Chapter 7
Smooth Vectors and Decay for $\mathrm{SL}_3(\mathbb{R})$

We will introduce in this chapter the notion of smooth vectors for a unitary representation. This notion will be important for the study of unitary representations of Lie groups in the following chapters. Already in this chapter we will use smooth vectors in our study of *effective decay* of matrix coefficients, and an effective Howe–Moore theorem for the group $\mathrm{SL}_3(\mathbb{R})$ (see also Theorem 1.80).

7.1 Smooth Vectors and Derivative Representations

7.1.1 Differential Operators and Smooth Vectors

Throughout this section we assume that G is a Lie group with Lie algebra $\mathfrak{g}$.

Definition 7.1 (Partial derivative). Let π be a unitary representation of the Lie group G. A vector $v \in \mathcal{H}_\pi$ has a *partial derivative* $\pi_\partial(\mathbf{a})v \in \mathcal{H}_\pi$ *in the direction* $\mathbf{a} \in \mathfrak{g}$ if

$$\pi_\partial(\mathbf{a})v = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\big(\pi(\exp(t\mathbf{a}))\big)v = \lim_{t\to 0}\frac{1}{t}\big(\pi(\exp(t\mathbf{a}))v - v\big)$$

exists in $\mathcal{H}_\pi$. We say that v is C^1*-smooth* if $\pi_\partial(\mathbf{a})v$ exists for all $\mathbf{a} \in \mathfrak{g}$, is C^r*-smooth* for some $r \geqslant 1$ if $\pi_\partial(\mathbf{a}_1)\cdots\pi_\partial(\mathbf{a}_r)v$ exists for all $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathfrak{g}$, and is *smooth* if v is C^r-smooth for all $r \geqslant 1$.

These notions will become more familiar after we see an example and establish some standard properties of derivatives and integrals in this context.

Example 7.2 (Smooth vectors for unitary representations of $\mathrm{SO}_2(\mathbb{R})$*).* Let

$$G = \mathrm{SO}_2(\mathbb{R}) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

and let

$$\mathbf{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g} = \mathfrak{so}_2(\mathbb{R}).$$

Furthermore, let π be a unitary representation of $\mathrm{SO}_2(\mathbb{R})$. Suppose first that $v \in \mathcal{H}_\pi$ is an eigenvector of weight $n \in \mathbb{Z}$ (that is, $\pi_{k_\theta} v = \mathrm{e}^{\mathrm{i}n\theta} v$ for all $k_\theta \in \mathrm{SO}_2(\mathbb{R})$). Then v is smooth, since

$$\pi_\partial(\mathbf{w})v = \lim_{t\to 0} \frac{1}{t}\bigl(\pi(\exp(t\mathbf{w}))v - v\bigr) = \lim_{t\to 0}\frac{1}{t}\bigl(\mathrm{e}^{\mathrm{i}nt} - 1\bigr)v = \mathrm{i}nv.$$

More generally, $v \in \mathcal{H}_\pi$ with eigenvector decomposition $v = \sum_{n\in\mathbb{Z}} v_n$ has a partial derivative $\pi_\partial(\mathbf{w})v$ if and only if $\sum_{n\in\mathbb{Z}} n^2 \|v_n\|^2 < \infty$, and in this case

$$\pi_\partial(\mathbf{w})v = \sum_{n\in\mathbb{Z}} \mathrm{i}nv_n.$$

Indeed, suppose first that $\sum_{n\in\mathbb{Z}} n^2\|v_n\|^2 < \infty$, which gives that

$$\pi_\partial(\mathbf{w})v = \lim_{t\to\infty} \sum_{n\in\mathbb{Z}} \underbrace{\frac{1}{t}\bigl(\mathrm{e}^{\mathrm{i}nt} - 1\bigr)}_{|\cdot|\leqslant n} v_n = \sum_{n\in\mathbb{Z}} \mathrm{i}nv_n$$

by (a trivial form of) dominated convergence. Suppose now that $\pi_\partial(\mathbf{w})v = \widetilde{v}$ exists, and assume that $\widetilde{v} = \sum_{n\in\mathbb{Z}} \widetilde{v}_n$ is the eigenvalue decomposition. For any $n \in \mathbb{Z}$ and $u \in \mathcal{H}_\pi$ with eigenvector decomposition $u = \sum_{m\in\mathbb{Z}} u_m$, we then have

$$\begin{aligned}
\langle \widetilde{v}_n, u\rangle = \langle \widetilde{v}_n, u_n\rangle = \langle \widetilde{v}, u_n\rangle &= \lim_{t\to 0}\frac{1}{t}\langle \pi(\exp(t\mathbf{w}))v - v, u_n\rangle \\
&= \lim_{t\to 0}\frac{1}{t}\bigl(\langle \pi(\exp(t\mathbf{w}))v, u_n\rangle - \langle v, u_n\rangle\bigr) \\
&= \lim_{t\to 0}\bigl(\langle v, \pi(\exp(-t\mathbf{w}))u_n\rangle - \langle v, u_n\rangle\bigr) \\
&= \lim_{t\to 0}\frac{1}{t}\bigl(\mathrm{e}^{\mathrm{i}nt} - 1\bigr)\langle v_n, u_n\rangle = \langle \mathrm{i}nv_n, u\rangle.
\end{aligned}$$

As this holds for all $u \in \mathcal{H}_\pi$, we see that $\widetilde{v}_n = \mathrm{i}nv_n$ and hence

$$\|\widetilde{v}\|^2 = \sum_{n\in\mathbb{Z}} n^2\|v_n\|^2 < \infty$$

as claimed.

Lemma 7.3 (Linearity, Fundamental Theorem). *Let $\mathbf{b}_1, \dots, \mathbf{b}_{\dim\mathfrak{g}}$ be a basis of the Lie algebra $\mathfrak{g}$ of the Lie group G. Let π be a unitary representation of G, and suppose that $v \in \mathcal{H}_\pi$ has the property that $\pi_\partial(\mathbf{b}_j)v$ exists for all $j = 1, \dots, \dim\mathfrak{g}$. Then $\pi_\partial(\mathbf{a})v$ exists for all $\mathbf{a} \in \mathfrak{g}$, depends linearly on $\mathbf{a}$, and satisfies*

$$\pi\left(\exp(t\mathbf{a})\right)v - v = \int_0^t \pi\left(\exp(s\mathbf{a})\right)\pi_\partial(\mathbf{a})v\,\mathrm{d}s \tag{7.1}$$

for all $t \in \mathbb{R}$ (with the usual sign conventions for Riemann integrals).

Proof. We start by proving (7.1) for the vector v as in the lemma and a direction $\mathbf{a} \in \mathfrak{g}$ for which $\pi_\partial(\mathbf{a})$ is already known to exist. For this, notice that $\pi(\exp(s\mathbf{a}))\pi_\partial(\mathbf{a})v$ depends continuously on s, which implies that the $\mathcal{H}_\pi$-valued weak integral on the right-hand side of (7.1) exists. Now fix some vector $w \in \mathcal{H}_\pi$ and notice that the derivative of the map $s \mapsto \langle \pi(\exp(s\mathbf{a}))v, w\rangle$ is given by

$$\begin{aligned}\lim_{t\to 0}&\frac{\langle \pi\left(\exp((s+t)\mathbf{a})\right)v, w\rangle - \langle \pi\left(\exp(s\mathbf{a})\right)v, w\rangle}{t}\\ &= \lim_{t\to 0}\frac{1}{t}\left\langle \pi\left(\exp(t\mathbf{a})\right)v - v, \pi\left(\exp(-s\mathbf{a})\right)w\right\rangle\\ &= \langle \pi_\partial(\mathbf{a})v, \pi\left(\exp(-s\mathbf{a})\right)w\rangle\\ &= \langle \pi\left(\exp(s\mathbf{a})\right)\pi_\partial(\mathbf{a})v, w\rangle,\end{aligned}$$

and so is also continuous in s. Hence, by the fundamental theorem of calculus (for $\mathbb{C}$-valued functions),

$$\begin{aligned}\langle \pi\left(\exp(t\mathbf{a})\right)v, w\rangle - \langle v, w\rangle &= \int_0^t \langle \pi\left(\exp(s\mathbf{a})\right)\pi_\partial(\mathbf{a})v, w\rangle\,\mathrm{d}s\\ &= \left\langle \int_0^t \pi\left(\exp(s\mathbf{a})\right)\pi_\partial(\mathbf{a})v\,\mathrm{d}s, w\right\rangle.\end{aligned}$$

As this holds for all $w \in \mathcal{H}_\pi$, we see that (7.1) holds for $\mathbf{a}$ (assuming only that $\pi_\partial(\mathbf{a})v$ exists).

By assumption, $\pi_\partial(\mathbf{b}_j)v$ exists for $j = 1, \ldots, d = \dim\mathfrak{g}$ so that (7.1) holds, in particular, already for $\mathbf{a} = \mathbf{b}_j$. For the proof of the first part of the lemma, we will combine (7.1) with the coordinate system of the second kind defined by

$$\Psi\colon \mathbb{R}^d \ni (t_1, t_2, \ldots, t_d) \longmapsto \exp(t_1\mathbf{b}_1)\exp(t_2\mathbf{b}_2)\cdots\exp(t_d\mathbf{b}_d) \in G.$$

Since the derivative of Ψ at 0 is the map $(s_1, \ldots, s_d) \mapsto s_1\mathbf{b}_1 + \cdots + s_d\mathbf{b}_d \in \mathfrak{g}$ and so is invertible, Ψ indeed defines a local diffeomorphism. For some $\mathbf{a} \in \mathfrak{g}$ we define smooth functions $t_j(t)$ for t close to 0 and $j = 1, \ldots, d$ by

$$(t_1(t), t_2(t), \ldots, t_d(t)) = \Psi^{-1}\left(\exp(t\mathbf{a})\right),$$

or equivalently by

$$\exp(t\mathbf{a}) = \exp(t_1(t)\mathbf{b}_1)\cdots\exp(t_d(t)\mathbf{b}_d).$$

Since the derivative of $t \mapsto \exp(t\mathbf{a})$ at $t = 0$ is $\mathbf{a} \in \mathfrak{g}$, we see that the derivative of $\Psi^{-1}(\exp(t\mathbf{a}))$ at $t = 0$ is equal to $(s_1, \dots, s_d) \in \mathbb{R}^d$ with

$$s_1\mathbf{b}_1 + \cdots + s_d\mathbf{b}_d = \mathbf{a}.$$

Equivalently, we have

$$\lim_{t\to 0} \frac{t_j(t)}{t} = s_j \tag{7.2}$$

for $j = 1, \dots, d$.

We now express $\pi(\exp(t\mathbf{a}))v - v$ as the telescoping sum

$$\sum_{j=1}^{d}\bigl(\pi\left(\exp(t_1\mathbf{b}_1)\cdots\exp(t_j\mathbf{b}_j)\right)v - \pi\left(\exp(t_1\mathbf{b}_1)\cdots\exp(t_{j-1}\mathbf{b}_{j-1})\right)v\bigr)$$
$$= \sum_{j=1}^{d}\pi\bigl(\exp(t_1\mathbf{b}_1)\cdots\exp(t_{j-1}\mathbf{b}_{j-1})\bigr)\bigl(\pi\left(\exp(t_j\mathbf{b}_j)\right)v - v\bigr)$$

and apply (7.1) for the directions $\mathbf{b}_1, \dots, \mathbf{b}_d$. This shows that $\pi(\exp(t\mathbf{a}))v - v$ equals

$$\sum_{j=1}^{d}\pi\bigl(\exp(t_1\mathbf{b}_1)\cdots\exp(t_{j-1}\mathbf{b}_{j-1})\bigr)\int_0^{t_j}\pi\left(\exp(s\mathbf{b}_j)\right)\pi_\partial(\mathbf{b}_j)v\,\mathrm{d}s$$
$$= \sum_{j=1}^{d}\int_0^{t_j}\pi\bigl(\exp(t_1\mathbf{b}_1)\cdots\exp(t_{j-1}\mathbf{b}_{j-1})\exp(s\mathbf{b}_j)\bigr)\pi_\partial(\mathbf{b}_j)v\,\mathrm{d}s.$$

We now divide by t and use (7.2) together with continuity of the representation to obtain

$$\lim_{t\to 0}\frac{1}{t}\left(\pi(\exp(t\mathbf{a}))v - v\right) = \sum_{j=1}^{d} s_j\pi_\partial(\mathbf{b}_j)v.$$

This proves that

$$\pi_\partial(\mathbf{a})v = s_1\pi_\partial(\mathbf{b}_1)v + \cdots + s_d\pi_\partial(\mathbf{b}_d)v$$

exists and depends linearly on $\mathbf{a} \in \mathfrak{g}$. Together with the first part of the proof, the lemma follows. □

Exercise 7.4. Suppose that π is a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$. Suppose $v \in \mathcal{H}_\pi$ has $v_{\mathbf{a}} \in \mathcal{H}_\pi$ as a weak derivative in the direction $\mathbf{a} \in \mathfrak{g}$ in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle\pi\left(\exp(t\mathbf{a})\right)v, w\rangle = \langle v_{\mathbf{a}}, w\rangle$$

for all $w \in \mathcal{H}_\pi$ (or just for a dense set of vectors). Show in this case that $v_{\mathbf{a}} = \pi_\partial(\mathbf{a})v$ is in fact the derivative of v in the sense of Definition 7.1.

Proposition 7.5 (Existence of smooth vectors). *Let π be a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$, $v \in \mathcal{H}_\pi$, and $\psi \in C_c^\infty(G)$. Then $\pi_*(\psi)v$ is smooth, and $\pi_\partial(\mathbf{a})\pi_*(\psi)v = \pi_*(\lambda_\partial(\mathbf{a})\psi)v$ for any $\mathbf{a} \in \mathfrak{g}$, where*[†]

$$\lambda_\partial(\mathbf{a})\psi(g) = \frac{\partial}{\partial t}\Big|_{t=0} \psi\left(\exp(-t\mathbf{a})g\right)$$

is the partial derivative with respect to the left regular representation. Moreover, for a smooth approximate identity (ψ_n) in C_c^∞ (see Proposition 1.42) we have

$$v = \lim_{n\to\infty} \pi_*(\psi_n)v$$

for any $v \in \mathcal{H}_\pi$ and so, in particular, the smooth vectors in $\mathcal{H}_\pi$ are dense.

Proof. Using the definition of the convolution operator we see that

$$\begin{aligned}\pi\left(\exp(t\mathbf{a})\right)\pi_*(\psi)v &= \int_G \pi\left(\exp(t\mathbf{a})\right)\psi(h)\pi_h v \,\mathrm{d}m(h)\\ &= \int_G \psi\left(\exp(-t\mathbf{a})g\right)\pi_g v\,\mathrm{d}m(g)\end{aligned}$$

by using the substitution $g = \exp(t\mathbf{a})h$. This gives

$$\begin{aligned}\frac{1}{t}\big(\pi(\exp(t\mathbf{a}))\pi_*(\psi)v - \pi_*(\psi)v\big) &= \frac{1}{t}\int_G \big(\psi\left(\exp(-t\mathbf{a})g\right) - \psi(g)\big)\pi_g v\,\mathrm{d}m(g)\\ &= \int_G \frac{\psi(\exp(-t\mathbf{a})g) - \psi(g)}{t}\pi_g v\,\mathrm{d}m(g)\end{aligned}$$

for all $t \in \mathbb{R}\smallsetminus\{0\}$. As $\psi \in C_c^\infty(G)$, we know that

$$\frac{\psi(\exp(-t\mathbf{a})g) - \psi(g)}{t} \longrightarrow \lambda_\partial(\mathbf{a})\psi(g) \tag{7.3}$$

as $t \to 0$, that this convergence is uniform in g, and that this convergence takes place inside a compact subset of G in the sense that the left-hand side vanishes for all $t \in [-1,1]\smallsetminus\{0\}$ outside the compact subset $\exp\left([-1,1]\mathbf{a}\right)\operatorname{supp}(\psi) \subseteq G$. In particular, the convergence also takes place in $L^1(G)$, and it follows that $\pi_\partial(\mathbf{a})\pi_*(\psi)v = \pi_*(\lambda_\partial(\mathbf{a})\psi)v$ exists. Applying this inductively to expressions of the form $\pi_\partial(\mathbf{a}_n)\cdots\pi_\partial(\mathbf{a}_1)\pi_*(\psi)v$ for $n \geqslant 1$ shows that $\pi_*(\psi)v$ is smooth.

Using an approximate identity in $C_c^\infty(G) \subseteq L^1(G)$ the proposition follows from Proposition 1.47. □

[†] This formula may look a bit unusual, as it corresponds to a right-invariant vector field on G (rather than a left-invariant vector field).

Definition 7.6 (Sobolev norm). Let π be a unitary representation of G, let $\mathbf{b}_1, \ldots, \mathbf{b}_{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g} = \operatorname{Lie} G$, and let $r \geqslant 0$ be an integer. The *degree r Sobolev norm* of a C^r-smooth vector $v \in \mathcal{H}_\pi$ (with respect to the fixed basis) is defined by

$$\mathcal{S}(v)^2 = \mathcal{S}_r(v)^2 = \|v\|^2 + \sum_{s=1}^{r} \sum_{j_1,\ldots,j_s=1}^{\dim \mathfrak{g}} \|\pi_\partial(\mathbf{b}_{j_1}) \cdots \pi_\partial(\mathbf{b}_{j_s}) v\|^2.$$

Essential Exercise 7.7 (Lipshitz bound). Let π be a unitary representation of the Lie group G and v a C^1-smooth vector. Show that

$$\|\pi_{\exp \mathbf{a}} v - v\| \leqslant \|\mathbf{a}\| \, \mathcal{S}(v)$$

for all $\mathbf{a} \in \mathfrak{g}$, where $\mathcal{S}$ is a degree-one Sobolev norm defined by an orthonormal basis of $\mathfrak{g}$.

Exercise 7.8. Extend Proposition 7.5, and show that $\mathcal{S}_r(\pi_*(\psi)v) \ll_\psi \|v\|$ (and express the implicit constant in terms of ψ).

Example 7.9 (Derivatives and smooth vectors for $\mathbb{R}^d$). We let $G = \mathbb{R}^d$ for some $d \in \mathbb{N}$, and will use the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$ of its Lie algebra $\mathfrak{g}$ (which is also $\mathbb{R}^d$). Let π be a unitary representation. Applying the spectral theorem (Corollary 2.12), we assume that $\mathcal{H}_\pi = L^2_\mu(X)$ for a finite (or σ-finite) measure μ on $X = \mathbb{R}^d \times \mathbb{N}$ and that π is defined by the multiplication representation†

$$\big(\pi_x v\big)(t,n) = \mathrm{e}^{2\pi\mathrm{i}(x \cdot t)} v(t,n)$$

for all $x \in \mathbb{R}^d$, $v \in L^2_\mu(X)$, and $(t,n) \in \mathbb{R}^d \times \mathbb{N}$. In this case we obtain, for a C^1-smooth vector v, that

$$\pi_\partial(\mathbf{e}_j) v = \lim_{s \to 0} \underbrace{\frac{\mathrm{e}^{2\pi\mathrm{i} s t_j} - 1}{s}}_{|\cdot| \leqslant 2\pi |t_j|} v(t,n) = 2\pi\mathrm{i} t_j v(t,n) \tag{7.4}$$

for $j = 1, \ldots, d$, so that

$$\mathcal{S}_1(v)^2 = \|v\|^2 + \sum_{j=1}^{d} \|M_{2\pi\mathrm{i}t_j} v\| \tag{7.5}$$

where $M_{2\pi\mathrm{i}t_j}$ is the multiplication operator on $L^2_\mu(X)$ defined by

$$\big(M_{2\pi\mathrm{i}t_j} v\big)(t,n) = 2\pi\mathrm{i} t_j v(t,n)$$

† In Corollary 2.12 we used a simplified notation and wrote M_g for the multiplication operator defined by the function $\widehat{G} \ni t \mapsto \langle g, t \rangle \in \mathbb{S}^1$. In the case of $x \in \mathbb{R}^d$ and $t \in \widehat{\mathbb{R}^d} \cong \mathbb{R}^d$ this function corresponds to $\mathbb{R}^d \ni t \mapsto \mathrm{e}^{2\pi\mathrm{i}(x \cdot t)}$ by Exercise 2.6 or Proposition 2.39.

for all $v \in L^2_\mu(X)$, $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, and $n \in \mathbb{N}$. Moreover, if $v \in L^2_\mu(X)$ and $j \in \{1, \dots, d\}$ have the property that $M_{2\pi \mathrm{i} t_j}(v)$ belongs to $L^2_\mu(X)$, then, by applying dominated convergence in (7.4), we see that $\pi_\partial(\mathbf{e}_j)(v)$ exists. If this holds for all $j \in \{1, \dots, d\}$, then v is C^1-smooth and the degree-one Sobolev norm is given by (7.5).

Now let $r \in \mathbb{N}$. Applying the above recursively to the partial derivatives, we see that $v \in L^2(\mathbb{R}^d)$ is C^r-smooth if and only if $pv \in L^2_\mu(X)$ where p is any polynomial in $\mathbb{C}[t_1, \dots, t_d]$ of degree at most r.

Finally, we wish to apply this to the regular representation λ of $\mathbb{R}^d$ on $L^2(\mathbb{R}^d)$, which will reveal the connection to Sobolev spaces (see, for example, [21, Ch. 5]). By the Plancherel formula (Theorem 2.15), the regular representation is isomorphic to the multiplication representation as above for the Lebesgue measure $\mu = m_{\mathbb{R}^d}$. Applying the above, we see that $v \in L^2(\mathbb{R}^d)$ is smooth for the regular representation if and only if $\check{v} \in L^2(\mathbb{R}^d)$ satisfies $p\check{v} \in L^2(\mathbb{R}^d)$ for any polynomial $p \in \mathbb{C}[t_1, \dots, t_d]$. Using the polynomial

$$p(t) = \prod_{j=1}^{d} (t_j^2 + 1)$$

which has $\frac{1}{p} \in L^2(\mathbb{R}^d)$, it follows that $\check{v} = \frac{1}{p}(p\check{v}) \in L^1(\mathbb{R}^d)$ and hence by Corollaries 2.12 and 2.5 that $v = \widehat{(\check{v})} \in C_0(\mathbb{R}^d)$. Now fix some $j \in \{1, \dots, d\}$ and note that the above also applies to $\lambda_\partial(\mathbf{e}_j)v \in C_0(\mathbb{R}^d)$. Here

$$\lambda_\partial(\mathbf{e}_j)v = \lim_{s \to 0} \frac{1}{s}\bigl(\lambda_{s\mathbf{e}_j} v - v\bigr),$$

which, for the isometric Fourier transform on L^2, becomes as above

$$M_\partial(\mathbf{e}_j)\check{v}(t) = \lim_{s \to 0} \frac{1}{s}\bigl(\mathrm{e}^{2\pi \mathrm{i} s t_j} - 1\bigr)\check{v}(t) = 2\pi \mathrm{i} t_j \check{v}(t)$$

in $L^2(\mathbb{R}^d)$ and for almost any $t \in \mathbb{R}^d$. We now multiply this once more by $p(t)$ and apply dominated convergence (by relying on the fact that $t \mapsto t_j p(t)\check{v}(t)$ lies in $L^2(\mathbb{R}^d)$) to see that

$$\lim_{s \to 0} p(t) \underbrace{\frac{1}{s}\bigl(\mathrm{e}^{2\pi \mathrm{i} s t_j} - 1\bigr)}_{|\cdot| \leqslant 2\pi |t_j|} \check{v}(t) = p(t) 2\pi \mathrm{i} t_j \check{v}(t)$$

converges in $L^2(\mathbb{R}^d)$. Multiplying by $\frac{1}{p} \in L^2(\mathbb{R}^d)$ gives convergence in $L^1(\mathbb{R}^d)$ by the Cauchy–Schwarz inequality. However, this gives, by the continuity bound in Corollary 2.5 applied to the forward Fourier transform

$$L^1(\widehat{G}) \ni F \longrightarrow \widehat{F} \in C_0(G)$$

that

$$\left\| \frac{1}{s}\big(\lambda_{s\mathbf{e}_j} v - v\big) - \lambda_\partial(\mathbf{e}_j)v \right\|_\infty \leqslant \left\| \frac{1}{s}\big(\mathrm{e}^{2\pi \mathrm{i} s t_j} - 1\big)\check{v}(t) - \widecheck{\lambda_\partial(\mathbf{e}_j)v} \right\|_1 \longrightarrow 0$$

as $s \to 0$. As this holds for all $j \in \{1, \dots, d\}$ and can be applied recursively to the partial derivatives of v, it follows that $v \in C^\infty(\mathbb{R}^d)$.

Once again the reasoning above can be reversed to see that $v \in L^2(\mathbb{R}^d)$ is smooth with respect to the regular representation if and only if $v \in C^\infty(\mathbb{R}^d)$ and its partial derivatives $\partial^\alpha v$ belong to $L^2(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ (see Exercise 7.10).

Exercise 7.10. Complete the proof of the last claim in Example 7.9.

Essential Exercise 7.11. Let π be a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$. Let $r \geqslant 1$ and $v \in \mathcal{H}_\pi$ be a C^r-smooth vector. Let $\mathcal{S}$ denote the degree r Sobolev norm. Show that $\mathcal{S}(\pi_g v) \ll_g \mathcal{S}(v)$, and that the implicit constant can be chosen to be uniformly bounded on compact subsets of G.

Exercise 7.12. Let π be a unitary representation of G, let $\mathbf{b}_1, \dots, \mathbf{b}_{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$, the Lie algebra of G, and let $\widetilde{\mathbf{b}}_1, \dots, \widetilde{\mathbf{b}}_{\dim \mathfrak{g}}$ be another basis of $\mathfrak{g}$. Let $r \geqslant 1$ and let $\mathcal{S}$ (respectively $\widetilde{\mathcal{S}}$) be the degree r Sobolev norm defined by $\mathbf{b}_1, \dots, \mathbf{b}_{\dim \mathfrak{g}}$ (resp. by $\widetilde{\mathbf{b}}_1, \dots, \widetilde{\mathbf{b}}_{\dim \mathfrak{g}}$). Show that we have $\mathcal{S}(v) \ll \widetilde{\mathcal{S}}(v) \ll \mathcal{S}(v)$ for any C^r-smooth $v \in \mathcal{H}_\pi$. Show also that if $r = 1$ and both bases are orthonormal with respect to an inner product on $\mathfrak{g}$, then $\mathcal{S}(v) = \widetilde{\mathcal{S}}(v)$ for any C^1-smooth $v \in \mathcal{H}_\pi$.

Exercise 7.13. (a) Let $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the isometry group of the plane as in Section 3.3.1. Let $\pi \in \widehat{G}$ be an irreducible representation. Find and prove a description of the space of smooth vectors in $\mathcal{H}_\pi$. Also show that any $v \in \mathcal{H}_\pi$ is smooth for the restriction of π to $H = \mathbb{R}^2$.
(b) Let G be the '$ax + b$' group as in Section 3.3.2, and let $\pi_+ \in \widehat{G}$ be the irreducible representation corresponding to the set $(0, \infty) \subseteq \mathbb{R} \cong \widehat{R}$. Show that any $f \in C_c^\infty\big((0,\infty)\big)$ is a smooth vector. Can you again characterize smoothness with an appropriate moment condition?
(c) Let G be the Heisenberg group as in Section 3.3.4, and let $\pi_\xi \in \widehat{G}$ be the irreducible representation corresponding to the central character χ_ξ determined by $\xi \in \mathbb{R}^\times$. Show that any $f \in C_c^\infty(\mathbb{R})$ is a smooth vector. Can you again characterize smoothness?

7.1.2 The Total Derivative

Definition 7.14 (Total derivative). Let π be a unitary representation of G. The *total derivative* of π is defined on every C^1-smooth vector $v \in \mathcal{H}_\pi$ as the linear map $T_\pi(v)$ in $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$ given by

$$T_\pi(v)\colon \mathbf{a} \longmapsto \pi_\partial(\mathbf{a})v$$

for $\mathbf{a} \in \mathfrak{g}$. After fixing a basis $\mathbf{b}_1, \dots, \mathbf{b}_{\dim \mathfrak{g}}$ of $\mathfrak{g}$ we can identify $T_\pi(v)$ with the tuple

$$\big(\pi_\partial(\mathbf{b}_1)v, \ldots, \pi_\partial(\mathbf{b}_{\dim \mathfrak{g}})v\big) \in \mathcal{H}_\pi^{\dim \mathfrak{g}} \cong \mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi).$$

Lemma 7.15 (Closed operator). *Let π be a unitary representation of G. Then the total derivative T_π with domain*

$$D_{T_\pi} = \{v \in \mathcal{H}_\pi \mid v \textit{ is } C^1\textit{-smooth}\}$$

is a densely defined closed operator.

Proof. Suppose that (v_n) in D_{T_π} is a sequence with

$$(v_n, T_\pi(v_n)) \longrightarrow (v, L) \in \mathcal{H}_\pi \times \mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$$

as $n \to \infty$, and let $\mathbf{a} \in \mathfrak{g}$. By Lemma 7.3 this implies that

$$\pi\left(\exp(t\mathbf{a})\right) v_n - v_n = \int_0^t \pi\left(\exp(s\mathbf{a})\right) \underbrace{\pi_\partial(\mathbf{a})v_n}_{=T_\pi(v_n)\mathbf{a}} \,\mathrm{d}s$$

for any $t \in \mathbb{R}$. Since $T_\pi(v_n) \to L$ in $\mathcal{H}_\pi^{\dim \mathfrak{g}}$ as $n \to \infty$ we have

$$T_\pi(v_n)(\mathbf{a}) \longrightarrow L(\mathbf{a})$$

as $n \to \infty$. Moreover, since the integral defines a continuous operator on $\mathcal{H}_\pi$ we also obtain from this that

$$\pi\left(\exp(t\mathbf{a})\right) v - v = \int_0^t \pi\left(\exp(s\mathbf{a})\right) L(\mathbf{a}) \,\mathrm{d}s.$$

Dividing this by $t \neq 0$ and using continuity of the representation π we arrive at

$$\pi_\partial(\mathbf{a})v = L(\mathbf{a})$$

for any $\mathbf{a} \in \mathfrak{g}$. However, this implies that $v \in D_{T_\pi}$ and $T_\pi(v) = L$, and hence the lemma. □

We will now bring this into connection with the *adjoint* representation $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ for $g \in G$, satisfying $\exp(\mathrm{Ad}_g(\mathbf{a})) = g \exp(\mathbf{a}) g^{-1}$ for $g \in G$ and $\mathbf{a} \in \mathfrak{g}$.

Lemma 7.16 (Chain rule). *Let π be a unitary representation of G. Then*

$$T_\pi(\pi_g v) = D\pi_g(T_\pi(v))$$

for every C^1-smooth vector $v \in \mathcal{H}_\pi$, where $D\pi$ is the continuous representation defined by $D\pi_g(L) = \pi_g \circ L \circ \mathrm{Ad}_{g^{-1}}$ for any linear map $L \colon \mathfrak{g} \to \mathcal{H}_\pi$. In other words, for a C^1-smooth vector $v \in \mathcal{H}_\pi$, $\mathbf{a} \in \mathfrak{g}$, and $g \in G$, we have

$$\pi_g \pi_\partial(\mathbf{a})v = \pi_\partial(\mathrm{Ad}_g \mathbf{a})\pi_g v \tag{7.6}$$

In particular, the vector space of C^r-smooth vectors is invariant under π_g for every $g \in G$ and $r \geqslant 1$.

Let us point out that Lemma 7.16 is indeed just a version of the chain rule adapted for unitary representations. Given a smooth vector $v \in \mathcal{H}_\pi$, Lie algebra element $\mathbf{a} \in \mathfrak{g}$, and group element $g \in G$, we wish to calculate the derivative $\pi_\partial(\mathrm{Ad}_g\,\mathbf{a})\pi_g v$ of $\pi_g v$ in the direction $\mathrm{Ad}_g\,\mathbf{a}$. Knowing that $\pi_\partial(\mathbf{a})v$ exists, we will use the commutative diagram

$$
\begin{array}{ccc}
v & \xrightarrow{\ \pi(\exp(t\mathbf{a}))\ } & \pi_{\exp(t\mathbf{a})}v \\
\big\downarrow{\scriptstyle \pi_g} & & \big\downarrow{\scriptstyle \pi_g} \\
\pi_g v & \xrightarrow{\ \pi\left(g\exp(t\mathbf{a})g^{-1}\right)\ } & \pi_{\exp(t\,\mathrm{Ad}_g\,\mathbf{a})}\pi_g v = \pi_{g\exp(t\mathbf{a})}v
\end{array}
$$

We note that continuity of the representation is defined as in Definition 1.1(3) but that we did not claim unitarity of the representation $D\pi$ (see Section 7.1.3).

Proof of Lemma 7.16. Let $g \in G$, $\mathbf{a} \in \mathfrak{g}$, and let $v \in \mathcal{H}_\pi$ be a C^1-smooth vector. Using the fact that π_g is bounded we have

$$
\begin{aligned}
\pi_\partial(\mathbf{a})\pi_g v &= \frac{\partial}{\partial t}\Big|_{t=0} \pi\left(\exp(t\mathbf{a})g\right)v \\
&= \lim_{t\to 0}\frac{1}{t}\pi_g\left(\pi\left(\exp(t\,\mathrm{Ad}_g^{-1}\,\mathbf{a})\right)v - v\right) = \pi_g\pi_\partial(\mathrm{Ad}_g^{-1}\,\mathbf{a})v.
\end{aligned}
$$

As this holds for any $\mathbf{a} \in \mathfrak{g}$ we see that $T_\pi \circ \pi_g = D\pi_g \circ T_\pi$ (where defined). The formulation in (7.6) is obtained by replacing $\mathbf{a}$ with $\mathrm{Ad}_g\,\mathbf{a}$.

To see that $D\pi$ defines a representation on $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$ let $g, h \in G$ and let $L \in \mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$, and calculate

$$
\begin{aligned}
D\pi_g\left(D\pi_h(L)\right) &= \pi_g \circ D\pi_h(L) \circ \mathrm{Ad}_{g^{-1}} \\
&= \pi_g \circ \pi_h \circ L \circ \mathrm{Ad}_{h^{-1}} \circ \mathrm{Ad}_{g^{-1}} = D\pi_{gh}(L).
\end{aligned}
$$

Moreover, notice that $D\pi_e(L) = L$.

As noted after Definition 7.14, we make the identification of $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$ with $\mathcal{H}_\pi^{\dim\mathfrak{g}}$ using a fixed basis of $\mathfrak{g} = \mathrm{Lie}\,G$. This identification gives the vector space $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$ the structure of a Hilbert space. With this, we also have

$$\begin{aligned}\big\|D\pi_g(L)\big\|^2 &= \sum_{j=1}^{\dim\mathfrak{g}} \|\pi_g\big(L(\mathrm{Ad}_g^{-1}\mathbf{b}_j)\big)\|^2 = \sum_{j=1}^{\dim\mathfrak{g}} \|L(\mathrm{Ad}_g^{-1}\mathbf{b}_j)\|^2 \\ &\leqslant \dim\mathfrak{g} \max_{j=1,\dots,\dim\mathfrak{g}} \left\| \sum_{k=1}^{\dim\mathfrak{g}} \left[\mathrm{Ad}_g^{-1}\right]_{kj} (L(\mathbf{b}_k)) \right\|^2 \\ &\leqslant (\dim\mathfrak{g})^3 \max_{j,k=1,\dots,\dim\mathfrak{g}} \left[\mathrm{Ad}_g^{-1}\right]_{kj}^2 \|(L(\mathbf{b}_k))\|^2 \ll_g \|L\|^2.\end{aligned}$$

In other words, $D\pi_g$ is a bounded operator on $\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)$, where $\left[\mathrm{Ad}_g^{-1}\right]_{kj}$ denotes the matrix entry of the matrix representing the linear map

$$\mathrm{Ad}_g^{-1}\colon \mathfrak{g} \longrightarrow \mathfrak{g}$$

in the basis $\mathbf{b}_1,\dots,\mathbf{b}_{\dim\mathfrak{g}}$. To see the continuity of the representation $D\pi$, let $L \in \mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)$, fix some $j \in \{1,\dots,\dim\mathfrak{g}\}$, and suppose (g_n) is a sequence in G with $g_n \to g$ as $n \to \infty$. Then

$$\begin{aligned}D\pi_{g_n}(L)(\mathbf{b}_j) = \pi_{g_n}\left(L\left(\mathrm{Ad}_{g_n}^{-1}\mathbf{b}_j\right)\right) &= \sum_{k=1}^{\dim\mathfrak{g}} \left[\mathrm{Ad}_{g_n}^{-1}\right]_{kj} \pi_{g_n}(L(\mathbf{b}_k)) \\ \longrightarrow \sum_{k=1}^{\dim\mathfrak{g}} \left[\mathrm{Ad}_g^{-1}\right]_{kj} \pi_g(L(\mathbf{b}_k)) &= \pi_g\left(L\,\mathrm{Ad}_g^{-1}\mathbf{b}_j\right) = D\pi_g(L)(\mathbf{b}_j)\end{aligned}$$

as $n \to \infty$, This gives $D\pi_{g_n}(L) \to D\pi_g(L)$ as $n \to \infty$ as required.

The final statement follows from the argument above and induction on the degree of smoothness $r \geqslant 1$. □

We finish this subsection with an interesting exercise, which requires the following definition.

Definition 7.17 (Adjoint operator). Let T be a densely defined closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. The adjoint operator T^* is defined on the domain

$$D_{T^*} = \left\{ w \in \mathcal{H}_2 \mid D_T \ni v \longmapsto \langle Tv, w\rangle_{\mathcal{H}_2} \text{ is bounded} \right\}$$

and satisfies

$$\langle Tv, w\rangle_{\mathcal{H}_2} = \langle v, T^*w\rangle_{\mathcal{H}_1}$$

for all $v \in D_T$ and $w \in D_{T^*}$.

We refer to [21, Lemma 13.3] for the properties of the adjoint operator.

Exercise 7.18. Show that $\pi_\partial(\mathbf{a})^* = -\pi_\partial(\mathbf{a})$ for any unitary representation π of G and element $\mathbf{a} \in \mathfrak{g}$.

7.1.3 Unitarity of the Derivative Representation

In this section we prove the following proposition which gives unitarity of the total derivative in some interesting cases.

Proposition 7.19 (Unitarity of D and Equivariance of $T_\pi^* T_\pi$). *Let G be a Lie group with Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$. Suppose that $\mathfrak{g}$ is equipped with an inner product with the property that Ad_g is orthogonal for any $g \in G$, and let π be a unitary representation of G. Use an orthonormal basis of $\mathfrak{g}$ to define the isomorphism $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi) = \mathcal{H}_\pi^{\dim \mathfrak{g}}$ and hence a Hilbert space structure on $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$. Then the derivative representation $D\pi$ on $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$ is unitary, and hence $T_\pi^* T_\pi$ is a densely defined closed equivariant operator from $\mathcal{H}_\pi$ to $\mathcal{H}_\pi$. If $\mathbf{b}_1, \dots, \mathbf{b}_{\dim G}$ is an orthonormal basis of $\mathfrak{g}$ and $v \in \mathcal{H}_\pi$ is C^2-smooth, then*

$$T_\pi^* T_\pi v = -\sum_{j=1}^{\dim G} \pi_\partial(\mathbf{b}_j)^2 v = \Omega v. \tag{7.7}$$

If π is irreducible, then there exists some $\alpha_\pi \geqslant 0$ with $\Omega v = \alpha_\pi v$ for all v in $\mathcal{H}_\pi$.

Notice that the Hilbert space structure of $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$ and the representation D becomes clearer after noting that $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi) \cong \mathcal{H}_\pi \otimes_{\mathbb{R}} \mathfrak{g}$ and that the latter carries a unitary representation since $\mathcal{H}_\pi$ carries a unitary representation and the real Hilbert space $\mathfrak{g}$ carries a natural representation of G that is assumed to be 'orthogonal'.

Allowing ourselves to consider formal products of Lie algebra elements (giving elements of the so-called universal enveloping algebra; see Section 9.1), we may write the differential operator Ω on $\mathcal{H}_\pi$ also as $-\pi_\partial\left(\sum_{j=1}^{\dim G} \mathbf{b}_j \circ \mathbf{b}_j\right)$.

Proof of Proposition 7.19. Let $\mathfrak{g} = \mathrm{Lie}\, G$ and π be as in the proposition and suppose that $\mathbf{b}_1, \dots, \mathbf{b}_{\dim G}$ is an orthonormal basis with respect to the assumed inner product on $\mathfrak{g}$. For $g \in G$ and $L \in \mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$ we have, by unitarity of π_g, that $\left\|D\pi_g L\right\|_{\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)}^2$ is equal to

$$\begin{aligned}
\sum_{j=1}^{\dim \mathfrak{g}} \left\|\pi_g L\big(\mathrm{Ad}_g^{-1} \mathbf{b}_j\big)\right\|_{\mathcal{H}_\pi}^2 &= \sum_{j=1}^{\dim \mathfrak{g}} \left\|L\big(\mathrm{Ad}_g^{-1} \mathbf{b}_j\big)\right\|_{\mathcal{H}_\pi}^2 \\
&= \sum_{j=1}^{\dim \mathfrak{g}} \left\|\sum_{k=1}^{\dim \mathfrak{g}} \big[\mathrm{Ad}_g^{-1}\big]_{kj} L(\mathbf{b}_k)\right\|_{\mathcal{H}_\pi}^2 \\
&= \sum_{j=1}^{\dim \mathfrak{g}} \sum_{k,\ell=1}^{\dim \mathfrak{g}} \big[\mathrm{Ad}_g^{-1}\big]_{kj} \big[\mathrm{Ad}_g^{-1}\big]_{\ell j} \left\langle L(\mathbf{b}_k), L(\mathbf{b}_\ell)\right\rangle_{\mathcal{H}_\pi},
\end{aligned}$$

where $[\mathrm{Ad}_g^{-1}]_{kj}$ again denotes the entries of the matrix representation of Ad_g^{-1} in the basis $\mathbf{b}_1, \ldots, \mathbf{b}_{\dim \mathfrak{g}}$. Reordering the summation in the last expression above we obtain

$$\begin{aligned}\left\|D\pi_g L\right\|^2_{\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)} &= \sum_{k,\ell=1}^{\dim \mathfrak{g}} \underbrace{\sum_{j=1}^{\dim \mathfrak{g}} \left[\mathrm{Ad}_g^{-1}\right]_{kj} \left[\mathrm{Ad}_g^{-1}\right]_{\ell j}}_{\delta_{k,\ell}} \langle L(\mathbf{b}_k), L(\mathbf{b}_\ell)\rangle_{\mathcal{H}_\pi} \\ &= \sum_{k=1}^{\dim \mathfrak{g}} \|L(\mathbf{b}_k)\|^2_{\mathcal{H}_\pi} = \|L\|_{\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)}\end{aligned}$$

as required.

This implies that $T_\pi^* T_\pi$ is equivariant where it is defined. To see this, we first recall that for $g \in G$ we have

$$T_\pi \pi_g = D\pi_g \circ T_\pi$$

by Lemma 7.16. Suppose now that v is in the domain of $T_\pi^* T_\pi$ and $g \in G$. Then, for all w in the domain of T_π, we have

$$\begin{aligned}\langle \pi_g T_\pi^* T_\pi v, w\rangle_{\mathcal{H}_\pi} = \langle T_\pi^* T_\pi v, \pi_g^{-1} w\rangle_{\mathcal{H}_\pi} &= \langle T_\pi v, T_\pi \pi_{g^{-1}} w\rangle_{\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)} \\ &= \langle T_\pi v, D_{\pi_{g^{-1}}} T_\pi w\rangle_{\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)} \\ &= \langle D_{\pi_g} T_\pi v, T_\pi w\rangle_{\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)} \\ &= \langle T_\pi \pi_g v, T_\pi w\rangle_{\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)}.\end{aligned}$$

However, this, by definition of the adjoint, implies that $T_\pi \pi_g v$ belongs to the domain of T_π^* and that

$$T_\pi^* T_\pi \pi_g v = \pi_g T_\pi^* T_\pi v.$$

This shows that the domain of $T_\pi^* T_\pi$ is invariant under π_g, and that

$$\pi_g T_\pi^* T_\pi \supseteq T_\pi^* T_\pi \pi_g$$

for all $g \in G$. Applying this for g^{-1} together with the invariance of the domain of $T_\pi^* T_\pi$, we actually obtain

$$\pi_g T_\pi^* T_\pi = T_\pi^* T_\pi \pi_g$$

for $g \in G$, as required. For the proof that $T_\pi^* T_\pi$ is densely defined and closed, we refer to [21, Th. 13.10] (see also Section 1.3.4 for similar arguments).

Now let $w_1, \ldots, w_{\dim \mathfrak{g}} \in \mathcal{H}_\pi$ be C^1-smooth vectors. We claim that the linear map L defined by

$$L\left(\sum_{j=1}^{\dim\mathfrak{g}} s_j\mathbf{b}_j\right)=\sum_{j=1}^{\dim\mathfrak{g}} s_j w_j$$

belongs to the domain $D_{T_\pi^*}$ of T_π^*, and

$$T_\pi^*(L)=-\sum_{j=1}^{\dim\mathfrak{g}}\pi_\partial(\mathbf{b}_j)w_j.$$

For this, let $v\in\mathcal{H}_\pi$ be C^1-smooth (that is, in the domain of T_π) and calculate

$$\begin{aligned}\left\langle-\sum_{j=1}^{\dim\mathfrak{g}}\pi_\partial(\mathbf{b}_j)w_j,v\right\rangle_{\mathcal{H}_\pi}&=-\sum_{j=1}^{\dim\mathfrak{g}}\left\langle\pi_\partial(\mathbf{b}_j)w_j,v\right\rangle_{\mathcal{H}_\pi}\\&=-\sum_{j=1}^{\dim\mathfrak{g}}\lim_{t\to0}\frac{1}{t}\left\langle\pi(\exp(t\mathbf{b}_j))w_j-w_j,v\right\rangle_{\mathcal{H}_\pi}\\&=-\sum_{j=1}^{\dim\mathfrak{g}}\lim_{t\to0}\frac{1}{t}\left\langle w_j,\pi(\exp(-t\mathbf{b}_j))v-v\right\rangle_{\mathcal{H}_\pi}\\&=\sum_{j=1}^{\dim\mathfrak{g}}\left\langle w_j,\pi_\partial(\mathbf{b}_j)v\right\rangle_{\mathcal{H}_\pi}=\langle L,T_\pi v\rangle_{\mathrm{End}(\mathfrak{g},\mathcal{H}_\pi)}.\end{aligned}$$

As $v\in D_{T_\pi}$ was arbitrary, this gives the claim and the claim implies (7.7). The final claim in the proposition follows from Schur's lemma (Corollary 1.36). □

7.1.4 The Casimir Operator for $\mathbf{SU_2}(\mathbb{R})$

We wish to study an example of Proposition 7.19 and explicitly calculate the constants α_π for all irreducible representations of $G=\mathrm{SU}_2(\mathbb{R})$ (which we already classified in Section 6.2). For this we will also use the basis $\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\in\mathrm{SU}_2(\mathbb{R})$ in (6.6). Moreover, we will also consider formal products $\mathbf{a}\circ\mathbf{b}\in\mathfrak{E}$, squares $\mathbf{a}^{\circ2}=\mathbf{a}\circ\mathbf{a}\in\mathfrak{E}$ in the so-called universal enveloping algebra $\mathfrak{E}$ of $\mathfrak{su}_2(\mathbb{R})$, the formal identity $\mathbb{1}_{\mathfrak{E}}$ of $\mathfrak{E}$, and the rules

$$\begin{cases}\pi_\partial(\mathbf{a}\circ\mathbf{b})=\pi_\partial(\mathbf{a})\pi_\partial(\mathbf{b}),\\ \quad\pi_\partial(\mathbb{1}_{\mathfrak{E}})=I\end{cases}$$

for all Lie algebra elements $\mathbf{a},\mathbf{b}\in\mathfrak{su}_2(\mathbb{R})$ (see also Section 9.1).

Corollary 7.20 (Casimir operator on $\mathrm{Sym}^n(\mathbb{C}^2)$). *For every $n\in\mathbb{N}_0$ the so-called*[(11)] Casimir element $\Omega=\mathbb{1}_{\mathfrak{E}}-(\mathbf{b}_1\circ\mathbf{b}_1+\mathbf{b}_2\circ\mathbf{b}_2+\mathbf{b}_3\circ\mathbf{b}_3)$ *acts* *on* $\mathrm{Sym}^n(\mathbb{C}^2)$ *by differentiation, and equals the scalar multiplication*

$$\pi_\partial(\Omega) = I - \pi_\partial(\mathbf{b}_1^{\circ 2} + \mathbf{b}_2^{\circ 2} + \mathbf{b}_3^{\circ 2}) = (n+1)^2 I$$

by the square of the dimension of $\mathrm{Sym}^n(\mathbb{C}^2)$.

We note that we added $\mathbb{1}_{\mathfrak{E}}$ in the definition of Ω to make the conclusion of the corollary easier to remember.

Proof of Corollary 7.20. We first note that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathfrak{su}_2(\mathbb{R})$ as defined in (6.6) form an orthonormal basis for the inner product defined by the quadratic form det. Moreover, as discussed in Section 6.2.2, $\mathrm{SU}_2(\mathbb{R})$ acts via the adjoint representation by orthogonal matrices on $\mathfrak{su}_2(\mathbb{R})$ with respect to this inner product. Thus $G = \mathrm{SU}_2(\mathbb{R})$ and $\mathfrak{su}_2(\mathbb{R})$ equipped with this inner product satisfy the assumptions in Proposition 7.19.

Let $n \in \mathbb{N}_0$. As the representation π on $\mathrm{Sym}^n(\mathbb{C}^2)$ is an irreducible representation of $\mathrm{SU}_2(\mathbb{R})$ by Theorem 6.6, we obtain from Proposition 7.19 that

$$-\pi_\partial(\mathbf{b}_1^{\circ 2} + \mathbf{b}_2^{\circ 2} + \mathbf{b}_3^{\circ 2}) = \alpha_n I$$

for some $\alpha_n \geqslant 0$.

To calculate α_n, we use the basis vectors $e_1^{\odot n} \in \mathrm{Sym}^n(\mathbb{C}^2)$. For $t \in \mathbb{R}$ we have

$$\exp(t\mathbf{b}_1) = \exp\begin{pmatrix} \mathrm{i}t & \\ & -\mathrm{i}t \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}t} & \\ & \mathrm{e}^{-\mathrm{i}t} \end{pmatrix}$$

and

$$\pi(\exp t\mathbf{b}_1)e_1^{\odot n} = \mathrm{e}^{\mathrm{i}nt} e_1^{\odot n},$$

which implies that

$$\pi_\partial(\mathbf{b}_1)e_1^{\odot n} = \mathrm{i}n e_1^{\odot n}$$

and

$$\pi_\partial(\mathbf{b}_1^{\circ 2})e_1^{\odot n} = -n^2 e_1^{\odot n}. \tag{7.8}$$

For $\mathbf{b}_2, \mathbf{b}_3$ we similarly have

$$\begin{aligned} \exp(t\mathbf{b}_2) &= \exp\begin{pmatrix} & \mathrm{i}t \\ \mathrm{i}t & \end{pmatrix} = \begin{pmatrix} \cos t & \mathrm{i}\sin t \\ \mathrm{i}\sin t & \cos t \end{pmatrix}, \\ \exp(t\mathbf{b}_3) &= \exp\begin{pmatrix} & -t \\ t & \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \\ \pi(\exp t\mathbf{b}_2)e_1^{\odot n} &= (\cos t e_1 + \mathrm{i}\sin t e_2)^{\odot n}, \end{aligned}$$

and

$$\pi(\exp t\mathbf{b}_3)e_1^{\odot n} = (\cos t e_1 + \sin t e_2)^{\odot n}.$$

Expanding the latter expressions using the binomial theorem, we can take the derivative with respect to t at $t = 0$ and notice that only one term is relevant to obtain

$$\pi_\partial(\mathbf{b}_2)e_1^{\odot n} = \mathrm{i}ne_1^{\odot(n-1)} \odot e_2$$

and

$$\pi_\partial(\mathbf{b}_3)e_1^{\odot n} = ne_1^{\odot(n-1)} \odot e_2.$$

We repeat this step and obtain

$$\begin{aligned}
\pi(\exp t\mathbf{b}_2)\pi_\partial(\mathbf{b}_2)e_1^{\odot n} &= \mathrm{i}n(\cos te_1 + \mathrm{i}\sin te_2)^{\odot(n-1)} \odot (\mathrm{i}\sin te_1 + \cos te_2)\\
&= \mathrm{i}n\big(\cos^{n-1} te_1^{\odot(n-1)} + \mathrm{i}(n-1)\cos^{n-2} t\sin te_1^{\odot(n-2)} \odot e_2 + \cdots\big)\\
&\qquad\qquad \odot (\mathrm{i}\sin te_1 + \cos te_2)\\
&= \mathrm{i}n\big(\mathrm{i}\sin t\cos^{n-1} te_1^{\odot n} + \mathrm{i}(n-1)\sin t\cos^{n-1} te_1^{\odot(n-2)} \odot e_2^{\odot 2} + \cdots\big)
\end{aligned}$$

and

$$\begin{aligned}
\pi(\exp t\mathbf{b}_3)\pi_\partial(\mathbf{b}_3)e_1^{\odot n} &= n\big(\cos te_1 + \sin te_2\big)^{\odot(n-1)} \odot \big(-\sin te_1 + \cos te_2\big)\\
&= n\big(\cos^{n-1} te_1^{\odot(n-1)} + (n-1)\cos^{n-2} t\sin te_1^{\odot(n-2)} \odot e_2 + \cdots\big)\\
&\qquad\qquad \odot \big(-\sin te_1 + \cos te_2\big)\\
&= n\big(-\sin t\cos^{n-1} te_1^{\odot n} + (n-1)\sin t\cos^{n-1} te_1^{\odot(n-2)} \odot e_2^{\odot 2} + \cdots\big),
\end{aligned}$$

which implies that

$$\pi_\partial(\mathbf{b}_2^{\circ 2})e_1^{\odot n} = -ne_1^{\odot n} - n(n-1)e_1^{\odot(n-2)} \odot e_2^{\odot 2}$$

and

$$\pi_\partial(\mathbf{b}_3^{\circ 2})e_1^{\odot n} = -ne_1^{\odot n} + n(n-1)e_1^{\odot(n-2)} \odot e_2^{\odot 2}.$$

Together with (7.8) this gives

$$-\pi_\partial(\mathbf{b}_1^{\circ 2} + \mathbf{b}_2^{\circ 2} + \mathbf{b}_3^{\circ 2})e_1^{\odot n} = (n^2 + 2n)e_1^{\odot n}.$$

Adding $e_1^{\odot n}$ to this, the corollary follows. □

7.2 Effective Decay, Definitions, and First Results

In the following we will be interested in concrete examples of closed linear groups. By a *closed linear group* we mean a closed subgroup G of $\mathrm{SL}_d(\mathbb{R})$ for some $d \geqslant 1$. The assumption that $G < \mathrm{SL}_d(\mathbb{R})$ instead of the seemingly more general $G < \mathrm{GL}_d(\mathbb{R})$ is harmless, as we can consider $\mathrm{GL}_d(\mathbb{R})$ itself as a closed subgroup of $\mathrm{SL}_{d+1}(\mathbb{R})$. One reason for the assumption is that it gives the following notion of norm on G more meaning. We fix the Hilbert–Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ on $\mathrm{Mat}_{d,d}(\mathbb{R})$ and will write $|g| = \|g\|_{\mathrm{HS}}$ for elements $g \in G$ of the closed linear group $G < \mathrm{SL}_d(\mathbb{R}) \subseteq \mathrm{Mat}_{d,d}(\mathbb{R})$. We also note that for the purposes of establishing effective decay, the notion of degree r Sobolev norms for unitary representations from Definition 7.6 will be important.

Definition 7.21 (Effective decay of matrix coefficients). Let G be a closed linear group with Lie algebra $\mathfrak{g}$, let π be a unitary representation of G, let $r \geqslant 0$, denote the degree r Sobolev norm on C^r-smooth vectors in $\mathcal{H}_\pi$ by $\mathcal{S}(\cdot)$, and write

$$\mathcal{H}_\pi^G = \{v \in \mathcal{H}_\pi \mid \pi_g v = v \text{ for all } g \in G\}$$

for the subspace of fixed vectors. We say that π has *effective decay of matrix coefficients* if there exists some $\kappa > 0$ such that

$$\left|\langle \pi_g v, w\rangle\right| \ll \|g\|^{-\kappa}\, \mathcal{S}(v)\, \mathcal{S}(w)$$

for all C^r-smooth $v, w \in (\mathcal{H}_\pi^G)^\perp$ and for all $g \in G$, where the implicit constant is allowed to depend on π, κ, and $r \geqslant 0$. We will call κ a *decay exponent*, and define the *almost decay exponent* κ_π of the unitary representation by

$$\kappa_\pi = \sup\bigl(\{\kappa > 0 \mid \kappa \text{ is a decay exponent}\} \cup \{0\}\bigr).$$

For semi-simple groups effective decay of matrix coefficients as defined above gives a formulation of effectiveness of the Howe–Moore theorem (see Section 1.7 and Theorem 1.80). Our aim is to show that many natural actions have this property, and we will give an example of a decay exponent in Section 7.3.

We emphasize that the above notions as defined depend on the fact that we consider closed subgroups $G \leqslant \mathrm{SL}_d(\mathbb{R})$. The reader troubled by this may fix a Riemannian metric on an abstract Lie group G and use instead of the norm $\|g\|$ for $g \in G$ the expression $\mathrm{e}^{-\mathsf{d}(g,e)}$ for $g \in G$, and use this to define a notion of *exponential decay of matrix coefficients*. We note, however, that this notion will now depend on the choice of the Riemannian metric (instead of the particular embedding). We have chosen the terminology above as it is much easier to generalize Definition 7.21 in its formulation for closed linear subgroups of $\mathrm{SL}_d(\mathbb{Q}_p)$ or $\mathrm{SL}_d\bigl(\mathbb{F}_p(\!(t)\!)\bigr)$ for a prime p. We also note that for the simply connected Lie group $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ with infinite centre (any kind of) decay of matrix coefficients cannot hold for irreducible representations due to Corollary 1.30. Finally, we will only be interested in $\mathrm{SL}_d(\mathbb{R})$ for $d \in \{2, 3\}$, and here there should be no doubt that the Hilbert–Schmidt norm $\|\cdot\|$ is a meaningful measuring tool for the size of the group elements.

7.2.1 Relationship to Spectral Gap

We will show here that effective decay of matrix coefficients implies spectral gap (see Section 4.2.1). We will also see in Section 7.3 that $\mathrm{SL}_3(\mathbb{R})$ satisfies the following stronger property.

Definition 7.22 (Uniform decay exponent). We say that a closed linear group G has a *uniform decay exponent* $\kappa > 0$ if κ is a decay exponent for any unitary representation of G, and both the Sobolev degree r and the implicit constant in Definition 7.21 can be chosen absolute.

Proposition 7.23 (Effective decay implies spectral gap). *A unitary representation of a closed linear group with effective decay of matrix coefficients has spectral gap. Moreover, any closed linear group G for which there exists a uniform decay exponent $\kappa > 0$ has property* (T).

Proof. Let G be a closed linear group, let $\kappa > 0$ be a decay exponent for a unitary representation π of G, and let $\mathcal{S}$ be the Sobolev norm as in Definition 7.21. Also let $\psi \in C_c^\infty(G)$ satisfy $\|\psi\|_1 = 1$ and $\psi \geqslant 0$. We define

$$A = \pi_*(\psi)^* \pi_g \pi_*(\psi)$$

for some $g \in G$ to be determined.

For $v, w \in \mathcal{H}_\pi$ we apply Exercise 7.8 to see that $\mathcal{S}\bigl(\pi_*(\psi)v\bigr) \ll_\psi \|v\|$ and $\mathcal{S}\bigl(\pi_*(\psi)w\bigr) \ll_\psi \|w\|$. By assumption, we also have

$$\bigl|\bigl\langle Av, w\bigr\rangle\bigr| = \bigl|\bigl\langle \pi_g \pi_*(\psi)v, \pi_*(\psi)w\bigr\rangle\bigr| \ll_\psi |g|^{-\kappa}\,\|v\|\|w\|$$

for all $v, w \in \bigl(\mathcal{H}_\pi^G\bigr)^\perp$. We now choose $g \in G$ sufficiently large to ensure that[†]

$$\bigl|\bigl\langle Av, w\bigr\rangle\bigr| \leqslant \tfrac{1}{2}\|v\|\|w\|$$

for all $v, w \in \bigl(\mathcal{H}_\pi^G\bigr)^\perp$, or, equivalently, so that

$$\|A\|_{(\mathcal{H}_\pi^G)^\perp} \leqslant \tfrac{1}{2}.$$

Now recall that by Section 1.4.3 — and (1.14) in particular — we have

$$A = \pi_*(\psi^*)\pi_g\pi_*(\psi) = \pi_*\bigl(\psi^* * \lambda_g \psi\bigr).$$

Thus Proposition 4.23 implies that π has spectral gap.

If G has a uniform decay exponent, then any unitary representation has spectral gap by the above. However, this implies that G has property (T). □

The following exercise shows that property (T) and possessing a positive uniform decay exponents are *not* equivalent in general.

[†] Note that if G is compact, then G has property (T) by Exercise 4.18 in any case.

Exercise 7.24. For the purpose of this exercise, use the fact that $\mathrm{SL}_3(\mathbb{Z})$ has property (T), which follows from the fact that $\mathrm{SL}_3(\mathbb{R})$ has property (T) (see Theorem 7.26), and the fact that $\mathrm{SL}_3(\mathbb{Z})$ is a lattice in $\mathrm{SL}_3(\mathbb{R})$ (see [21, Sec. 10.3]). Show that the natural action of $\mathrm{SL}_3(\mathbb{Z})$ on $\mathbb{T}^3 \cong \mathbb{R}^3/\mathbb{Z}^3$ is ergodic, and that it does not have (effective) decay of matrix coefficients.

7.2.2 Eigenvectors of $\mathrm{SO}_2(\mathbb{R})$ for Representations of $\mathrm{SL}_2(\mathbb{R})$

We define the subgroups

$$A = \left\{ a_t = \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\} < \mathrm{SL}_2(\mathbb{R})$$

and

$$K = \mathrm{SO}_2(\mathbb{R}) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} < \mathrm{SL}_2(\mathbb{R}),$$

and will frequently use the following terminology.

For a unitary representation π of $\mathrm{SL}_2(\mathbb{R})$, we say that a vector $v \in \mathcal{H}_\pi$ is a K-*eigenvector* if there exists some $n \in \mathbb{Z}$ with

$$\pi_{k_\theta} v = \mathrm{e}^{\mathrm{i} n\theta} v$$

for all $k_\theta \in K$. We will also refer to n as the *weight* of the K-eigenvector. Moreover, in the cases where $\mathcal{H}_\pi$ is clearly a space of functions, we will also call any K-eigenvector a K-eigenfunction.

Using 'Fourier series' we now show that for establishing effective decay of matrix coefficients for $\mathrm{SL}_2(\mathbb{R})$, it suffices to study K-eigenvectors. We will use this observation repeatedly.

Proposition 7.25 (Upgrade to smooth vectors for $\mathrm{SL}_2(\mathbb{R})$). *Let π be a unitary representation of $\mathrm{SL}_2(\mathbb{R})$, $c > 0$, and $\kappa > 0$ so that*

$$\left|\langle \pi_{a_t} v, w\rangle\right| \leqslant c\mathrm{e}^{-\kappa t}\|v\|\|w\| \tag{7.9}$$

for all $t \in \mathbb{R}$ and all K-eigenvectors $v, w \in \mathcal{H}_\pi$. Suppose also that $B \in \mathrm{B}(\mathcal{H}_\pi)$ is a bounded operator that commutes with π_k for all $k \in \mathrm{SO}_2(\mathbb{R})$. Then we have

$$\left|\langle \pi_g B v, w\rangle\right| \ll c\|B\|_{\mathrm{op}}\, |g|^{-\kappa}\, \mathcal{S}(v)\, \mathcal{S}(w)$$

for all $g \in \mathrm{SL}_2(\mathbb{R})$ and all C^1-smooth vectors $v, w \in \mathcal{H}_\pi$, where the implicit constant is absolute.

Proof. If $v \in \mathcal{H}_\pi$ is C^1-smooth, then the decomposition of $v = \sum_{m\in\mathbb{Z}} v_m$ into K-eigenvectors not only converges in $\mathcal{H}_\pi$ (which it always does) but in fact converges absolutely. To see this, let $\mathbf{w} \in \mathfrak{sl}_2(\mathbb{R})$ denote the element in the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ corresponding to $\mathrm{SO}_2(\mathbb{R})$ such that

$$\pi_\partial(\mathbf{w})v_m = \lim_{\theta\to 0}\frac{\pi_{k_\theta}v_m - v_m}{\theta} = \lim_{\theta\to 0}\frac{\mathrm{e}^{\mathrm{i}m\theta}-1}{\theta}v_m = \mathrm{i}mv_m$$

and

$$\pi_\partial(\mathbf{w})v = \sum_{m\in\mathbb{Z}}\mathrm{i}mv_m$$

by Example 7.2. Since $\mathbf{w} \in \mathfrak{sl}_2(\mathbb{R})$ can be expressed as a sum of basis vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ in $\mathfrak{sl}_2(\mathbb{R})$, the same holds for $\pi_\partial(\mathbf{w})v$, $\pi_\partial(\mathbf{b}_1)v$, $\pi_\partial(\mathbf{b}_2)v$, and $\pi_\partial(\mathbf{b}_3)v$ (see Lemma 7.3). Therefore,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} m^2\|v_m\|^2 = \|\pi_\partial(\mathbf{w})v\|^2 &\ll \big(\|\pi_\partial(\mathbf{b}_1)v\| + \|\pi_\partial(\mathbf{b}_2)v\| + \|\pi_\partial(\mathbf{b}_3)v\|\big)^2\\ &\ll \mathcal{S}(v)^2 < \infty\end{aligned}$$

by the triangle inequality, and

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\|v_m\| &= \|v_0\| + \sum_{m\in\mathbb{Z}\smallsetminus\{0\}}\tfrac{1}{m}m\|v_m\|\\ &\leqslant \|v_0\| + \Bigg(\sum_{m\in\mathbb{Z}\smallsetminus\{0\}}\tfrac{1}{m^2}\Bigg)^{\frac12}\Bigg(\sum_{m\in\mathbb{Z}\smallsetminus\{0\}}m^2\|v_m\|^2\Bigg)^{\frac12} \ll \mathcal{S}(v)\end{aligned}$$

by the Cauchy–Schwarz inequality in $\ell^2(\mathbb{Z}\smallsetminus\{0\})$ and the fact that the sequence $\big(\frac{1}{m^2}\big)_{m\in\mathbb{N}}$ is summable.

Also note that B and π_{k_θ} for $k_\theta \in K$ map any K-eigenvector to a K-eigenvector with the same weight.

With this, we can now finish the proof using the Cartan decomposition

$$g = k_\theta a_t k_{\theta'}$$

of $g \in \mathrm{SL}_2(\mathbb{R})$ with $t \geqslant 0$ and $k_\theta, k_{\theta'} \in K$ satisfying $|g| = |a_t| \asymp \mathrm{e}^t$. Indeed, we obtain from our assumption applied separately to each summand below that

$$\begin{aligned}\big|\langle\pi_g Bv, w\rangle\big| &= \Big|\sum_{m,n}\langle\pi_{a_t}\pi_{k_{\theta'}}Bv_m, \pi_{k_\theta}^* w_n\rangle\Big|\\ &\ll c\,\underbrace{\mathrm{e}^{-\kappa t}}_{\ll |g|^{-\kappa}}\sum_{m,n}\underbrace{\|Bv_m\|}_{\leqslant\|B\|_{\mathrm{op}}\|v_m\|}\;\|w_n\|\\ &\ll c\|B\|_{\mathrm{op}}\,|g|^{-\kappa}\,\mathcal{S}(v)\,\mathcal{S}(w)\end{aligned}$$

as claimed. □

7.3 A Uniform Decay Exponent for $\mathrm{SL}_3(\mathbb{R})$*

We start with the following accessible case. We note that the decay exponent $\frac{3}{8}$ below is not optimal, and refer to work of Oh [54] for more general and sharp results.

Theorem 7.26 (Effective decay for $\mathrm{SL}_3(\mathbb{R})$). *The group $\mathrm{SL}_3(\mathbb{R})$ has a uniform decay exponent: If π is a unitary representation of $\mathrm{SL}_3(\mathbb{R})$ and the vectors $v, w \in \bigl(\mathcal{H}_\pi^{\mathrm{SL}_3(\mathbb{R})}\bigr)^\perp$ are C^1-smooth, then*

$$\bigl|\langle \pi_g v, w\rangle\bigr| \ll \lfloor g \rceil^{-\frac{3}{8}}\, \mathcal{S}(v)\, \mathcal{S}(w)$$

for all $g \in \mathrm{SL}_3(\mathbb{R})$, where the implicit constant is absolute and $\mathcal{S}$ is a degree-one Sobolev norm. In particular, $\mathrm{SL}_3(\mathbb{R})$ has property (T).

7.3.1 Eigenvectors of $\mathrm{SO}_2(\mathbb{R})$

We will use the terminology of Section 7.2.2 for the restriction of a unitary representation π of $\mathrm{SL}_3(\mathbb{R})$ to the subgroup

$$\mathrm{ASL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} \mid g \in \mathrm{SL}_2(\mathbb{R}), x \in \mathbb{R}^2 \right\} < \mathrm{SL}_3(\mathbb{R}),$$

containing

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} g & 0 \\ & 1 \end{pmatrix} \mid g \in \mathrm{SL}_2(\mathbb{R}) \right\}$$

and so also $K = \mathrm{SO}_2(\mathbb{R})$. We will also make use of the normal abelian closed subgroup

$$H \lhd \mathrm{ASL}_2(\mathbb{R})$$

defined by

$$H = \left\{ h_x = \begin{pmatrix} I & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{R}^2 \right\} \cong \mathbb{R}^2,$$

as well as the elements

$$a_t = \begin{pmatrix} \mathrm{e}^t & 0 & 0 \\ 0 & \mathrm{e}^{-t} & 0 \\ & & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \tag{7.10}$$

for $t \in \mathbb{R}$. Since $H \lhd \mathrm{ASL}_2(\mathbb{R})$, we may use the results in Section 3.1 for $\pi|_{\mathrm{ASL}_2(\mathbb{R})}$ for the proof of the following first step towards Theorem 7.26. For this, we first note that for $g \in \mathrm{SL}_2(\mathbb{R})$ we will denote the inner automorphism of $\mathrm{ASL}_2(\mathbb{R})$ defined by

$$\begin{pmatrix} g & 0 \\ & 1 \end{pmatrix} \in \mathrm{ASL}_2(\mathbb{R})$$

by θ_g, so

$$\theta_g(h_x) = \begin{pmatrix} g & 0 \\ & 1 \end{pmatrix} \begin{pmatrix} I & x \\ & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ & 1 \end{pmatrix} = \begin{pmatrix} I & gx \\ & 1 \end{pmatrix} = h_{gx}$$

for all $x \in \mathbb{R}^2$. In particular, the dual automorphism $\widehat{\theta}_g$ on $\widehat{H} \cong \mathbb{R}^2$ is given by the linear map defined by g^{t}, and the action of $g \in \mathrm{SL}_2(\mathbb{R})$ on this dual group $\mathbb{R}^2$ from Section 3.1 is defined by $(g^{\mathrm{t}})^{-1}$.

Lemma 7.27 (Eigenvectors). *Let π be as in Theorem 7.26, and suppose that $v, w \in \big(\mathcal{H}_\pi^{\mathrm{SL}_3(\mathbb{R})}\big)^\perp$ are K-eigenvectors with*

$$\mathrm{SO}_2(\mathbb{R}) = K < \mathrm{ASL}_2(\mathbb{R}) < \mathrm{SL}_3(\mathbb{R}).$$

Then the diagonal matrices in (7.10) satisfy

$$\big|\langle \pi_{a_t} v, w \rangle\big| \ll \mathrm{e}^{-\frac{|t|}{2}} \|v\| \|w\|$$

for all $t \in \mathbb{R}$, where the implicit constant is absolute.

Proof. Let $v \in \mathcal{H}$ be a K-eigenvector of weight $n \in \mathbb{Z}$, so that

$$\pi_{k_\theta} v = \mathrm{e}^{\mathrm{i} n \theta} v \tag{7.11}$$

for all $k_\theta \in \mathrm{SO}_2(\mathbb{R}) < \mathrm{ASL}_2(\mathbb{R})$. Now notice that (7.11) and sesqui-linearity of the inner product implies $\mu_{\pi_{k_\theta} v} = \mu_v$ (also see Proposition 2.50(2)). By Proposition 3.1, this implies that the spectral measure μ_v is invariant under the rotation k_θ for all $\theta \in \mathbb{R}$.

Since $v \in (\mathcal{H}_\pi^{\mathrm{SL}_3(\mathbb{R})})^\perp$, we have $\mu_v(\{0\}) = 0$ by Exercise 1.78 (see also the hint on p. 475) and the same holds for μ_w. We assume that $t > 0$ (switching the roles of v and w then gives the general case). We define the subsets

$$B_{\mathrm{horizontal}} = \{(x_1, x_2) \mid \big|\tfrac{x_2}{x_1}\big| \leqslant \mathrm{e}^{-t}\}$$

and

$$B_{\mathrm{vertical}} = \{(x_1, x_2) \mid \big|\tfrac{x_2}{x_1}\big| \geqslant \mathrm{e}^{t}\},$$

which are the two sectors in $\mathbb{R}^2$ illustrated in Figure 7.1.

Next we use these and the functional calculus for $\pi|_H$ to define the associated projection operator $\mathit{\Pi}_B$ for

$$B \in \{B_{\mathrm{horizontal}}, \mathbb{R}^2 \smallsetminus B_{\mathrm{horizontal}}, B_{\mathrm{vertical}}, \mathbb{R}^2 \smallsetminus B_{\mathrm{vertical}}\}$$

and split v and w into two parts according to these two sectors in $\mathbb{R}^2$,

$$v = v_{\mathrm{main}} + v_{\mathrm{horizontal}}$$

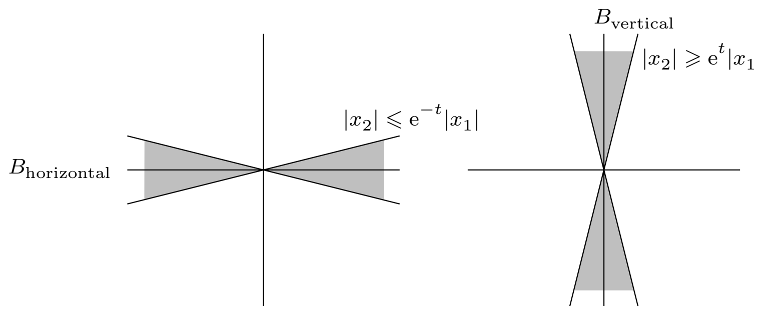

Fig. 7.1: The sets used for decomposing v and w.

with $v_{\mathrm{main}} = \mathit{\Pi}_{\mathbb{R}^2 \smallsetminus B_{\mathrm{horizontal}}}(v)$, $v_{\mathrm{horizontal}} = \mathit{\Pi}_{B_{\mathrm{horizontal}}}(v)$ and

$$w = w_{\mathrm{main}} + w_{\mathrm{vertical}},$$

with $w_{\mathrm{main}} = \mathit{\Pi}_{\mathbb{R}^2 \smallsetminus B_{\mathrm{vertical}}}(w)$ and $w_{\mathrm{vertical}} = \mathit{\Pi}_{B_{\mathrm{vertical}}}(w)$.

We now use the fact that μ_v is invariant under rotation, $\mu_v(\{0\}) = 0$, and $B_{\mathrm{horizontal}}$ consists of two sectors with internal angle $\ll \mathrm{e}^{-t}$. Using Exercise 2.57 (see the hint on p. 477) we obtain from these the bound

$$\|v_{\mathrm{horizontal}}\|^2 = \mu_v(B_{\mathrm{horizontal}}) \ll \mathrm{e}^{-t} \|v\|^2, \tag{7.12}$$

and similarly

$$\|w_{\mathrm{vertical}}\|^2 = \mu_w(B_{\mathrm{vertical}}) \ll \mathrm{e}^{-t} \|w\|^2. \tag{7.13}$$

Also, by Exercise 2.57 we have

$$\mu_{v_{\mathrm{main}}} = \mu_v|_{\mathbb{R}^2 \smallsetminus B_{\mathrm{horizontal}}},$$

which together with Proposition 3.1 gives

$$\mu_{\pi_{a_t} v_{\mathrm{main}}} = (a_t^{-1})_* \mu_v|_{\mathbb{R}^2 \smallsetminus B_{\mathrm{horizontal}}}$$

since the transpose of a_t is a_t itself. A simple calculation now reveals that the set $a_t^{-1}(\mathbb{R}^2 \smallsetminus B_{\mathrm{horizontal}})$ agrees with B_{vertical} with the exception of the boundaries, which are null sets. It follows that

$$\mu_{\pi_{a_t} v_{\mathrm{main}}} \perp \mu_{w_{\mathrm{main}}}$$

which implies that $\pi_{a_t} v_{\mathrm{main}} \perp w_{\mathrm{main}}$ by Proposition 2.50(7). Together with (7.12) and (7.13), this gives

$$\begin{aligned}\bigl|\langle\pi_{a_t}v,w\rangle\bigr| &\leqslant \underbrace{\bigl|\langle\pi_{a_t}v_{\mathrm{main}},w_{\mathrm{main}}\rangle\bigr|}_{=0}+\bigl|\langle\pi_{a_t}v_{\mathrm{main}},w_{\mathrm{vertical}}\rangle\bigr|+\bigl|\langle\pi_{a_t}v_{\mathrm{horizontal}},w\rangle\bigr|\\ &\ll \mathrm{e}^{-\frac{t}{2}}\|v\|\|w\|\end{aligned}$$

by the Cauchy–Schwarz inequality. □

7.3.2 Bootstrapping to the General Case

Lemma 7.27 will allow us to use the following lemma with the value $\kappa=\frac{1}{2}$.

Lemma 7.28 (Smooth vectors for SL_3). *Let π be a unitary representation of $\mathrm{SL}_3(\mathbb{R})$ and assume that*

$$\bigl|\langle\pi_{a_t}v,w\rangle\bigr|\leqslant c\mathrm{e}^{-\kappa t}\|v\|\|w\| \tag{7.14}$$

holds for some constants $c,\kappa>0$ and all $\mathrm{SO}_2(\mathbb{R})$-eigenfunctions v,w lying in $\bigl(\mathcal{H}_\pi^{\mathrm{SL}_3(\mathbb{R})}\bigr)^\perp$. Then

$$|\langle\pi_a v,w\rangle|\ll c\mathrm{e}^{-\frac{1}{2}\kappa|t_2-t_1|}\,\mathcal{S}(v)\,\mathcal{S}(w)$$

for all C^1-smooth vectors $v,w\in\bigl(\mathcal{H}_\pi^{\mathrm{SL}_3(\mathbb{R})}\bigr)^\perp$, where

$$a=\begin{pmatrix}\mathrm{e}^{t_1}&&\\&\mathrm{e}^{t_2}&\\&&\mathrm{e}^{t_3}\end{pmatrix} \tag{7.15}$$

for $t_1,t_2,t_3\in\mathbb{R}$ with $t_1+t_2+t_3=0$.

Proof. We will use Proposition 7.25 and its notation for π restricted to

$$\bigl(\mathcal{H}_\pi^{\mathrm{SL}_3(\mathbb{R})}\bigr)^\perp.$$

For this, we notice that

$$a=\begin{pmatrix}\mathrm{e}^{t_1}&&\\&\mathrm{e}^{t_2}&\\&&\mathrm{e}^{t_3}\end{pmatrix}=\underbrace{\begin{pmatrix}\mathrm{e}^{t_1+\frac{1}{2}t_3}&&\\&\mathrm{e}^{t_2+\frac{1}{2}t_3}&\\&&1\end{pmatrix}}_{=a_{t_1+\frac{1}{2}t_3}}\underbrace{\begin{pmatrix}\mathrm{e}^{-\frac{1}{2}t_3}&&\\&\mathrm{e}^{-\frac{1}{2}t_3}&\\&&\mathrm{e}^{t_3}\end{pmatrix}}_{=b}$$

where $a_{t_1+\frac{1}{2}t_3}\in\mathrm{ASL}_2(\mathbb{R})$ is as defined in (7.10) and $b\in\mathrm{SL}_3(\mathbb{R})$ commutes with K. We set $B=\pi_b$ and apply Proposition 7.25. We note that its conclusion holds even if $\mathcal{S}(\cdot)$ denotes the Sobolev norm with respect to the unitary action π of $\mathrm{SL}_3(\mathbb{R})$. Indeed, we may include a basis of $\mathfrak{sl}_2(\mathbb{R})$ in the basis of $\mathfrak{sl}_3(\mathbb{R})$ in the definition of the Sobolev norm, and then use Exercise 7.12 to see that this assumption does not matter since we allow ourselves to change the implicit constant. Therefore

$$|\langle \pi_a v, w\rangle| = \left|\left\langle \pi_{a_{t_1+\frac{1}{2}t_3}} Bv, w\right\rangle\right| \ll c\mathrm{e}^{-\kappa|t_1+\frac{1}{2}t_3|}\,\mathcal{S}(v)\,\mathcal{S}(w)$$

for all C^1-smooth vectors $v, w \in \left(\mathcal{H}_\pi^{\mathrm{SL}_3(\mathbb{R})}\right)^\perp$. Since $t_1 + \frac{1}{2}t_3 = \frac{1}{2}(t_1 - t_2)$, this gives the lemma. □

Proof of Theorem 7.26. By Lemma 7.27, we have (7.14) for $\kappa = \frac{1}{2}$. By conjugation with permutation matrices, we have that Lemma 7.28 also shows

$$|\langle \pi_a v, w\rangle| \ll \mathrm{e}^{-\frac{1}{4}|t_3-t_1|}\,\mathcal{S}(v)\,\mathcal{S}(w) \tag{7.16}$$

and

$$|\langle \pi_a v, w\rangle| \ll \mathrm{e}^{-\frac{1}{4}|t_3-t_2|}\,\mathcal{S}(v)\,\mathcal{S}(w)$$

for all diagonal matrices a, in the notation of (7.15). Hence we may choose the best of these three estimates. Suppose without loss of generality that

$$t_3 \geqslant t_2 \geqslant t_1,$$

so that the best estimate is given by (7.16). Now notice that $|a|$ is essentially equal to e^{t_3} in the sense that

$$\mathrm{e}^{t_3} \leqslant |a| \ll \mathrm{e}^{t_3},$$

and $t_1 + t_2 + t_3 = 0$ gives

$$\mathrm{e}^{t_1-t_3} \leqslant \mathrm{e}^{\frac{1}{2}(t_1+t_2)-t_3} = \mathrm{e}^{-\frac{3}{2}t_3} \ll |a|^{-\frac{3}{2}}.$$

Combining this with (7.16), it follows that Theorem 7.26 holds for diagonal matrices.

To generalize the estimate to an arbitrary $g \in \mathrm{SL}_3(\mathbb{R})$ we use the Cartan decomposition $g = kak'$ for $k, k' \in \mathrm{SO}_3(\mathbb{R})$ and a diagonal a (see the footnote on p. 59). Notice that $|g| = |a|$, and that $\mathcal{S}(\pi_k v) \ll \mathcal{S}(v)$ uniformly for k in $\mathrm{SO}_3(\mathbb{R})$ (see Exercise 7.11). Therefore

$$\begin{aligned}\left|\left\langle \pi_g v, w\right\rangle\right| &= \left|\left\langle \pi_a(\pi_{k'} v), \pi_k^{-1} w\right\rangle\right| \\ &\ll |a|^{-\frac{3}{8}}\,\mathcal{S}(\pi_{k'} v)\,\mathcal{S}(\pi_k^{-1} w) \\ &\ll |g|^{-\frac{3}{8}}\,\mathcal{S}(v)\,\mathcal{S}(w),\end{aligned}$$

as required. The last claim now follows from Proposition 7.23. Going through the implicit constants appearing in the proof, starting with Lemma 7.27 and

Proposition 7.25, one verifies that the implicit constant above is also absolute. □

Exercise 7.29. Extend the argument used in the proof of Theorem 7.26 to give an effective estimate on the decay of matrix coefficients for $G = \mathrm{SL}_d(\mathbb{R})$ with $d \geqslant 3$ (for example by again using subgroups of the form $\mathrm{ASL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ leading to the exponent $\frac{1}{4}\frac{d}{d-1}$).

Exercise 7.30. Recall that $\mathrm{SL}_3(\mathbb{R})$ has a double cover $\widetilde{\mathrm{SL}_3(\mathbb{R})}$ (since the fundamental group of $\mathrm{SO}_3(\mathbb{R})$ is $\mathbb{Z}/2\mathbb{Z}$). Generalize Definition 7.21 and Theorem 7.26 to $\widetilde{\mathrm{SL}_3(\mathbb{R})}$ by using the isogeny $\imath\colon \widetilde{\mathrm{SL}_3(\mathbb{R})} \to \mathrm{SL}_3(\mathbb{R})$ to define the size of $g \in \widetilde{\mathrm{SL}_3(\mathbb{R})}$ by $\lceil g \rceil = \lceil \imath(g) \rceil$.