbf{b},[\mathbf{a},\mathbf{c}]]$$

for all $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathfrak{gl}_n(\mathbb{R})$, which can also be written as

$$\mathrm{ad}_{[\mathbf{a},\mathbf{b}]}=[\mathrm{ad}_{\mathbf{a}},\mathrm{ad}_{\mathbf{b}}]=\mathrm{ad}_{\mathbf{a}}\circ\mathrm{ad}_{\mathbf{b}}-\mathrm{ad}_{\mathbf{b}}\circ\mathrm{ad}_{\mathbf{a}}$$

for all $\mathbf{a},\mathbf{b}\in\mathfrak{gl}_n(\mathbb{R})$, where $\mathrm{ad}_{\bullet}$ is defined in (6.1).

We note that these facts generalize without too much effort to closed linear subgroups G of $\mathrm{GL}_n(\mathbb{R})$, and that these cases will suffice for all of our discussions.

Essential Exercise 6.1. (a) Show that any continuous homomorphism

$$g_{\cdot}\colon\mathbb{R}\ni t\longmapsto g_t\in\mathrm{GL}_n(\mathbb{R})$$

is differentiable.
(b) Show that the map $\mathfrak{gl}_n(\mathbb{R})\ni\mathbf{m}\mapsto\big(\mathbb{R}\ni t\mapsto g_t=\exp(t\mathbf{m})\big)$ is a bijection between elements $\mathbf{m}$ of the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ and differentiable one-parameter subgroups as in (a).

6.1.2 The Lie Groups SU(2) and SL(2)

We recall the definition of the real Lie group

$$\mathrm{SL}_2(\mathbb{R})=\Big\{g=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{Mat}_{22}(\mathbb{R})\mid\det g=ad-bc=1\Big\},$$

called the *special linear group in 2 dimensions*, and its Lie algebra

$$\mathfrak{sl}_2(\mathbb{R})=\{\mathbf{m}\in\mathrm{Mat}_{22}(\mathbb{R})\mid\operatorname{tr}\mathbf{m}=0\},$$

which contains all elements of the form

$$\mathbf{m}=\begin{pmatrix}a&b\\c&-a\end{pmatrix}$$

for $a,b,c\in\mathbb{R}$. As in any Lie group, the Lie algebra has the property that $\exp\big(\mathfrak{sl}_2(\mathbb{R})\big)$ is an open neighbourhood of the identity $I\in\mathrm{SL}_2(\mathbb{R})$ and so can be used to describe $\mathrm{SL}_2(\mathbb{R})$ locally.

We will frequently use the basis of $\mathfrak{sl}_2(\mathbb{R})$ given by

$$\mathbf{a}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\mathbf{e}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\mathbf{f}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad(6.2)$$

which we will refer to as the $\mathfrak{sl}_2$-triple. These three elements satisfy the following relations

$$\left.\begin{aligned} [\mathbf{a},\mathbf{e}] &= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=2\mathbf{e}\\ [\mathbf{a},\mathbf{f}] &= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=-2\mathbf{f}\\ [\mathbf{e},\mathbf{f}] &= \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\mathbf{a}. \end{aligned}\right\} \tag{6.3}$$

In other words, with respect to the map $\mathrm{ad}_{\mathbf{a}} = [a,\cdot]\colon \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$, $\mathbf{e}$ is an eigenvector with eigenvalue 2, $\mathbf{f}$ is an eigenvector with eigenvalue -2, $\mathbf{a}$ is an eigenvector with eigenvalue 0, and $\mathbf{e}$ and $\mathbf{f}$ together generate $\mathbf{a}$. As a consequence, the number 2 will be quite prevalent in the representation theory of $\mathrm{SL}_2(\mathbb{R})$. Much of what we wish to discuss here will use these simple relations. We also note that we have been using versions of the relations (6.3) within the group $\mathrm{SL}_2(\mathbb{R})$ already in Section 1.7.

We will also use the complex Lie group

$$\mathrm{SL}_2(\mathbb{C}) = \left\{g = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \mathrm{Mat}_{22}(\mathbb{C}) \mid \det g = ad - bc = 1\right\},$$

and, more specifically, its complex Lie algebra

$$\mathfrak{sl}_2(\mathbb{C}) = \{\mathbf{m} \in \mathrm{Mat}_{22}(\mathbb{C}) \mid \mathrm{tr}\,\mathbf{m} = 0\},$$

which also has the Lie algebra elements $\mathbf{a}$, $\mathbf{e}$, and $\mathbf{f}$ as a basis over $\mathbb{C}$. A more formal way of stating this is to say that there is an isomorphism

$$\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}, \tag{6.4}$$

where $\mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ is the complex Lie algebra obtained by bilinearly extending the Lie bracket map from $\mathbb{R}$ to $\mathbb{C}$. Because of the isomorphism in (6.4) we also say that $\mathfrak{sl}_2(\mathbb{R})$ is a *real form* or, more specifically, the *split real form* of $\mathfrak{sl}_2$. We now turn our attention to the only other real form (up to isomorphism).

The *special unitary group*

$$\mathrm{SU}_2(\mathbb{R}) = \{g \in \mathrm{Mat}_{22}(\mathbb{C}) \mid g^*g = I, \det g = 1\},$$

is a real Lie group with Lie algebra

$$\mathfrak{su}_2(\mathbb{R}) = \{\mathbf{m} \in \mathrm{Mat}_{22}(\mathbb{C}) \mid \mathbf{m}^* + \mathbf{m} = 0, \mathrm{tr}\,\mathbf{m} = 0\}.$$

We note that these are not a complex Lie group and Lie algebra since the adjoint operation is semi-linear over $\mathbb{C}$. The elements of $\mathfrak{su}_2(\mathbb{R})$ have the form

$$\mathbf{m} = \begin{pmatrix} a\mathrm{i} & b\mathrm{i}-c \\ b\mathrm{i}+c & -a\mathrm{i} \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$ and we may use the basis

$$\mathbf{b}_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \ \mathbf{b}_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \mathbf{b}_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.5}$$

with the relations

$$\left.\begin{aligned} [\mathbf{b}_1, \mathbf{b}_2] &= \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = 2\mathbf{b}_3 \\ [\mathbf{b}_2, \mathbf{b}_3] &= \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = 2\mathbf{b}_1 \\ [\mathbf{b}_3, \mathbf{b}_1] &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} - \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2\mathbf{b}_2. \end{aligned}\right\} \tag{6.6}$$

Since the elements in (6.5) are also a basis of $\mathfrak{sl}_2(\mathbb{C})$ over $\mathbb{C}$, we obtain once again an isomorphism

$$\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Once more we say that $\mathfrak{su}_2(\mathbb{R})$ is a real form of $\mathfrak{sl}_2(\mathbb{C})$, which is called the *compact real form* since $\mathrm{SU}_2(\mathbb{R})$ is compact.

To better understand $\mathfrak{su}_2(\mathbb{R})$ and $\mathrm{SU}_2(\mathbb{R})$, we may also present these in equivalent ways as in the next lemma.

Lemma 6.2 (Two isomorphisms). (a) *The vector space $\mathbb{R}^3$ equipped with the cross product is a Lie algebra isomorphic to $\mathfrak{su}_2(\mathbb{R})$.*
(b) *The sphere $\mathbb{S}^3 \subseteq \mathbb{H}$ inside the four-dimensional Hamiltonian quaternions $\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ forms a real Lie group with respect to multiplication so that $\mathbb{S}^3 \cong \mathrm{SU}_2(\mathbb{R})$ as real Lie groups.*

Proof of Lemma 6.2(a). The cross product map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} y\gamma - z\beta \\ z\alpha - x\gamma \\ x\beta - y\alpha \end{pmatrix}$$

for

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathbb{R}^3$$

is bilinear and antisymmetric, just as a Lie bracket is. Also note that we have the relations $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, and $e_3 \times e_1 = e_2$. We define a linear map $\varphi\colon \mathbb{R}^3 \to \mathfrak{su}_2(\mathbb{R})$ by

$$\varphi(e_1) = \tfrac{1}{2}\mathbf{b}_1,$$
$$\varphi(e_2) = \tfrac{1}{2}\mathbf{b}_2,$$
$$\varphi(e_3) = \tfrac{1}{2}\mathbf{b}_3.$$

Then φ is a linear isomorphism which satisfies $\varphi(a \times b) = [\varphi(a), \varphi(b)]$. In fact this follows first for a and b being any two basis vectors in positive order by dividing the relations in (6.6) by 4. By antisymmetry of the cross product and the Lie bracket this extends to any two basis vectors. Finally, by bi-linearity of the cross product and the Lie bracket this extends to all $a, b \in \mathbb{R}^3$. As $\mathfrak{su}_2(\mathbb{R})$ is a Lie algebra, the same therefore holds for $\mathbb{R}^3$ with the cross product, proving (a). □

PROOF OF LEMMA 6.2(b). We recall first that the Hamiltonian quaternions are defined by

$$\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are formal symbols satisfying the four relations†

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -1, \\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k}, \\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i}, \text{ and} \\ \mathbf{ki} = -\mathbf{ik} &= \mathbf{j}. \end{aligned}$$

We may identify $\mathbb{H}$ with

$$\left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C} \right\} \subseteq \mathrm{Mat}_{22}(\mathbb{C}).$$

In fact, a direct calculation shows that the matrices $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ in (6.5) satisfy the relations between $\mathbf{i}, \mathbf{j}, \mathbf{k}$ above, so that

$$\varphi\colon \mathbb{H} \ni x + a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \longmapsto xI + a\mathbf{b}_1 + b\mathbf{b}_2 + c\mathbf{b}_3 = \begin{pmatrix} x + a\mathrm{i} & b\mathrm{i} - c \\ b\mathrm{i} + c & x - a\mathrm{i} \end{pmatrix} \in \mathrm{Mat}_{2,2}(\mathbb{C})$$

is an algebra isomorphism to a sub-algebra of $\mathrm{Mat}_{2,2}(\mathbb{C})$.

The norm operator defined by

$$N(x + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = x^2 + a^2 + b^2 + c^2 = \det \varphi(x + a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

for $x, a, b, c \in \mathbb{R}$ satisfies $N(\mathbf{gh}) = N(\mathbf{g})N(\mathbf{h})$ for all $\mathbf{g}, \mathbf{h} \in \mathbb{H}$. It follows that

$$\mathbb{S}^3 = \{\mathbf{g} \in \mathbb{H} \mid N(\mathbf{g}) = 1\}$$

is a Lie group under multiplication, which is mapped under φ to

† These relations are easy to reconstruct because of their symmetry under cyclic permutations.

$$\varphi(\mathbb{S}^3) = \left\{ g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C}, \det g = |z|^2 + |w|^2 = 1 \right\}.$$

Note that

$$g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \in \varphi(\mathbb{S}^3)$$

satisfies

$$g^* g = \begin{pmatrix} \overline{z} & \overline{w} \\ -w & z \end{pmatrix} \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} = \begin{pmatrix} |z|^2 + |w|^2 & 0 \\ 0 & |z|^2 + |w|^2 \end{pmatrix} = I$$

and so $\varphi(\mathbb{S}^3) \subseteq \mathrm{SU}_2(\mathbb{R})$. Conversely, if

$$g = \begin{pmatrix} z & z_1 \\ w & w_1 \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R}),$$

then $\left\| \begin{pmatrix} z \\ w \end{pmatrix} \right\| = 1$ and

$$0 = \left\langle \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} -\overline{w} \\ \overline{z} \end{pmatrix} \right\rangle$$

implies that $\begin{pmatrix} z_1 \\ w_1 \end{pmatrix} = \alpha \begin{pmatrix} -\overline{w} \\ \overline{z} \end{pmatrix}$ for some $\alpha \in \mathbb{C}$. Taking the determinant this gives

$$1 = \det g = \det \begin{pmatrix} z & -\alpha\overline{w} \\ w & \alpha\overline{z} \end{pmatrix} = \alpha|z|^2 + \alpha|w|^2 = \alpha.$$

This proves that $\varphi(\mathbb{S}^3) = \mathrm{SU}_2(\mathbb{R})$, and hence part (b) of the lemma. □

The following lemma describes the above Lie groups from a topological point of view.

Lemma 6.3 (Topological properties). (a) *The groups* $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SU}_2(\mathbb{R})$, *and* $\mathrm{SL}_2(\mathbb{C})$ *are connected.*
(b) *The Lie groups* $\mathrm{SU}_2(\mathbb{R})$ *and* $\mathrm{SL}_2(\mathbb{C})$ *are simply connected.*
(c) *The universal cover* $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ *of* $\mathrm{SL}_2(\mathbb{R})$ *is a* $\mathbb{Z}$*-cover of* $\mathrm{SL}_2(\mathbb{R})$.

Proof. For $\mathrm{SU}_2(\mathbb{R})$ both (a) and (b) follow easily from Lemma 6.2(b) since $\mathrm{SU}_2(\mathbb{R}) \cong \mathbb{S}^3$ is connected and simply connected.

The facts concerning $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ follow from the Iwasawa decomposition of these groups, as we will now show.

For $\mathrm{SL}_2(\mathbb{R})$, let

$$A = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \;\middle|\; t > 0 \right\},$$

$$U = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$

and $K = \mathrm{SO}_2(\mathbb{R})$. Then every element $g \in \mathrm{SL}_2(\mathbb{R})$ has a unique decomposition $g = kau$ with $k \in K$, $a \in A$, and $u \in U$ by the Iwasawa decomposition (see Exercise 6.4). This gives a homeomorphism $\mathrm{SL}_2(\mathbb{R}) \cong KAU \cong \mathbb{S}^1 \times \mathbb{R}^2$ which proves the claims in (a) and (c) for $\mathrm{SL}_2(\mathbb{R})$.

For $\mathrm{SL}_2(\mathbb{C})$ we have A as before,

$$U = \left\{ \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{C} \right\},$$

and $K = \mathrm{SU}_2(\mathbb{R}) \cong \mathbb{S}^3$, which gives a homeomorphism

$$\mathrm{SL}_2(\mathbb{C}) = KAU \cong \mathbb{S}^3 \times \mathbb{R}^3,$$

and hence the remaining claims in the lemma. $\square$

Exercise 6.4 (Euclidean meaning of Iwasawa decomposition). Give a proof of the Iwasawa decomposition for $\mathrm{SL}_2(\mathbb{R})$ (or of $\mathrm{SL}_d(\mathbb{R})$ for $d \geqslant 2$) and $\mathrm{SL}_2(\mathbb{C})$ using the Gram–Schmidt orthonormalization procedure.

6.1.3 A Quick Review of Lie Algebra Representations

Let us start by recalling that for a representation

$$\rho\colon G \ni g \longmapsto \rho_g \in \mathrm{GL}(W)$$

on a finite-dimensional vector space W of a (real or complex) Lie group G we can take the derivative of ρ at the identity to obtain the map

$$\mathrm{D}\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(W) = \mathrm{Hom}(W),$$

see Exercise 6.5. This defines a representation of the Lie algebra $\mathfrak{g}$ of G on the vector space W, or, equivalently, $\mathrm{D}\rho$ is a Lie algebra homomorphism satisfying

$$\mathrm{D}\rho\big([\mathbf{b}, \mathbf{c}]\big) = [\mathrm{D}\rho(\mathbf{b}), \mathrm{D}\rho(\mathbf{c})] \tag{6.7}$$

for all $\mathbf{b}, \mathbf{c} \in \mathfrak{g}$.

To see (6.7), we will use the fact that the exponential map satisfies, together with the derivative $\mathrm{D}\rho$, the categorical property

$$\rho \circ \exp = \exp \circ \mathrm{D}\rho, \tag{6.8}$$

see Exercise 6.5. Now fix $g \in G$ and $\mathbf{c} \in \mathfrak{g}$. Then, by definition, the adjoint representation applied to g and $\mathbf{c}$ is given by

$$\mathrm{Ad}_g(\mathbf{c}) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g \exp(t\mathbf{c}) g^{-1}.$$

For the element ρ_g and the Lie algebra element $\mathrm{D}\rho(\mathbf{c})$ for some $\mathbf{c} \in \mathfrak{g}$, this gives

$$\begin{aligned}\mathrm{Ad}_{\rho_g}(\mathrm{D}\rho(\mathbf{c})) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_g \exp(\mathrm{D}\rho(t\mathbf{c}))\rho_g^{-1} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_g \rho(\exp(t\mathbf{c}))\rho_g^{-1} && \text{(by (6.8))} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho\big(g \exp(t\mathbf{c}) g^{-1}\big) = \mathrm{D}\rho\,\mathrm{Ad}_g(\mathbf{c}), && (6.9)\end{aligned}$$

where we also used the homomorphism property of ρ and the chain rule for differentiation. If now $\mathbf{b}, \mathbf{c} \in \mathfrak{g}$ then we can set $g = \exp(t\mathbf{b})$ and use the defining property

$$[\mathbf{b}, \mathbf{c}] = \mathrm{ad}_{\mathbf{b}}(\mathbf{c}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\exp(t\mathbf{b})}(\mathbf{c})$$

in the same way. In fact we have

$$\begin{aligned}[\mathrm{D}\rho(\mathbf{b}), \mathrm{D}\rho(\mathbf{c})] &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\exp(\mathrm{D}\rho(t\mathbf{b}))}\big(\mathrm{D}\rho(\mathbf{c})\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\rho(\exp(t\mathbf{b}))}\big(\mathrm{D}\rho(\mathbf{c})\big) && \text{(by (6.8))} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{D}\rho\,\mathrm{Ad}_{\exp(t\mathbf{b})}(\mathbf{c}) = \mathrm{D}\rho[\mathbf{b}, \mathbf{c}]. && \text{(by (6.9))}\end{aligned}$$

Hence the study of finite-dimensional representations of G leads to the study of finite-dimensional representations of $\mathfrak{g}$. Below we will also denote the representation of $\mathfrak{g}$ induced from a representation ρ of G by ρ.

Let W be a finite-dimensional vector space carrying a representation ρ of a Lie group G, and let $V \subseteq W$ be a subspace. If V is invariant under $\rho(G)$, then by taking derivatives we see that V is also invariant under $\rho(\mathfrak{g})$. If G is connected, the reverse also holds. This follows, since $\rho(\mathbf{c})V \subseteq V$ for $\mathbf{c} \in \mathfrak{g}$ implies by (6.8) that

$$\rho\big(\exp(\mathbf{c})\big)V = \exp\big(\rho(\mathbf{c})\big)V \subseteq V$$

for all $\mathbf{c} \in \mathfrak{g}$, which then extends to $\rho_g V \subseteq V$ for all $g \in G^o = G$ (since the subgroup generated by $\exp(\mathfrak{g})$ is open).

In particular, for the connected groups $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$, and $\mathrm{SU}_2(\mathbb{R})$ the notions of irreducibility for finite-dimensional representations of the Lie group or of the Lie algebra coincide.

As discussed above, any finite-dimensional representation of G gives rise to a representation of its Lie algebra $\mathfrak{g}$. However, the converse to this is a bit harder to prove and only states that every finite-dimensional representation of $\mathfrak{g}$ gives rise to a finite-dimensional representation of the universal cover $\widetilde{G}$ of G. In the case of $G = \mathrm{SL}_2(\mathbb{C})$ and $G = \mathrm{SU}_2(\mathbb{R})$ this and Lemma 6.3 explains the correspondence between the irreducible representations of G and its Lie algebra in Theorem 6.6 below. However, for $G = \mathrm{SL}_2(\mathbb{R})$ this correspondence is a special feature.

Essential Exercise 6.5. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\rho\colon G \to \mathrm{GL}(W)$ be a finite-dimensional representation of G.
(a) Show that ρ is differentiable (in doing so, use the fact that $\exp\colon \mathfrak{g} \to G$ is smooth and has a local inverse near the identity).
(b) Show the categorical property (6.8) of the exponential map.

6.1.4 Irreducible Representations of the Lie Algebra

We now specialize to the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and classify its irreducible representations in the next theorem, which is crucial for the representation theory of all semi-simple Lie groups and the classification of semi-simple Lie groups.

Theorem 6.6 (Irreducible representations of $\mathfrak{sl}_2$). *The irreducible finite-dimensional representations of* $\mathfrak{sl}_2(\mathbb{C})$ *(respectively of* $\mathfrak{sl}_2(\mathbb{R})$ *or* $\mathfrak{su}_2(\mathbb{R})$*) are in a natural one-to-one correspondence with the elements of* $\mathbb{N}_0$*. In fact for every* $n \in \mathbb{N}_0$ *the representations of* $\mathrm{SL}_2(\mathbb{C})$ *on the* symmetric tensor product

$$\mathrm{Sym}^n(\mathbb{C}^2) = \Big\{\sum_{k=0}^{n} \alpha_k e_1^{\odot k} \odot e_2^{\odot(n-k)} \mid \alpha_0, \ldots, \alpha_n \in \mathbb{C}\Big\}$$

gives rise to an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ *of dimension* $(n+1)$*. By restriction, we also obtain irreducible representations of* $\mathrm{SL}_2(\mathbb{R})$*, or* $\mathrm{SU}_2(\mathbb{R})$*, and of the Lie algebras* $\mathfrak{sl}_2(\mathbb{R})$ *and* $\mathfrak{su}_2(\mathbb{R})$*. Any irreducible finite-dimensional representation of* $\mathfrak{sl}_2(\mathbb{C})$*, of* $\mathfrak{sl}_2(\mathbb{R})$*, or of* $\mathfrak{su}_2(\mathbb{R})$ *is isomorphic to one of these.*

Before we begin the proof of the theorem we first wish to describe the representations on $\mathrm{Sym}^n(\mathbb{C}^2)$ in more detail.

Given a finite-dimensional vector space W and a representation ρ of a group G on it, one can define the symmetric tensor product

$$\mathrm{Sym}^n(W) = \langle w_1 \odot w_2 \odot \cdots \odot w_n \mid w_1, \ldots, w_n \in W\rangle$$

as the linear hull of all formal commuting products of n vectors in W with the product map

$$W^n \ni (w_1, \ldots, w_n) \longmapsto w_1 \odot \cdots \odot w_n \in \mathrm{Sym}^n(W)$$

being multi-linear. In fact, using the n-fold tensor product $\bigotimes_{j=1}^n W$, we define

$$\operatorname{Sym}^n(W) = \bigotimes_{j=1}^{n} W / \langle w_1 \otimes \cdots \otimes w_n - w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)} \mid \sigma \in \mathrm{S}_n \rangle,$$

where S_n again denotes the symmetric group of $\{1, \ldots, n\}$. For $n = 0$ we define $\operatorname{Sym}^0(W) = \mathbb{C}$. We refer to Hungerford [38] for more details. Moreover, any linear map $A \in \operatorname{Hom}(W, W)$ can be used to induce a linear map $\operatorname{Sym}^n(A) \in \operatorname{Hom}\big(\operatorname{Sym}^n(W), \operatorname{Sym}^n(W)\big)$ with

$$\operatorname{Sym}^n(A)(w_1 \odot \cdots \odot w_n) = (Aw_1) \odot \cdots \odot (Aw_n)$$

for all $w_1, \ldots, w_n \in W$.

Returning to the setting of Theorem 6.6, we note that $\mathbb{C}^2$ carries the standard representation ρ defined by the linear action $\rho_g \colon \mathbb{C}^2 \ni v \mapsto gv \in \mathbb{C}^2$ of $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$, or $\mathrm{SU}_2(\mathbb{R})$. This defines the representation on $\operatorname{Sym}^n(\mathbb{C}^2)$ for all $n \in \mathbb{N}$, where $n = 1$ corresponds to the standard representation. In the special case $n = 0$ we use the trivial representation on $\operatorname{Sym}^n(\mathbb{C}^2) = \mathbb{C}$.

If we instead let W be the vector space of linear maps on $\mathbb{C}^2$, then $\operatorname{Sym}^n(W)$ becomes the space $\operatorname{Pol}_n(\mathbb{C}^2)$ of homogeneous polynomials of degree n on $\mathbb{C}^2$. The isomorphism $\operatorname{Sym}^n(\mathbb{C}^2) \cong \operatorname{Sym}^n(W)$ would also follows from the proof of the theorem below, but let us indicate briefly where it comes from. In fact, we will show that the standard representation on $\mathbb{C}^2$ and the representation ρ on W are isomorphic. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

and then send the linear map f defined by $f(X_1, X_2) = \alpha X_1 + \beta X_2$ to

$$f \circ g^{-1} \colon \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \longmapsto g^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \longmapsto (\alpha, \beta) g^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

In terms of the basis X_1, X_2 (dual to the standard basis e_1, e_2 of $\mathbb{C}^2$), this corresponds to the map

$$\begin{aligned} \rho_g = \big(g^{-1}\big)^{\mathrm{t}} &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{\mathrm{t}} \\ &= \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \end{aligned}$$

or $\rho_g = kgk^{-1}$ for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, where $k = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

This shows that the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ and its dual on W are isomorphic, which is a special property of $\mathrm{SL}_2(\mathbb{C})$ due to the special rule for calculating g^{-1}. This isomorphism can be used to find the isomorphism between $\operatorname{Sym}^n(\mathbb{C}^2)$ and $\operatorname{Sym}^n(W) = \operatorname{Pol}_n(\mathbb{C}^2)$.

Proof of irreducibility. Let us now work with the representation on the space $\mathrm{Sym}^n(\mathbb{C}^2)$ of $G = \mathrm{SL}_2(\mathbb{C})$ or $G = \mathrm{SL}_2(\mathbb{R})$. The trivial representation corresponding to $n = 0$ is clearly irreducible.

We now assume that $n \geqslant 1$. For $\mathbf{a} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $t \in \mathbb{R}$ we have

$$\exp(t\mathbf{a}) = \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix}$$

and $\rho\bigl(\exp(t\mathbf{a})\bigr)e_1 = \mathrm{e}^t e_1$ and $\rho\bigl(\exp(t\mathbf{a})\bigr)e_2 = \mathrm{e}^{-t} e_2$ for the representation on $\mathbb{C}^2$ and hence

$$\begin{aligned}\mathrm{Sym}^n\bigl(\rho\bigl(\exp(t\mathbf{a})\bigr)\bigr)\bigl(e_1^{\odot k} \odot e_2^{\odot(n-k)}\bigr) &= \bigl(\mathrm{e}^t e_1\bigr)^{\odot k} \odot \bigl(\mathrm{e}^{-t} e_2\bigr)^{\odot(n-k)} \\ &= \mathrm{e}^{kt-(n-k)t} e_1^{\odot k} \odot e_2^{\odot(n-k)}\end{aligned}$$

for $k = 0, \dots, n$. This shows that $\exp(t\mathbf{a})$ acts diagonally on $\mathrm{Sym}^n(\mathbb{C}^2)$ with eigenvalues $\mathrm{e}^{-nt}, \mathrm{e}^{(-n+2)t}, \dots, \mathrm{e}^{(n-2)t}, \mathrm{e}^{nt}$. Taking the derivative, we also obtain that the Lie algebra element $\mathbf{a}$ acts diagonally via

$$\mathrm{Sym}^n(\mathbf{a}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Sym}^n\bigl(\rho\bigl(\exp(t\mathbf{a})\bigr)\bigr),$$

with eigenvalues $-n, -n+2, \dots, n-2, n$.

For $\mathbf{e} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ and $t \in \mathbb{R}$ we have $\exp(t\mathbf{e}) = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ and $\rho(\exp(t\mathbf{e}))e_1 = e_1$ and $\rho(\exp(t\mathbf{e}))e_2 = te_1 + e_2$ for the representation on $\mathbb{C}^2$ and hence

$$\begin{aligned}\mathrm{Sym}^n\bigl(\rho(\exp(t\mathbf{e}))\bigr)\bigl(e_1^{\odot k} \odot e_2^{\odot(n-k)}\bigr) &= e_1^{\odot k}(te_1 + e_2)^{\odot(n-k)} \\ &= e_1^{\odot k}\left(e_2^{\odot(n-k)} + te_1\binom{n-k}{1} e_2^{\odot(n-k-1)} + \cdots\right),\end{aligned}$$

where the dots indicate the terms of order two and higher with respect to the variable t. Taking the derivative at $t = 0$, this gives

$$\mathrm{Sym}^n(\mathbf{e})(e_1^{\odot k} \odot e_2^{\odot(n-k)}) = (n-k)e_1^{\odot(k+1)} \odot e_2^{\odot(n-k-1)} \tag{6.10}$$

for $k = 0, \dots, n$, which for $k = n$ should simply be read as $\mathrm{Sym}^n(\mathbf{e})(e_1^{\odot n}) = 0$. Similarly,

$$\mathrm{Sym}^n(\mathbf{f})(e_1^{\odot k} \odot e_2^{\odot(n-k)}) = ke_1^{\odot(k-1)} \odot e_2^{\odot(n-k+1)} \tag{6.11}$$

for all $k = 0, \dots, n$, which for $k = n$ should be read as $\mathrm{Sym}^n(\mathbf{e})(e_1^{\odot n}) = 0$.

Suppose now that $V \subseteq \mathrm{Sym}^n(\mathbb{C}^2)$ is non-trivial and invariant under Sym^n. Since $\mathrm{Sym}^n(\mathbf{a})$ is diagonal with $n+1$ different eigenvalues, it follows that

$$e_1^{\odot k} \odot e_2^{\odot(n-k)} \in V$$

for some $k \in \{0, 1, \ldots, n\}$. However, using (6.10) and (6.11) we also see that

$$e_1^{\odot(k+1)} \odot e_2^{\odot(n-k-1)} \in V$$

if $k < n$, and

$$e_1^{\odot(k-1)} e_2^{\odot(n-k+1)} \in V$$

if $k > 0$. Iterating this shows that $V = \mathrm{Sym}^n(\mathbb{C}^2)$ contains all basis vectors, and irreducibility of our representation on $\mathrm{Sym}^n(\mathbb{C}^2)$ follows. □

As already visible in the proof above the eigenvectors of $\rho(\mathbf{a})$ and their eigenvalues play an important role in the theory. Hence they deserve a special name: the eigenvalues of $\rho(\mathbf{a})$ are called *weights* and the corresponding eigenvectors are called *weight vectors*. Moreover, the numbers 2 and -2 are called the *roots* and the vectors $\mathbf{e}, \mathbf{f} \in \mathfrak{sl}_2(\mathbb{C})$ the *root vectors*.

Proof of completeness. We now will show that for $G = \mathrm{SL}_2(\mathbb{C})$ and for $G = \mathrm{SL}_2(\mathbb{R})$ the list of irreducible finite-dimensional representations above is complete. For this we let W carry an arbitrary finite-dimensional representation ρ of the Lie algebra $\mathfrak{g}$ of G. Later we will assume that ρ is irreducible, but the initial part of the construction is more general. Since W is finite-dimensional, $\rho(\mathbf{a}) \in \mathrm{Hom}(W)$ must have at least one weight, that is an eigenvalue of $\rho(\mathbf{a})$. Let us assume that $\lambda_0 \in \mathbb{C}$ is a weight with the property that $\Re\lambda_0$ is maximal in the set $\{\Re\lambda \mid \lambda \text{ is a weight}\}$. Let $w_0 \in W$ be a weight vector for weight λ_0. We will be using $\rho(\mathbf{f})$ and the following more general claim to find more weight vectors with weights $\lambda_0 - 2$, $\lambda_0 - 4$,... which will lead to a complete classification of W.

Fundamental calculation. If $v \in W$ is a weight vector for weight λ, then we claim that $\rho(\mathbf{e})v$ and $\rho(\mathbf{f})v$ are either weight vectors for weight $\lambda + 2$ and $\lambda - 2$ respectively, or equal to zero.

The proof of the claim is rather simple. Indeed, using the defining property (6.7) of a Lie algebra homomorphism and the defining properties of the $\mathfrak{sl}_2$-triple in (6.3), we have

$$\begin{aligned}\rho(\mathbf{a})\rho(\mathbf{e})v &= \big(\rho(\mathbf{a})\rho(\mathbf{e}) - \rho(\mathbf{e})\rho(\mathbf{a})\big)v + \rho(\mathbf{e})\rho(\mathbf{a})v\\ &= \big[\rho(\mathbf{a}), \rho(\mathbf{e})\big]v + \rho(\mathbf{e})\lambda v\\ &= \rho\big([\mathbf{a}, \mathbf{e}]\big)v + \lambda\rho(\mathbf{e})v\\ &= \rho(2\mathbf{e})v + \lambda\rho(\mathbf{e})v = (\lambda + 2)\rho(\mathbf{e})v.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\rho(\mathbf{a})\rho(\mathbf{f})v &= [\rho(\mathbf{a}), \rho(\mathbf{f})]v + \rho(\mathbf{f})\rho(\mathbf{a})v\\ &= \rho\big([\mathbf{a}, \mathbf{f}]\big)v + \lambda\rho(\mathbf{f})v = (\lambda - 2)\rho(\mathbf{f})v.\end{aligned}$$

Construction of eigenvectors. Let $w_0 \in W$ be a weight vector with maximal real part of its weight λ_0 as above. By the fundamental calculation, $\rho(\mathbf{e})w_0$ would have weight $\lambda_0 + 2$, which by maximality of $\Re\lambda$ implies that

$$\rho(\mathbf{e})w_0 = 0. \tag{6.12}$$

On the other hand we can define

$$w_1 = \rho(\mathbf{f})w_0,\ w_2 = \rho(\mathbf{f})w_1,\ \ldots \tag{6.13}$$

and obtain weight vectors for weights $\lambda_0 - 2, \lambda_0 - 4, \ldots$. As eigenvectors for different eigenvalues are always linearly independent and $\dim W < \infty$ it follows that there must exist some $n \geqslant 0$ with $w_n = \rho(\mathbf{f})^n w_0 \neq 0$ but

$$\rho(\mathbf{f})w_n = 0. \tag{6.14}$$

An invariant subspace. We let $V = \langle w_0, \ldots, w_n \rangle \subseteq W$ and claim that V is invariant under ρ. Since V is generated by eigenvectors for $\rho(\mathbf{a})$, V is clearly invariant under $\rho(\mathbf{a})$. Moreover, by the construction in (6.13) and the property (6.14) we have $\rho(\mathbf{f})w_k \in V$ for all $k = 0, \ldots, n$ and hence $\rho(\mathbf{f})V \subseteq V$. It remains to study $\rho(\mathbf{e})$, where we will prove by induction that

$$\rho(\mathbf{e})w_k \begin{cases} = 0 & \text{for } k = 0; \\ \in \mathbb{C}w_{k-1} & \text{for } k > 0 \end{cases} \tag{6.15}$$

for $k = 0, \ldots, n$. Indeed, we know this for $k = 0$ by (6.12). If now (6.15) is already known for some $k \in \{0, \ldots, n-1\}$, then by construction

$$w_{k+1} = \rho(\mathbf{f})w_k$$

and

$$\begin{aligned} \rho(\mathbf{e})w_{k+1} &= \rho(\mathbf{e})\rho(\mathbf{f})w_k \\ &= [\rho(\mathbf{e}), \rho(\mathbf{f})]w_k + \rho(\mathbf{f})\rho(\mathbf{e})w_k \\ &= \rho(\mathbf{a})w_k + \rho(\mathbf{f})\rho(\mathbf{e})w_k \in \mathbb{C}w_k + \rho(\mathbf{f})\mathbb{C}w_{k-1} = \mathbb{C}w_k, \end{aligned}$$

by the inductive assumption. This shows the inductive step.

Assuming irreducibility. Suppose now in addition that ρ is irreducible. Since $V \subseteq W$ is non-trivial and invariant under ρ, it follows that $V = W$ has the basis $w_0, \ldots, w_n$ consisting of weight vectors for $\rho(\mathbf{a})$ for the weights

$$\lambda_0, \lambda_0 - 2, \ldots, \lambda_0 - 2n.$$

In particular, $\rho(\mathbf{a})$ is diagonalizable with these eigenvalues, and the trace of $\rho(\mathbf{a})$ is

$$\sum_{k=0}^{n} (\lambda_0 - 2k) = (n+1)\lambda_0 - 2\sum_{k=0}^{n} k = (n+1)\lambda_0 - (n+1)n.$$

Since $\rho(\mathbf{a}) = \rho\big([\mathbf{e},\mathbf{f}]\big) = [\rho(\mathbf{e}),\rho(\mathbf{f})]$, we also know that $\operatorname{tr}\rho(\mathbf{a}) = 0$, and so $\lambda = n \in \mathbb{N}_0$.

An isomorphism. We now combine the arguments above with our discussion of $\operatorname{Sym}^n(\mathbb{C}^2)$ by constructing the graph of an isomorphism within

$$\widetilde{W} = \operatorname{Sym}^n(\mathbb{C}^2) \oplus W.$$

In fact the vector $v_0 = (e_1^{\odot n}, w_0) \in \widetilde{W}$ is a weight vector of weight n satisfying $\operatorname{Sym}^n(\mathbf{e}) \oplus \rho(\mathbf{e})v_0 = 0$ just as in the case of our original vector $w_0 \in W$. Applying the same argument as before, we produce weight vectors $v_0, v_1, \ldots, v_n \in \widetilde{W}$ such that $V = \langle v_0, v_1, \ldots, v_n\rangle$ is invariant. Moreover, $\dim V = n+1 = \dim \operatorname{Sym}^n(\mathbb{C}^2) = \dim W$ and the projections of V onto $\operatorname{Sym}^n(\mathbb{C}^2)$ respectively onto W are surjective, which follows from (6.11) for $\operatorname{Sym}^n(\mathbb{C}^2)$ and since $w_0, w_1, \ldots, w_n$ is a basis of W. This shows that $V = \operatorname{Graph}(\varPhi)$ for some linear isomorphism $\varPhi\colon \operatorname{Sym}^n(\mathbb{C}^2) \to W$ and invariance of V implies that ϕ is an isomorphism of the representations. □

The maximal weight as it appeared in the proof above is called the *highest weight* and the corresponding weight vectors are called *highest weight vectors*.

Exercise 6.7. Use the arguments from the proof of Theorem 6.6 to show that for any finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ a highest weight vector for highest weight λ_0 always generates an irreducible subrepresentation of dimension $\lambda_0 + 1$.

We refer to Fulton and Harris [29] for an accessible treatment of the theory of highest weight vectors for more general semi-simple groups. We will see similar mechanisms for creating more eigenvectors out of an initial eigenvector also for unitary representations later.

To summarize, we have proved Theorem 6.6 in the two cases $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, where we did not see any difference in the arguments as the $\mathfrak{sl}_2$-triple belonged to $\mathfrak{sl}_2(\mathbb{R})$ and all subspaces of the representation space are assumed to be complex subspaces. As we will now show, the extension to $\mathfrak{su}_2(\mathbb{R})$ does not require much except for the right insight.

Proposition 6.8 (Complexification). *Any finite-dimensional (and, as always, complex) representation*

$$\rho\colon \mathfrak{su}_2(\mathbb{R}) \longrightarrow \operatorname{Hom}(W)$$

(or $\rho\colon \mathfrak{sl}_2(\mathbb{R}) \to \operatorname{Hom}(W)$) can be extended in a unique way to a $\mathbb{C}$-linear representation $\rho_{\mathbb{C}}\colon \mathfrak{sl}_2(\mathbb{C}) \to \operatorname{Hom}(W)$. A subspace $V \subseteq W$ is invariant under $\rho(\mathfrak{su}_2(\mathbb{R}))$ (respectively $\rho(\mathfrak{sl}_2(\mathbb{R}))$) if and only if it is invariant under $\rho_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C}))$. In particular, they have the same list of irreducible finite-dimensional representations.

Proof. Since $\operatorname{Hom}(W)$ is a complex vector space, $\mathfrak{su}_2(\mathbb{R})$ is a real vector space, and $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ by the discussion in Section 6.1.2, every $\mathbb{R}$-linear map $\rho\colon \mathfrak{su}_2(\mathbb{R}) \to \mathrm{Hom}(W)$ has a uniquely defined $\mathbb{C}$-linear extension $\rho_{\mathbb{C}}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Hom}(W)$ satisfying

$$\rho_{\mathbb{C}}(\mathbf{b}_1 + \mathrm{i}\mathbf{b}_2) = \rho(\mathbf{b}_1) + \mathrm{i}\rho(\mathbf{b}_2)$$

for all $\mathbf{b}_1, \mathbf{b}_2 \in \mathfrak{su}_2(\mathbb{R})$. Moreover, if

$$\rho\bigl([\mathbf{b}_1, \mathbf{b}_2]\bigr) = [\rho(\mathbf{b}_1), \rho(\mathbf{b}_2)]$$

for all $\mathbf{b}_1, \mathbf{b}_2 \in \mathfrak{su}_2(\mathbb{R})$ then this property extends by bilinearity of $[\cdot,\cdot]$ and linearity of $\rho_{\mathbb{C}}$ from $\mathfrak{su}_2(\mathbb{R})$ to all $\mathbf{b}_1, \mathbf{b}_2 \in \mathfrak{sl}_2(\mathbb{C})$.

The remaining claims of the proposition follow from this. □

We have shown that $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{R})$, and $\mathfrak{su}_2(\mathbb{R})$ as well as $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{SL}_2(\mathbb{R})$, and $\mathrm{SU}_2(\mathbb{R})$ have the same list of irreducible finite-dimensional representations, which concludes the proof of Theorem 6.6. In this discussion, the Lie algebra $\mathfrak{su}_2(\mathbb{R})$ and its Lie group $\mathrm{SU}_2(\mathbb{R})$ were the odd ones out as they required extra effort (because $\mathfrak{su}_2(\mathbb{R})$ does not contain an $\mathfrak{sl}_2$-triple). However, in the next section the special properties of $\mathrm{SU}_2(\mathbb{R})$ are used to prove an important property of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{R})$.

Exercise 6.9. Show that $\mathfrak{so}_n(\mathbb{R})$ and $\mathfrak{so}_{n-k,k}(\mathbb{R})$ with $k \in \{1, \ldots, n-2\}$ are real forms of $\mathfrak{so}_n(\mathbb{C})$, and conclude that Proposition 6.8 holds in the same way for these Lie algebras.

6.1.5 The Weyl Unitary Trick

Proposition 6.10 (Semi-simplicity of representations of $\mathfrak{su}_2(\mathbb{R})$). *Assume that W is a finite-dimensional representation of $\mathfrak{su}_2(\mathbb{R})$. If a complex subspace V of W is invariant under $\mathfrak{su}_2(\mathbb{R})$, then there exists an invariant complementary subspace V' so that $W = V \oplus V'$.*

Proof. As explained in Section 6.1.3, W is also a representation space for $\mathrm{SU}_2(\mathbb{R})$ and V is invariant under $\mathrm{SU}_2(\mathbb{R})$. Fix some inner product on W and apply Proposition 5.35. Hence we may assume that W is a unitary representation of $\mathrm{SU}_2(\mathbb{R})$ and we may define $V' = V^{\perp}$ with respect to this inner product. □

Theorem 6.11 (Finite-dimensional representations of $\mathfrak{sl}_2$). *For finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{R})$, and $\mathfrak{su}_2(\mathbb{R})$ we have the following properties.*

(a) *(Semi-simplicity) Any invariant subspace has an invariant complement.*
(b) *(Description) The representation is a finite direct sum of irreducible representations as described in Theorem 6.6.*

PROOF. For $\mathfrak{su}_2(\mathbb{R})$ part (a) is precisely the statement in Proposition 6.10. Part (b) follows from this by induction on the dimension. For $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{R})$ we combine Proposition 6.8 with the above. □

The argument above using compactness of $\mathrm{SU}_2(\mathbb{R}) \subseteq \mathrm{SL}_2(\mathbb{C})$ can in fact be used for all semi-simple real and complex Lie groups since there always exists a compact form[(12)] that can take the role of $\mathrm{SU}_2(\mathbb{R})$.

6.2 Harmonic Analysis of $\mathbf{SU_2}(\mathbb{R})$ and Quotients*

6.2.1 Peter–Weyl Theorem for $\mathbf{SU_2}(\mathbb{R})$

†We now start the in-depth discussion of the harmonic analysis on $\mathrm{SU}_2(\mathbb{R})$ and related spaces. Schur orthogonality and the Peter–Weyl theorem (Theorems 5.38 and 5.42) give a complete description of $L^2(G)$ for a compact group G, assuming a complete description of $\widehat{G}$. In Theorem 6.6 we have obtained the description of all irreducible finite-dimensional representations of $\mathrm{SU}_2(\mathbb{R})$. Combining these two (and calculating the inner product on $\mathrm{Sym}^n(\mathbb{C}^2)$) gives the following result. For this, we will be using the coordinate system $(z,w) \in \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$ for the elements

$$g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R}). \tag{6.16}$$

Corollary 6.12 (Peter–Weyl for $\mathrm{SU}_2(\mathbb{R})$). *The functions*

$$g \longmapsto \sqrt{n+1}\,\pi^{(n)}_{k,\ell}(g)$$

defined by

$$\pi^{(n)}_{k,\ell}(g) = \sqrt{k!(n-k)!\ell!(n-\ell)!} \times \sum_{\substack{i\in\{0,\dots,n-k\}\\ j\in\{0,\dots,k\}\\ i+j=\ell}} \frac{(-1)^{k-j}}{i!j!(n-k-i)!(k-j)!} z^{n-k-i} w^i \overline{z}^j \overline{w}^{k-j}$$

for $k,\ell \in \{0,\dots,n\}$ are the normalized matrix coefficients associated to the irreducible representation on the $(n+1)$-dimensional space $\mathrm{Sym}^n(\mathbb{C}^2)$ *for n in $\mathbb{N}_0$ (and using a convenient choice of orthonormal basis of* $\mathrm{Sym}^n(\mathbb{C}^2)$*). By also varying n in $\mathbb{N}_0$ we obtain an orthonormal basis of $L^2(\mathrm{SU}_2(\mathbb{R}))$.*

† This section discusses the first non-trivial and quite important example of a compact simple group. In order to be completely explicit, the discussion is quite heavy on concrete formulas. As indicated in the title, these are not important for most of the subsequent discussions.

Proof. As already hinted at before the corollary, we have done all the work required for the corollary apart from determining the inner product on $\mathrm{Sym}^n(\mathbb{C}^2)$ for $n \in \mathbb{N}_0$. We note that the inner product is uniquely determined up to a positive scalar by the irreducibility of the representations and Schur's lemma (Theorem 1.29).

The case $n = 0$ corresponds to the trivial representation, and setting in addition k and ℓ to be 0 gives $\sqrt{1}\pi^{(0)}_{0,0} = 1$. The case $n = 1$ corresponds to the standard representation on $\mathbb{C}^2$ and the standard inner product on $\mathbb{C}^2$ makes the action unitary (by definition of $\mathrm{SU}_2(\mathbb{R})$). In this case we use the standard basis

$$w_0 = e_1 = \begin{pmatrix}1\\0\end{pmatrix}, \ \ w_1 = e_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

and obtain $\pi^{(1)}_{0,0}(g) = z$, $\pi^{(1)}_{0,1}(g) = w$, $\pi^{(1)}_{1,0}(g) = -\overline{w}$, $\pi^{(1)}_{1,1}(g) = \overline{z}$ for g as in (6.16). Setting $n = 1$ and using $k, \ell \in \{0, 1\}$ in the formula in the corollary, we obtain the same four functions on $\mathrm{SU}_2(\mathbb{R})$. By Theorem 5.38, the functions $\sqrt{2}\pi^{(1)}_{k,\ell}$ are then orthonormal for $k, \ell \in \{0, 1\}$.

So suppose now that $n \geqslant 2$. Then the vectors

$$\widetilde{w_k} = e_1^{\odot(n-k)} \odot e_2^{\odot k}$$

are eigenvectors for the elements

$$\begin{pmatrix} z & \\ & \overline{z}\end{pmatrix} \in \mathrm{SU}_2(\mathbb{R})$$

for all $z \in \mathbb{S}^1$ for eigenvalues $z^{n-k}\overline{z}^k = z^{n-2k}$ for $k = 0, \ldots, n$. As the eigenvalues are distinct, they must be pairwise orthogonal with respect to the desired inner product $\langle\cdot,\cdot\rangle$ on $\mathrm{Sym}^n(\mathbb{C}^2)$. We claim that the inner product can be chosen so that the vectors

$$w_k = \binom{n}{k}^{\frac{1}{2}} e_1^{\odot(n-k)} \odot e_2^{\odot k} \tag{6.17}$$

are an orthonormal basis of $\mathrm{Sym}^n(\mathbb{C}^2)$. This could be checked directly (for example, by showing that the representation of $\mathfrak{su}_2(\mathbb{R})$ only takes on anti-Hermitian matrices with respect to that basis). However, we will give a more conceptual argument for this.

We consider $\bigotimes_{j=1}^n(\mathbb{C}^2)$ and apply Proposition 5.14 (inductively extended to n factors) to define the unitary inner tensor product representation ρ of $\mathrm{SU}_2(\mathbb{R})$ on $\bigotimes_{j=1}^n(\mathbb{C}^2)$ so that

$$\rho_g(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = (gu_1) \otimes (gu_2) \otimes \cdots \otimes (gu_n) \tag{6.18}$$

for all $u_1, \ldots, u_n \in \mathbb{C}^2$. Next note that there is a canonical equivariant map

$$\mathrm{Com}\colon \bigotimes_{j=1}^{n}(\mathbb{C}^2) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2)$$

that sends any tensor product to its commutative counterpart, that is,

$$\bigotimes_{j=1}^{n}(\mathbb{C}^2) \ni u_1 \otimes u_2 \otimes \cdots \otimes u_n \longmapsto u_1 \odot u_2 \odot \cdots \odot u_n \in \mathrm{Sym}^n(\mathbb{C}^2)$$

for all $u_1, \ldots, u_n \in \mathbb{C}^2$. In fact, as was already mentioned, $\mathrm{Sym}^n(\mathbb{C}^2)$ is defined as the quotient of $\bigotimes_{j=1}^{n}(\mathbb{C}^2)$ by the subspace generated by

$$u_1 \otimes u_2 \otimes \cdots \otimes u_n - u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)}$$

for all $u_1, u_2, \ldots, u_n \in \mathbb{C}^2$ and permutations $\sigma \in \mathrm{S}_n$.

Also note that the permutation group S_n acts unitarily on $\bigotimes_{j=1}^{n}(\mathbb{C}^2)$ by setting

$$\lambda_\sigma(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)} \tag{6.19}$$

for all $\sigma \in \mathrm{S}_n$ and $u_1, \ldots, u_n \in \mathbb{C}^2$ and then linearly extending this action. Moreover, using (6.18) and (6.19) it is easy to see that $\lambda_\sigma \rho_g = \rho_g \lambda_\sigma$ for all $\sigma \in \mathrm{S}_n$ and $g \in \mathrm{SU}_2(\mathbb{R})$. Therefore

$$V = \left\{ v \in \bigotimes_{j=1}^{n}(\mathbb{C}^2) \mid \lambda_\sigma(v) = v \text{ for all } \sigma \in \mathrm{S}_n \right\}$$

is invariant under ρ, and is non-trivial since $e_1 \otimes e_1 \otimes \cdots \otimes e_1 \in V$. Recall that $\bigotimes_{j=1}^{n}(\mathbb{C}^2)$ has

$$e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} \tag{6.20}$$

for $j_1, \ldots, j_n \in \{1, 2\}$ as an orthonormal basis, and note that λ_σ for $\sigma \in \mathrm{S}_n$ maps any such basis vector to another such basis vector. It follows from this that V is generated by the vectors

$$v_k = \sum_{\substack{j_1,\ldots,j_n \in \{1,2\} \\ n-k \text{ times } 1, \\ k \text{ times } 2}} e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} \tag{6.21}$$

for $k = 0, \ldots, n$. In fact, if one of the basis vectors $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n}$ with $|\{\ell \mid j_\ell = 1\}| = n - k$ appears with a coefficient c in the expansion of some $v \in V$ with respect to the basis in (6.20), then all other basis vectors appearing in the sum (6.21) are images of the original basis vector under some $\sigma \in \mathrm{S}_n$. Hence these all appear with the same coefficient c in the vector v. Using this argument for all $k = 0, \ldots, n$ we deduce that v is a linear combination of $v_0, v_1, \ldots, v_n$.

Now note that the vector v_k in (6.21) has $\binom{n}{k}$ summands which are mutually orthogonal unit vectors in $\bigotimes_{j=1}^{n}(\mathbb{C}^2)$. Hence the vectors

$$\widetilde{v}_k = \binom{n}{k}^{-\frac{1}{2}} v_k$$

for $k = 0, \dots, n$ are an orthonormal basis of V. Applying the equivariant map $\mathrm{Com}|_V\colon V \to \mathrm{Sym}^n(\mathbb{C}^2)$ we see that

$$\mathrm{Com}\, \widetilde{v}_k = \binom{n}{k}^{-\frac{1}{2}} \mathrm{Com}\, v_k = \binom{n}{k}^{-\frac{1}{2}} \binom{n}{k} e_1^{\odot(n-k)} \odot e_2^{\odot k} = w_k$$

for $k = 0, \dots, n$. Since the $\widetilde{v}_k$ for $k = 0, \dots, n$ form an orthonormal basis, this proves the claim that the basis in (6.17) is an orthonormal basis with respect to an inner product on $\mathrm{Sym}^n(\mathbb{C}^2)$ that makes ρ a unitary representation $\pi^{(n)}$ of $\mathrm{SU}_2(\mathbb{R})$.

We now calculate the matrix coefficients $\pi_{k,\ell}^{(n)} = \varphi_{w_k,w_\ell}$ for the basis vectors w_k, w_ℓ and $k, \ell = 0, \dots, n$. Using the notation (6.16) once again, we have $ge_1 = ze_1 + we_2$ and $ge_2 = -\overline{w}e_1 + \overline{z}e_2$ and hence

$$\begin{aligned}
\pi_g^{(n)} w_k &= \binom{n}{k}^{\frac{1}{2}} (ze_1 + we_2)^{\odot(n-k)} \odot (-\overline{w}e_1 + \overline{z}e_2)^{\odot k} \\
&= \binom{n}{k}^{\frac{1}{2}} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} z^{n-k-i} w^i e_1^{\odot(n-k-i)} \odot e_2^{\odot i} \right) \\
&\qquad\qquad \odot \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \overline{w}^{k-j} \overline{z}^j e_1^{\odot(k-j)} \odot e_2^{\odot j} \right) \\
&= \binom{n}{k}^{\frac{1}{2}} \sum_{i=0}^{n-k} \sum_{j=0}^{k} \binom{n-k}{i} \binom{k}{j} (-1)^{k-j} z^{n-k-i} w^i \overline{z}^j \overline{w}^{k-j} e_1^{\odot(n-i-j)} \odot e_2^{\odot(i+j)}.
\end{aligned}$$

Taking the inner product with

$$w_\ell = \binom{n}{\ell}^{\frac{1}{2}} e_1^{\odot(n-\ell)} e_2^{\odot \ell}$$

selects only those terms in the sum above with $i + j = \ell$. In fact, multiplying and dividing $\pi_g^{(n)} w_k$ by

$$\binom{n}{\ell}^{-\frac{1}{2}}$$

and using $\|w_\ell\| = 1$, we obtain

$$\begin{aligned}\pi^{(n)}_{k,\ell} &= \langle \pi^{(n)}_g w_k, w_\ell\rangle\\ &= \binom{n}{k}^{\frac12}\binom{n}{\ell}^{-\frac12}\sum_{\substack{i\in\{0,\dots,n-k\}\\ j\in\{0,\dots,k\}\\ i+j=\ell}}\binom{n-k}{i}\binom{k}{j}(-1)^{k-j}z^{n-k-i}w^i\overline{z}^j\overline{w}^{k-j}\\ &= \sqrt{k!(n-k)!\ell!(n-\ell)!}\sum_{\substack{i\in\{0,\dots,n-k\}\\ j\in\{0,\dots,k\}\\ i+j=\ell}}\frac{(-1)^{k-j}}{i!j!(n-k-i)!(k-j)!}z^{n-k-i}w^i\overline{z}^j\overline{w}^{k-j}.\end{aligned}$$

Together with Theorem 5.42, this concludes the proof. □

6.2.2 Peter–Weyl Theorem for $\mathbf{SO_3}(\mathbb{R})$

We briefly explain in this section how Corollary 6.12 also gives rise to a Peter–Weyl theorem for the group $\mathrm{SO}_3(\mathbb{R})$. We will not, however, give the orthonormal basis explicitly in terms of the coordinates of $\mathrm{SO}_3(\mathbb{R})$.

The connection between $\mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SO}_3(\mathbb{R})$ is given by the following lemma. We recall that an isogeny between two semi-simple Lie groups is a finite-to-one surjection.

Lemma 6.13 (Isogeny for $\mathrm{SU}_2(\mathbb{R})$). *We have*

$$\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{R})/C,$$

where $C = \{\pm I\}$ is the centre of $\mathrm{SU}_2(\mathbb{R})$.

PROOF. We recall from Lemma 6.2 that $\mathrm{SU}_2(\mathbb{R}) \cong \mathbb{S}^3 \subseteq \mathbb{H}$, and so in particular $\mathrm{SU}_2(\mathbb{R})$ is simply connected (see also Lemma 6.3). We claim now that $\mathrm{SO}_3(\mathbb{R})$ is also a connected three-dimensional Lie group. To see that it is connected, we let $g \in \mathrm{SO}_3(\mathbb{R})$. Then g has a real eigenvector v with eigenvalue 1. In fact, if all eigenvalues are real, then $g = I$ or the eigenvalues must equal -1, -1, and 1. If there is a non-real complex eigenvalue λ then the eigenvalues must be λ, $\overline{\lambda}$, and 1. Hence in either case $g = I$ or g can be viewed as a rotation about some axis in $\mathbb{R}^3$. It follows that g belongs to a one-parameter subgroup and hence to the connected component of $I \in \mathrm{SO}_3(\mathbb{R})$. As $g \in \mathrm{SO}_3(\mathbb{R})$ was arbitrary, we deduce that $\mathrm{SO}_3(\mathbb{R})$ is connected.

The Lie algebra of

$$\mathrm{SO}_3(\mathbb{R}) = \{g \in \mathrm{SL}_3(\mathbb{R}) \mid g^{\mathrm{t}} g = I\}$$

is given by

$$\mathfrak{so}_3(\mathbb{R}) = \{\mathbf{m} \in \mathfrak{sl}_3(\mathbb{R}) \mid \mathbf{m}^{\mathrm{t}} + \mathbf{m} = 0\},$$

and so consists of all matrices of the form

$$\mathbf{m} = \begin{pmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & -\gamma \\ \beta & \gamma & 0 \end{pmatrix}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$. It follows that $\mathrm{SO}_3(\mathbb{R})$ is a three-dimensional connected Lie group as claimed.

To define the homomorphism $\mathrm{SU}_2(\mathbb{R}) \to \mathrm{SO}_3(\mathbb{R})$, we identify the vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$$

with the Lie algebra element

$$\mathbf{m} = \begin{pmatrix} a\mathrm{i} & b\mathrm{i} - c \\ b\mathrm{i} + c & -a\mathrm{i} \end{pmatrix} \in \mathfrak{su}_2(\mathbb{R}), \tag{6.22}$$

and define

$$\rho_g(\mathbf{m}) = g\mathbf{m}g^{-1} = g\mathbf{m}g^*$$

for all $g \in \mathrm{SU}_2(\mathbb{R})$ and $\mathbf{m} \in \mathfrak{su}_2(\mathbb{R})$. Since

$$\mathfrak{su}_2(\mathbb{R}) = \{\mathbf{m} \in \mathfrak{gl}_2(\mathbb{C}) \mid \mathbf{m}^* = -\mathbf{m}, \operatorname{tr}\mathbf{m} = 0\},$$

it follows that $\rho_g(\mathfrak{su}_2(\mathbb{R})) \subseteq \mathfrak{su}_2(\mathbb{R})$. Moreover, for $\mathbf{m}$ as in (6.22), we have

$$\det \mathbf{m} = a^2 - (b\mathrm{i} - c)(b\mathrm{i} + c) = a^2 + b^2 + c^2$$

and $\det\big(\rho_g \mathbf{m}\big) = \det \mathbf{m}$ for all $g \in \mathrm{SU}_2(\mathbb{R})$. This shows that the adjoint representation ρ defines a homomorphism $\rho\colon \mathrm{SU}_2(\mathbb{R}) \to \mathrm{SO}_3(\mathbb{R})$.

Suppose now that $\rho_g = I$, so $g\mathbf{m}g^{-1} = \mathbf{m}$ for all $\mathbf{m} \in \mathfrak{su}_2(\mathbb{R})$. Since $\mathfrak{su}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{su}_2(\mathbb{C})$ we see that $g\mathbf{m} = \mathbf{m}g$ for all $\mathbf{m} \in \mathfrak{sl}_2(\mathbb{C})$. For

$$g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

and

$$\mathbf{m} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we deduce that

$$g\mathbf{m} = \begin{pmatrix} z & \overline{w} \\ w & -\overline{z} \end{pmatrix}, \quad \mathbf{m}g = \begin{pmatrix} z & -\overline{w} \\ -w & -\overline{z} \end{pmatrix}, \tag{6.23}$$

so $w = 0$. Similarly, by using

$$\mathbf{m} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we obtain

$$g\mathbf{m} = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad \mathbf{m}g = \begin{pmatrix} 0 & \overline{z} \\ 0 & 0 \end{pmatrix}$$

and hence $z = \overline{z} \in \mathbb{R}$. To summarize, if $g \in \ker \rho$ then $g = I$ or $g = -I$. The converse of this statement is clear. The lemma follows from this: Since both $\mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SO}_3(\mathbb{R})$ are three-dimensional and ρ has finite kernel, $\rho\big(\mathrm{SU}_2(\mathbb{R})\big)$ is also three-dimensional, which together with connectedness of $\mathrm{SO}_3(\mathbb{R})$ implies that $\rho\big(\mathrm{SU}_2(\mathbb{R})\big) = \mathrm{SO}_3(\mathbb{R})$. □

Corollary 6.14 (Peter–Weyl for $\mathrm{SO}_3(\mathbb{R})$). *The irreducible representation $\pi^{(n)}$ on $\mathrm{Sym}^n(\mathbb{C}^2)$ gives rise to a unitary representation of*

$$\mathrm{SO}_3(\mathbb{R}) = {}^{\mathrm{SU}_2(\mathbb{R})}/C$$

if and only if n is even. In particular, the normalized matrix coefficients

$$\sqrt{n+1}\pi^{(n)}_{k,\ell}$$

for $k, \ell \in \{0, \ldots, n\}$ and $n \in 2\mathbb{N}_0$ give rise to an orthonormal basis of $L^2(\mathrm{SO}_3(\mathbb{R}))$.

Proof. If π is an irreducible unitary representation of $\mathrm{SO}_3(\mathbb{R})$, then the isomorphism $\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{R})/C$ can be used to consider π also as an irreducible unitary representation of $\mathrm{SU}_2(\mathbb{R})$, which we again denote by π. By Theorem 6.6, π is isomorphic to the representation $\pi^{(n)}$ on $\mathrm{Sym}^n(\mathbb{C}^2)$ for some $n \in \mathbb{N}_0$. By construction, we have $\pi^{(n)}(-I) = (-I)^n = I$, which implies that n is even.

On the other hand, if $n \in 2\mathbb{N}$ then $\pi^{(n)}(-I) = I$, which shows that $\pi^{(n)}$ descends to a unitary representation of $\mathrm{SO}_3(\mathbb{R})$.

The final claim follows from the Peter–Weyl theorem (Theorem 5.42) as in the proof of Corollary 6.12. □

6.2.3 The Unitary Representation on $L^2(\mathbb{S}^2)$

By the discussion in the previous section we know that $\mathrm{SU}_2(\mathbb{R})$ acts naturally on the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. We equip $\mathbb{S}^2$ with the natural surface area measure and obtain a measure-preserving action of $\mathrm{SU}_2(\mathbb{R})$ on $\mathbb{S}^2$. This in turn gives rise to a unitary representation of $\mathrm{SU}_2(\mathbb{R})$ on $L^2(\mathbb{S}^2)$ as in Proposition 1.3. We wish to use the description of irreducible representations of $\mathrm{SU}_2(\mathbb{R})$ in Theorem 6.6 to describe how $L^2(\mathbb{S}^2)$ splits into irreducible components.

Corollary 6.15 (Decomposition of $L^2(\mathbb{S}^2)$). *Using the unitary representation of* $\mathrm{SU}_2(\mathbb{R})$ *on* $L^2(\mathbb{S}^2)$ *as described above, we have*

$$L^2(\mathbb{S}^2) = \bigoplus_{n\in 2\mathbb{N}} \mathrm{Sym}^n(\mathbb{C}^2).$$

In other words, every irreducible unitary representation of $\mathrm{SU}_2(\mathbb{R})$ *of even highest weight (equivalently, every irreducible representation of* $\mathrm{SO}_3(\mathbb{R})$*) appears in* $L^2(\mathbb{S}^2)$ *with multiplicity one.*

Proof. We will combine the description of $L^2(\mathrm{SU}_2(\mathbb{R}))$ in Corollary 6.12 with the isomorphism

$$\mathbb{S}^2 \cong \mathrm{SU}_2(\mathbb{R})/T \tag{6.24}$$

where

$$T = \left\{ \begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} \;\middle|\; \alpha \in \mathbb{S}^1 \right\}$$

is the diagonal subgroup in $\mathrm{SU}_2(\mathbb{R})$.

For the proof of (6.24) we recall from Section 6.2.2 that $\mathrm{SU}_2(\mathbb{R})$ factors onto $\mathrm{SO}_3(\mathbb{R})$ and acts on $\mathfrak{su}_2(\mathbb{R}) \cong \mathbb{R}^3$ while preserving the quadratic form det. Let

$$v_0 = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix}$$

and note that $gv_0g^{-1} = v_0$ if and only if $g \in T$ (see the argument in (6.23)). Moreover, as $\mathrm{SO}_3(\mathbb{R})$ acts transitively on $\mathbb{S}^2$ and $\mathrm{SU}_2(\mathbb{R})$ factors onto $\mathrm{SO}_3(\mathbb{R})$ by this action, we obtain $\mathbb{S}^2 = \rho(\mathrm{SU}_2(\mathbb{R}))v_0$, and (6.24) follows. More precisely, $gT \in \mathrm{SU}_2(\mathbb{R})/T$ corresponds to $gv_0g^{-1} \in \mathbb{S}^2$ under the isomorphism and the action of $\mathrm{SU}_2(\mathbb{R})$ corresponds to left multiplication on $\mathrm{SU}_2(\mathbb{R})$.

Also recall that the Haar measure on a homogeneous space is unique up to positive proportionality, which implies that the surface area measure described above the corollary agrees up to a positive multiple with the push-forward of the Haar measure on $\mathrm{SU}_2(\mathbb{R})$ onto $\mathrm{SU}_2(\mathbb{R})/T$.

To summarise, we may consider $L^2(\mathbb{S}^2)$ as the subspace $\mathcal{V}$ of $L^2(\mathrm{SU}_2(\mathbb{R}))$ consisting of all functions on $\mathrm{SU}_2(\mathbb{R})$ that are right-invariant under T. In other words,

$$L^2(\mathbb{S}^2) \cong \mathcal{V} = \{f \in L^2(\mathrm{SU}_2(\mathbb{R})) \mid f \text{ has weight } 0 \text{ for } T\}.$$

Here we say that $f \in L^2(\mathrm{SU}_2(\mathbb{R}))$ has weight $m \in \mathbb{Z}$ for T if $f(gt) = \alpha^m f(g)$ for all diagonal elements

$$t = \begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} \in T$$

and almost every $g \in \mathrm{SU}_2(\mathbb{R})$.

We now use the notation

$$g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

again for elements of $\mathrm{SU}_2(\mathbb{R})$, and the orthonormal basis comprising the functions $\sqrt{n+1}\pi_{k,\ell}^{(n)}$ for $n \in \mathbb{N}_0$ and $k, \ell \in \{0, \dots, n\}$ of $L^2(\mathrm{SU}_2(\mathbb{R}))$ from Corollary 6.12. Using the concrete formula for $\pi_{k,\ell}^{(n)}$ and the notation for g and t above, we see that

$$gt = \begin{pmatrix} \alpha z & -\overline{\alpha w} \\ \alpha w & \overline{\alpha z} \end{pmatrix}$$

and

$$\begin{aligned}\pi_{k,\ell}^{(n)}(gt) &= (k!(n-k)!\ell!(n-\ell)!)^{\frac{1}{2}} \\ &\quad \times \sum_{\substack{i \in \{0,\dots,n-k\} \\ j \in \{0,\dots,k\} \\ i+j=\ell}} \frac{(-1)^{k-j}}{i!j!(n-k-i)!(k-j)!} (\alpha z)^{n-k-i} (\alpha w)^i (\overline{\alpha z})^j (\overline{\alpha w})^{k-j} \\ &= \alpha^{n-2k} \pi_{k,\ell}^{(n)}(g).\end{aligned}$$

In other words, the orthonormal basis consists of eigenvectors for the right-regular representation restricted to $T \cong \mathbb{S}^1$. Hence we can obtain an orthonormal basis of $\mathcal{V}$ by using only those normalized matrix coefficients $\sqrt{n+1}\pi_{k,\ell}^{(n)}$ with weight $n - 2k = 0$. Hence we have shown that $\mathcal{V}$ has the orthonormal basis consisting of the functions $\sqrt{2k+1}\pi_{k,\ell}^{(2k)}$ with $k \in \mathbb{N}_0$ and $\ell \in \{0, \dots, 2k\}$, and the left-regular representation of $\mathrm{SU}_2(\mathbb{R})$ on

$$\mathcal{V}_{2k} = \left\langle \pi_{k,\ell}^{(2k)} \mid \ell \in \{0, \dots, 2k\} \right\rangle$$

is isomorphic to $\mathrm{Sym}^{2k}(\mathbb{C}^2)$ by the argument used in the last part of the proof of the Peter–Weyl theorem (Theorem 5.42). This gives the corollary. □

6.2.4 Conjugacy Classes and Characters of $\mathbf{SU_2}(\mathbb{R})$

We wish to finish the discussion of the compact group $\mathrm{SU}_2(\mathbb{R})$ by describing $\mathrm{SU}_2(\mathbb{R})^\sharp$ and calculating the characters of $\mathrm{SU}_2(\mathbb{R})$.

Proposition 6.16 (Sato–Tate measure). *The trace* $\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R}) \to [-2, 2]$ *descends to a homeomorphism*

$$\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R})^\sharp \longrightarrow [-2, 2].$$

The push-forward of the Haar measure is given by

$$\mathrm{tr}_* \, m_{\mathrm{SU}_2(\mathbb{R})}\big((a,b)\big) = \frac{1}{\pi}\int_a^b \sqrt{1-\tfrac{t^2}{4}}\,\mathrm{d}t,$$

and is called the Sato–Tate measure.

Proof. For $A, B \in \mathrm{Mat}_{d,d}(\mathbb{C})$ recall that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, since

$$\mathrm{tr}(AB) = \sum_{i=1}^{d} (AB)_{i,i} = \sum_{i,j=1}^{d} A_{i,j}B_{j,i} = \sum_{j,i=1}^{d} B_{i,j}A_{j,i}.$$

In particular, the map $\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R})^\sharp \to \mathbb{R}$ defined by $\mathrm{tr}([g]) = \mathrm{tr}(g)$ is a well-defined continuous map. As the eigenvalues of any $g \in \mathrm{SU}_2(\mathbb{R})$ are of the form $\alpha, \overline{\alpha}$ for some $\alpha \in \mathbb{S}^1$, we see that $\mathrm{tr}(g)$ lies in $[-2,2]$. Varying $\alpha \in \mathbb{S}^1$ and using

$$t = \begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R}),$$

we also see that $\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R})^\sharp \to [-2,2]$ is surjective.

We claim that the trace map is also injective on $\mathrm{SU}_2(\mathbb{R})^\sharp$. So suppose that $\mathrm{tr}([g_1]) = \mathrm{tr}([g_2])$ for some $[g_1],[g_2] \in \mathrm{SU}_2(\mathbb{R})^\sharp$. Then the characteristic polynomials of g_1 and g_2 agree (since these are determined for 2×2 matrices by the trace and determinant). It follows that the eigenvalues $\alpha, \overline{\alpha}$ of g_1 and g_2 are equal. Since $g_1 \in \mathrm{SU}_2(\mathbb{R})$ there exists an orthonormal basis $u_1, u_2 \in \mathbb{C}^2$ consisting of eigenvectors for α, resp. $\overline{\alpha}$. Multiplying u_2 by a scalar of absolute value one if necessary, we may assume $\det h = 1$ where $h = (u_1, u_2)$. It follows that $h \in \mathrm{SU}_2(\mathbb{R})$ and

$$g_1 = h \begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} h^{-1}.$$

In other words,

$$[g_1] = \left[\begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix}\right],$$

and by symmetry between g_1, g_2 also $[g_1] = [g_2]$, as required.

As both $\mathrm{SU}_2(\mathbb{R})^\sharp$ and $[-2,2]$ are compact and tr is continuous, it follows that $\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R})^\sharp \to [-2,2]$ is a homeomorphism.

It remains to prove the explicit description of the image of Haar measure. For this, we again identify $\mathrm{SU}_2(\mathbb{R})$ with $\mathbb{S}^3 \subseteq \mathbb{H} \cong \mathbb{R}^4$. With this the Haar measure can be defined using the four-dimensional Lebesgue measure $m_{\mathbb{R}^4}$. In fact for $B \subseteq \mathbb{S}^3$ we define

$$m_{\mathbb{S}^3}(B) = m_{\mathbb{R}^4}\big(\{rv \mid r \in [0,1], v \in B\}\big) \tag{6.25}$$

and, since $\mathbb{S}^3$ acts linearly as a unimodular transformation on $\mathbb{R}^4$, it follows that $m_{\mathbb{S}^3}$ defines a measure on $\mathbb{S}^3$ with total measure $m_{\mathbb{S}^3}(\mathbb{S}^3) = m_{\mathbb{R}^4}\big(B_1^{\mathbb{R}^4}\big)$. This turns the description of $\mathrm{tr}_* \, m_{\mathbb{S}^3}$ into an exercise in multi-dimensional calculus.

In fact we will use four-dimensional spherical coordinates defined by

$$S\colon \begin{pmatrix} r \\ \theta \\ \phi \\ \psi \end{pmatrix} \longmapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta\cos\phi \\ r\sin\theta\sin\phi\cos\psi \\ r\sin\theta\sin\phi\sin\psi \end{pmatrix}$$

with total derivative

$$\begin{pmatrix} \cos\theta & -r\sin\theta & 0 & 0 \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi & 0 \\ \sin\theta\sin\phi\cos\psi & r\cos\theta\sin\phi\cos\psi & r\sin\theta\cos\phi\cos\psi & -r\sin\theta\sin\phi\sin\psi \\ \sin\theta\sin\phi\sin\psi & r\cos\theta\sin\phi\sin\psi & r\sin\theta\cos\phi\sin\psi & r\sin\theta\sin\phi\cos\psi \end{pmatrix}$$

and Jacobian determinant

$$r^3\sin^2\theta\sin\phi\det\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi & 0 \\ \sin\theta\sin\phi\cos\psi & \cos\theta\sin\phi\cos\psi & \cos\phi\cos\psi & -\sin\psi \\ \sin\theta\sin\phi\sin\psi & \cos\theta\sin\phi\sin\psi & \cos\phi\sin\psi & \cos\psi \end{pmatrix}.$$

Expanding the remaining determinant along the first row, we see that it is given by

$$\mathit{\Delta}\cos^2\theta + \mathit{\Delta}\sin^2\theta = \mathit{\Delta}$$

where

$$\begin{aligned} \mathit{\Delta} &= \det\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi\cos\psi & \cos\phi\cos\psi & -\sin\psi \\ \sin\phi\sin\psi & \cos\phi\sin\psi & \cos\psi \end{pmatrix} \\ &= \cos^2\phi\det\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} + \sin^2\phi\det\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} = 1. \end{aligned}$$

A convenient domain for the spherical coordinates is

$$U = (0,\infty)\times(0,\pi)\times(0,\pi)\times(0,2\pi),$$

and spherical coordinates define a diffeomorphism S from U to a full measure open set $V \subseteq \mathbb{R}^4$.

Now let $a < b$ be in $[-2,2]$, and define $\theta_a = \arccos\frac{a}{2}$ and $\theta_b = \arccos\frac{b}{2}$ so that

$$\begin{aligned}
\operatorname{tr}^{-1}(a,b) &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{S}^3 \;\middle|\; \frac{a}{2} < x_1 < \frac{b}{2} \right\} \\
&= \left\{ \begin{pmatrix} \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi\cos\psi \\ \sin\theta\sin\phi\sin\psi \end{pmatrix} \;\middle|\; \theta \in (\theta_b, \theta_a), \phi \in (0,\pi), \psi \in (0, 2\pi) \right\}.
\end{aligned}$$

Using the description of the Haar measure $m_{\mathbb{S}^3}$ in (6.25), this leads to

$$\begin{aligned}
\operatorname{tr}_* m_{\mathbb{S}^3}\big((a,b)\big) &= \int_0^1 r^3 \,\mathrm{d}r \int_{\theta_b}^{\theta_a} \sin^2\theta \,\mathrm{d}\theta \int_0^\pi \sin\phi \,\mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\psi \\
&= \tfrac{1}{4} \cdot 2 \cdot 2\pi \int_{\theta_b}^{\theta_a} \sin^2\theta \,\mathrm{d}\theta.
\end{aligned}$$

Instead of calculating the latter integral, we wish to rewrite it as an integral over $t \in [a,b]$ using $t = 2\cos\theta$ and $\mathrm{d}t = -2\sin\theta\,\mathrm{d}\theta$. This gives

$$\operatorname{tr}_* m_{\mathbb{S}^3}\big((a,b)\big) = \frac{\pi}{2} \int_a^b \sqrt{1 - \tfrac{t^2}{4}} \,\mathrm{d}t.$$

Normalizing the measure to be a probability gives the proposition. □

Using the identification between $\mathrm{SU}_2(\mathbb{R})^\sharp$ and the interval $[-2,2]$ we now describe the characters of $\mathrm{SU}_2(\mathbb{R})$. By Theorem 6.6 the irreducible representations of $\mathrm{SU}_2(\mathbb{R})$ are given by the nth symmetric tensor products $\mathrm{Sym}^n(\mathbb{C}^2)$ of the standard representation for all $n \in \mathbb{N}_0$. We define $t = z + \overline{z} \in [-2,2]$ with $z \in \mathbb{S}^1$. Then the eigenvalues for the group element

$$\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R})$$

and the basis vector $e_1^{\odot(n-k)} \odot e_2^{\odot k}$ is given by $z^{n-k}\overline{z}^k = z^{n-2k}$ for $k \in \{0,\ldots,n\}$ and $n \in \mathbb{N}_0$. Hence the character χ associated to $\mathrm{Sym}^n(\mathbb{C}^2)$ is given by

$$\chi_n(t) = \chi_n\left(\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix}\right) = \sum_{k=0}^{n} z^{n-2k}$$

for every $n \in \mathbb{N}_0$. Using the variable $t \in [-2,2] \cong \mathrm{SU}_2(\mathbb{R})^\sharp$, the first few characters are given by

$$\begin{aligned}
&\chi_0(t) = 1, \\
&\chi_1(t) = t, \\
&\chi_2(t) = z^2 + 1 + z^{-2} = (z + \overline{z})^2 - 1 = t^2 - 1,
\end{aligned}$$

and

$$\chi_3(t) = z^3 + z + z^{-1} + z^{-3} = (z + \overline{z})^3 - 2(z + \overline{z}) = t^3 - 2t.$$

We conclude by linking the general character to certain classical polynomials. Using the notation $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{S}^1$ with $\theta \in [0, 2\pi)$, the eigenvalues of $\pi^{(n)}$ are given by

$$z^{n-2k} = \mathrm{e}^{\mathrm{i}(n-2k)\theta}$$

for $k = 0, \ldots, n$, and so

$$\begin{aligned} \chi_n\left(\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix}\right) &= \mathrm{e}^{\mathrm{i}n\theta} + \mathrm{e}^{\mathrm{i}(n-2)\theta} + \cdots + \mathrm{e}^{-\mathrm{i}n\theta} \\ &= \mathrm{e}^{-\mathrm{i}n\theta} \frac{\mathrm{e}^{\mathrm{i}2(n+1)\theta} - 1}{\mathrm{e}^{\mathrm{i}2\theta} - 1} \\ &= \frac{\mathrm{e}^{\mathrm{i}(n+1)\theta} - \mathrm{e}^{-\mathrm{i}(n+1)\theta}}{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}} = \frac{\sin\bigl((n+1)\theta\bigr)}{\sin\theta} \end{aligned}$$

by the geometric series summation formula (using $q = \mathrm{e}^{\mathrm{i}2\theta}$). Expressing this in terms of

$$t = \operatorname{tr}\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix} = z + \overline{z} = 2\cos\theta$$

gives a formula for the character $\chi_n(t)$. Using instead the variable

$$T = \tfrac{t}{2} = \cos\theta,$$

this would give rise to the *Chebyshev polynomials of the second kind*.

Exercise 6.17. Describe $\mathrm{SO}_3(\mathbb{R})^\sharp$ and the characters of $\mathrm{SO}_3(\mathbb{R})$.

6.3 Summary and Outlook

The main purpose of this chapter was to provide a concrete example of the unitary dual of a compact abelian group, which was done using $\mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SO}_3(\mathbb{R})$.

To achieve this, we actually classified all irreducible finite-dimensional representations of the complex Lie group $\mathrm{SL}_2(\mathbb{C})$, respectively of its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by differentiation. As we will see in Chapter 7, generalizing this derivative representation to unitary representations is more delicate than it is in finite dimensions.