Appendix C: Adeles and Local Fields

A particularly important collection of locally compact abelian groups come from *local fields*; in this appendix we show how local fields (the real numbers, for example) arise as completions of global fields (the rational numbers, for example) and develop the adelic approach to certain arithmetic properties of global fields, using methods from analysis. A convenient source for the background needed is Cassells [33]; some of the proofs are sketched here in special cases to illustrate the ideas. Complete proofs may be found in the monographs of Weil [203] or Ramakrishnan and Valenza [167].

C.1 Global Fields and their Places

Definition C.1. *A* local field *is a non-discrete locally compact field.*

If K is a local field and $a \in K^{\times}$, then the map $x \mapsto ax$ is a continuous automorphism of the additive group $(K,+)$, and we define $\mathrm{mod}_K(a)$, the *module* of a, by the property

$$m_K(aA) = \mathrm{mod}_K(a) m_K(A)$$

for any Borel set A with $m_K(A) < \infty$ (where as usual m_K denotes the Haar measure on the locally compact group $(K,+)$).

Thus, for example, $\mathrm{mod}_{\mathbb{R}}(a) = |a|$, $\mathrm{mod}_{\mathbb{C}}(a) = |a|^2$, and $\mathrm{mod}_{\mathbb{Q}_p}(p) = \frac{1}{p}$.

It is easily checked that the module of the image $I = \{n1_{\mathbb{K}}\}$ of the integers in a local field $\mathbb{K}$ has the following possible behaviors:

(1) $\mathrm{mod}_{\mathbb{K}}(n1_{\mathbb{K}}) = |n|^{\nu}$, in which case $\nu = 1$ and $\mathbb{K} = \mathbb{R}$, or $\nu = 2$ and $\mathbb{K} = \mathbb{C}$;
(2) $\mathrm{mod}_{\mathbb{K}}(n1_{\mathbb{K}}) \leqslant 1$, and is somewhere less than 1, in which case there is a prime p for which $\mathrm{mod}_{\mathbb{K}}(n1_{\mathbb{K}}) = p^{-\mathrm{ord}_p(n)}$;
(3) $\mathrm{mod}_{\mathbb{K}}(n1_{\mathbb{K}}) = 0$ for some n, in which case $\mathbb{K}$ has positive characteristic.

In the last two cases, $\mathbb{K}$ is called *ultrametric*, and the metric on $\mathbb{K}$ satisfies the ultrametric inequality

$$\mathsf{d}(x,y) \leqslant \max\{\mathsf{d}(x,z), \mathsf{d}(z,y)\}.$$

Definition C.2. *A field is called an $\mathbb{A}$-field if it is an algebraic number field (that is, a finite extension of $\mathbb{Q}$), or a finite algebraic extension of $\mathbb{F}_p(t)$.*

An $\mathbb{A}$-field is countable, and therefore its additive group is a locally compact group in the discrete topology. In this Appendix we describe how $\mathbb{A}$-fields may be studied via their locally compact completions, which are local fields.

Definition C.3. *Let k be an $\mathbb{A}$-field, and let $\psi : k \to K$ be an injection into a local field K. Then the pair (ψ, K) is called a* completion *of k if the image $\psi(k)$ is a dense subfield of K. Completions (ψ, K) and (ψ', K') are* equivalent *if there is an isomorphism of fields $\theta : K \to K'$ for which the diagram in Figure* C.2 *commutes. Finally, a* place *of k is an equivalence class of completions.*

$$\begin{tikzcd}
 & k \arrow[dl, "\psi"'] \arrow[dr, "\psi'"] & \\
K \arrow[rr, "\theta"] & & K'
\end{tikzcd}$$

Fig. C.2. Equivalent completions.

A place $v \ni (\psi, K)$ of an $\mathbb{A}$-field k is *real* if $K \cong \mathbb{R}$, *imaginary* if $K \cong \mathbb{C}$, *infinite* in either of those cases, and *finite* in all other cases.

If $\theta : K \to K'$ is an isomorphism of fields between two completions (ψ, K) and (ψ', K') in the same place v of k, then $\mathrm{mod}_{K'} \circ \theta = \mathrm{mod}_K$.

Lemma C.4. *Completions (ψ, K) and (ψ', K') lie in the same place of k if and only if $\mathrm{mod}_K \circ \psi = \mathrm{mod}_{K'} \circ \psi'$.*

Proof. If (ψ, K) and (ψ', K') lie in the same place, then

$$\mathrm{mod}_K \circ \psi = \mathrm{mod}_{K'} \circ \psi'$$

by the diagram in Figure C.2. Conversely, if $\mathrm{mod}_K \circ \psi = \mathrm{mod}_{K'} \circ \psi'$ for two completions K, K' of k, then both of those completions must belong to the place v containing the completion of k in the metric

$$\mathsf{d}_v(x,y) = (\mathrm{mod}_K \circ \psi(x-y))^{\nu}$$

for $\nu = 1$ (if K is ultrametric) or $\nu = \frac{1}{2}$ (if K is not ultrametric). $\square$

The places of $\mathbb{Q}$ comprise:

(1) An infinite place containing the completion $\mathbb{Q} \to \mathbb{R}$, equivalently $\mathbb{R} = \mathbb{Q}_\infty$ is the completion of $\mathbb{Q}$ with respect to the valuation $|x|_\infty = |x|$.
(2) For each prime p, the completion $\mathbb{Q} \to \mathbb{Q}_p$, equivalently $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the valuation $|x|_p = p^{-\operatorname{ord}_p(x)}$ for $x \in \mathbb{Q}$.

The places of $\mathbb{F}_p(t)$ comprise:

(1) No infinite places.
(2) One place corresponding to a valuation with $|T|_v > 1$.
(3) For each irreducible polynomial $q \in \mathbb{F}_p[t]$, a place corresponding to a valuation with $|q|_v < 1$.

For any global field K we write $\mathsf{P}(K)$ for the set of all places of K, $\mathsf{P}_\infty(K)$ for the set of infinite places, and $\mathsf{P}_{\mathrm{fin}}(K)$ for the set of finite places. For any finite place v of K, the subset

$$r_v = \{x \in K_v \mid |x|_v \leqslant 1\}$$

is an open compact subring (by the maximal inequality) of K_v. Elements of r_v are called integral in K_v; in the case $K = \mathbb{Q}$ and $v = p$ we write $\mathbb{Z}_p$ for r_v, to denote the ring of p-adic integers.

The fields $\mathbb{Q}$ and $\mathbb{F}_p(t)$ are the prime motivating examples, and an understanding of the places of their generalizations (algebraic number fields and rational function fields, respectively) may be built up from them.

If k is an $\mathbb{A}$-field with an embedding $\lambda : k \to K$ into a local field, and $k_0 \subseteq k$ is an infinite subfield, then $\overline{\lambda(k_0)}$ (closure in the topology of K) is a locally compact field in K, and in particular is itself a local field. Moreover, if

$$[k : k_0] < \infty$$

then $\overline{\lambda(k)}$ is the field generated by $\lambda(k)$ over $\overline{\lambda(k_0)}$. This is clear since k is a finite dimensional vector space over k_0. We apply this remark to see the following.

Lemma C.5. *Let k be an $\mathbb{A}$-field with k' a finite algebraic extension of k, w a place of k' and $\lambda : k' \to k'_w$ an embedding into k'_w. Then k'_w is a finite algebraic extension of $\overline{\lambda(k)} \subseteq k'_w$, and the pair $(\lambda|_k, \overline{\lambda(k)})$ determines a place v of k.*

In this situation the place w of k' is said to *lie above* the place v of k, v is said to *lie below* w, and we write $w|v$.

Lemma C.6. *Let k be an $\mathbb{A}$-field, let k' be a finite algebraic extension of k, and let v be a place of k. Then there is at least one, and only finitely many, places w of k' with w above v.*

Proof. Let ξ be an element of k', so there is a polynomial $p \in k[t]$ with

$$p(\xi) = 0.$$

Since there is an embedding of k into k_v we may view p as a polynomial in $k_v[t]$, which therefore defines an extension of k_v. Repeating this finitely often (for each element of a k-basis of k') we produce a finite algebraic extension of k_v containing an isomorphic copy of k', which therefore determines a place w of k' lying above v by construction.

To see that there can only be finitely many places of k' above v, let K be an algebraic closure of k_v, and let $K_0 \subseteq K$ be an algebraic closure of k. If $w|v$ then $(k')_w$ is an algebraic extension of k_v, so there is an isomorphic embedding $\phi : k'_w \hookrightarrow K$. Let $\psi = \phi|_{k'}$ be the restriction of ϕ to k', giving a map $\psi : k' \to K_0$. Conversely, given such a map $\psi : k' \to K_0$ we obtain an embedding of k' into a finite extension (specifically, into the smallest field containing $\psi(k')$ and k_v) of k_v. Thus the map ψ determines and is determined by a place of k'. Now there are only finitely many distinct maps $\psi : k' \to K_0$ since a zero of a polynomial in k' must go to a zero of the same polynomial in K_0, so there are only finitely many places w above v. □

Thus an $\mathbb{A}$-field has only finitely many infinite places; at least one if the characteristic is zero, and none if the characteristic is positive.

Lemma C.7. *Let k be an $\mathbb{A}$-field. Then for any $x \in k$, we have $|x|_v \leqslant 1$ for all but finitely many $v \in \mathsf{P}(k)$.*

Proof. Assume first that the characteristic of k is zero. For $k = \mathbb{Q}$ this is clear, so assume that $k \supsetneq \mathbb{Q}$ and let $x \in k$. It follows that there is some polynomial identity

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

with coefficients $a_i \in \mathbb{Q}$. Let P be the set of prime numbers appearing in the denominators of the coefficients (so $|a_i|_p \leqslant 1$ for $1 \leqslant i \leqslant n$, for all $p \notin P$) and let q be a prime not in P. Since $|a_i|_q \leqslant 1$ for $1 \leqslant i \leqslant n$, so x is integral over $\mathbb{Z}_q$, and hence if v is a place of k above p then $x \in r_v$ by the ultrametric inequality. It follows that $|x|_v > 1$ only if v lies above p for some $p \in P$, and P is finite with a finite number of places above each element by Lemma C.6. The case of positive characteristic is similar. □

C.2 Adeles and the Approximation Theorem

Adeles provide a convenient setting for treating all the places of a global field simultaneously. For many of the applications in dynamical systems, it is convenient to have a slightly more flexible version of the basic results(55).

If S is a set of places of K with $\mathsf{P}_\infty(K) \subseteq S \subseteq \mathsf{P}(K)$ then there is an associated ring of S-integers

$$R_S = \{x \in K \mid |x|_v \leqslant 1 \text{ for all } v \notin S\}.$$

Notice that if $S = \mathsf{P}(K)$ then $R_S = K$.

Example C.8. If $K = \mathbb{Q}$ then $P_\infty(K) = \{\infty\}$ corresponding to the single infinite place containing the completion with respect to the valuation $|x|_\infty = |x|$, the usual absolute value, and $P_{\mathrm{fin}}(K) = \{2, 3, 5, \dots\}$ is the set of rational primes, with the prime p denoting the single finite place containing the completion corresponding to the valuation $|x|_p = p^{-\operatorname{ord}_p(x)}$. Thus R_S denotes the ring of rationals $\frac{a}{b}$ with $a \in \mathbb{Z}$ and b only divisible by primes appearing in S.

Definition C.9. *Let k be an $\mathbb{A}$-field and let S be a set of places with*

$$\mathsf{P}_\infty(k) \subseteq S \subseteq \mathsf{P}(k).$$

For each finite set P of places with $\mathsf{P}_\infty(k) \subseteq P \subseteq S$, define

$$k_{\mathbb{A}}(P) = \prod_{v \in P} k_v \times \prod_{v \in S \smallsetminus P} r_v = \{(x_v)_{v \in S} \mid |x_v|_v \leqslant 1 \text{ for } v \notin P\}$$

*(a locally compact ring), and define the S-*adele *ring of k to be the ring*

$$k_{\mathbb{A},S} = \bigcup_{\substack{P \text{ finite}, \\ \mathsf{P}_\infty(k) \subseteq P \subseteq S}} k_{\mathbb{A}}(P)$$

with the topology defined by requiring that each $k_{\mathbb{A}}(P)$ is an open subring. We also write

$$k_{\mathbb{A}} = k_{\mathbb{A},\mathsf{P}(k)}$$

for the adele ring *of k.*

Notice that we may also write

$$k_{\mathbb{A},S} = \{(x_v)_{v \in S} \mid |x_v|_v \leqslant 1 \text{ for all but finitely many } v \in S\};$$

for S infinite it follows that $k_{\mathbb{A},S}$ is not an open subset of $\prod_{v \in S} k_v$ in the product topology. Moreover, for S infinite, the ring $\prod_{v \in S} k_v$ is not locally compact in the product topology, whereas $k_{\mathbb{A},S}$ always is: the set

$$\{(x_v)_{v \in S} \mid |x_v|_v \leqslant 1 \text{ for all } v \in S\}$$

is a compact neighborhood of the additive identity. Despite the fact that $k_{\mathbb{A},S}$ is not a direct product when S is infinite, we do have for any $v \in S$ a decomposition

$$k_{\mathbb{A},S} \cong k_v \times k_{\mathbb{A},S \smallsetminus \{v\}},$$

and k_v is a *quasi-factor* of $k_{\mathbb{A},S}$, which may be identified with

$$\{(x_w)_{w \in S} \mid x_w = 0 \text{ for all } w \neq v\}.$$

This construction is often referred to as a *restricted direct product* of the groups $\mathbb{Q}_p$ with respect to the compact subgroups $\mathbb{Z}_p$ (see Tate [193] for a convenient account of the construction in greater generality).

For each $v \in \mathsf{P}(K)$ there is an embedding $K \to K_v$; we will therefore think of elements of K as elements of K_v. By Lemma C.7 there is an embedding

$$\begin{aligned} \delta : R_S &\longrightarrow k_{\mathbb{A},S} \\ x &\longmapsto (x, x, x, \dots) \end{aligned}$$

of the ring of S-integers into the ring of S-adeles. We define the ring of integers of K to be the ring

$$O_K = \bigcap_{v \in \mathsf{P}_{\mathrm{fin}}(K)} \{x \in K \mid |x|_v \leqslant 1\}.$$

It may be shown that O_K is the integral closure of $\mathbb{Z} \subseteq K$ (if K has characteristic zero) and is the integral closure of $\mathbb{F}_p(t)$ (if K has positive characteristic).

Theorem C.10 (Approximation theorem). *For any $\mathbb{A}$-field k and set S with $\mathsf{P}_\infty(k) \subseteq S \subseteq \mathsf{P}(k)$,*

$$k_{\mathbb{A},S} = \delta(R_S) + k_{\mathbb{A},S}(\mathsf{P}_\infty(k)). \tag{C.1}$$

Moreover, $\delta(R_S)$ is discrete in $k_{\mathbb{A},S}$ and the quotient $k_{\mathbb{A},S}/\delta(R_S)$ is compact.

Proof. We indicate the proof in the case $k = \mathbb{Q}$; the case $k = \mathbb{F}_p(t)$ is very similar. The general case is built up from these cases by studying the behavior of places under finite extensions. Assume therefore that $k = \mathbb{Q}$ and S is a set of places of $\mathbb{Q}$ containing ∞, and for each prime $p \in S \smallsetminus \{\infty\}$ let

$$\mathbb{Q}^{(p)} = \{\xi \in R_S \mid |\xi|_q \leqslant 1 \text{ for all } q \in S \smallsetminus \{\infty, p\}\} = \{p^n a \mid n < 0, a \in \mathbb{Z}\}.$$

It follows that

$$\mathbb{Q}^{(p)} \cap \mathbb{Z}_p = \mathbb{Z}$$

and

$$\mathbb{Q}_p = \mathbb{Q}^{(p)} + \mathbb{Z}_p.$$

Now let $A_\infty = \mathbb{R} \times \prod_{p \in S \smallsetminus \{\infty\}} \mathbb{Z}_p = \mathbb{Q}_{\mathbb{A},S}(\mathsf{P}_\infty(\mathbb{Q}))$. An element of $\mathbb{Q}_{\mathbb{A},S}$ differs from $\prod_{p \in S \smallsetminus \{\infty\}} \mathbb{Z}_p$ in only finitely many terms, and thus only differs by an element of R_S. It follows that

$$\mathbb{Q}_{\mathbb{A},S} = \delta(R_S) + A_\infty, \tag{C.2}$$

$$\delta(R_S) \cap A_\infty = \mathbb{Z},$$

and A_∞ is an open subgroup of $\mathbb{Q}_{\mathbb{A},S}$. Now $\delta(\mathbb{Z})$ is a discrete subgroup of A_∞ (since $\mathbb{Z}$ is discrete in $\mathbb{R}$). Since $\delta(R_S) \cap A_\infty = \delta(\mathbb{Z})$, it follows that $\delta(R_S)$ is discrete in $\mathbb{Q}_{\mathbb{A},S}$.

If $C = [-\frac{1}{2}, \frac{1}{2}] \times \prod_{p \in S \smallsetminus \{\infty\}} \mathbb{Z}_p$ then $C + \delta(\mathbb{Q}) = \mathbb{Q}_{\mathbb{A},S}$ by (C.2), since

$$C + \delta(\mathbb{Z}) = A_\infty.$$

The set C is a compact subset of $\mathbb{Q}_{\mathbb{A},S}$ and contains a representative of each coset of $\delta(R_S)$ in $\mathbb{Q}_{\mathbb{A},S}$, so $\mathbb{Q}_{\mathbb{A},S}/\delta(R_S)$ is compact in the quotient topology. □

C.3 Duality for Global Fields

Local fields are isomorphic to their character groups (see Weil [203, Sect. II.5]), and such an isomorphism is effected by dilating any given non-trivial character. Adele rings have the same property, and this gives a description of the character group of global fields, and more generally of rings of S-integers.

Example C.11. A simple illustration of the structure used to describe the character group of the ring of S-integers is given by the simplest case $k = \mathbb{Q}$ and $S = \{\infty\}$, so $R_S = \mathbb{Z}$. In this case $k_{\mathbb{A},S} = \mathbb{R}$, so Theorem C.10 simply records the usual embedding $\delta : \mathbb{Z} \to \mathbb{R}$ and the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\delta} \mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z} \to 0$$

with the compact quotient $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ and the canonical quotient map π being reduction modulo 1. This may be thought of as a realization of $\mathbb{R}$ as a covering space for the circle with fiber given by the integers. Define a basic character $\chi \in \widehat{\mathbb{R}}$ by $\chi(t) = \mathrm{e}^{2\pi \mathrm{i} t}$, so that $\ker(\chi) = \mathbb{Z} \subseteq \mathbb{R}$. This gives an isomorphism

$$\begin{aligned} \theta : \mathbb{R} &\longrightarrow \widehat{\mathbb{R}} \\ s &\longmapsto \chi_s \end{aligned}$$

where $\chi_s(t) = \mathrm{e}^{2\pi \mathrm{i} st}$. The subgroup $\mathbb{Z} \leqslant \mathbb{R}$ is discrete and therefore closed; moreover under the isomorphism θ we have

$$\mathbb{Z}^{\perp} = \{\chi_s \mid \chi_s \equiv 1 \text{ on } \mathbb{Z}\} \cong \{s \in \mathbb{R} \mid \mathrm{e}^{2\pi \mathrm{i} sk} = 1 \text{ for } k \in \mathbb{Z}\} = \mathbb{Z}.$$

Thus

$$\widehat{\mathbb{R}}/\mathbb{Z}^{\perp} \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{T} \cong \widehat{\mathbb{Z}},$$

giving a construction of the character group of the ring $\mathbb{Z}$ of S-integers.

Example C.11 suggest that a description for $\widehat{R_S}$ can be given once we find an LCA group X with the following properties:

- There is an embedding $R_S \to X$ realizing R_S as a discrete subgroup of X with compact quotient.
- There is an isomorphism $X \to \widehat{X}$ sending the annihilator of the embedded subgroup isomorphic to R_S to the image of R_S under the isomorphism.

In order to extend Example C.11 to this setting two additional ingredients are needed. First, we need to replace the explicit isomorphism $\mathbb{R} \to \widehat{\mathbb{R}}$ with a similar map for any local field. Second, we need to handle the fact that $k_{\mathbb{A},S}$ is not a direct product of local fields when S is infinite. Once again we state the result in the general case and indicate how the proof works for the case of the rational numbers.

Theorem C.12. *Let k be an $\mathbb{A}$-field and let S be a set of places with*

$$\mathsf{P}_\infty(k) \subseteq S \subseteq \mathsf{P}(k).$$

Then there is an isomorphism between $k_{\mathbb{A},S}$ and $\widehat{k_{\mathbb{A},S}}$ with the property that the annihilator of $\delta(R_S)$ corresponds to $\delta(R_S)$.

Just as in Example C.11, it follows that $\widehat{R_S} \cong k_{\mathbb{A},S}/\delta(R_S)$.

Proof of Theorem C.12. We indicate the proof for $k = \mathbb{Q}$; the case of $\mathbb{F}_p(t)$ is similar, and the general case follows by viewing k as a finite-dimensional vector space over one of these. Thus $k = \mathbb{Q}$ and S is a subset of the rational primes $\{2, 3, 5, \dots\}$.

As in the proof of Theorem C.10, let

$$A_\infty = \mathbb{R} \times \prod_{p \in S \smallsetminus \{\infty\}} \mathbb{Z}_p = \mathbb{Q}_{\mathbb{A},S}(\mathsf{P}_\infty(\mathbb{Q})),$$

and define a character on the LCA group A_∞ by

$$\chi\left((x_p)_{p \in S}\right) = \mathrm{e}^{-2\pi \mathrm{i} x_\infty}.$$

Notice that $\chi \in \delta(\mathbb{Z})^\perp$ so χ extends uniquely to a character $\widetilde{\chi} \in \widehat{\mathbb{Q}_{\mathbb{A},S}}$ if we also require that $\chi \in \delta(R_S)^\perp$. The character $\widetilde{\chi}$ defines a local character $\widetilde{\chi}_p$ on $\mathbb{Q}_p$ for each quasi-factor $\mathbb{Q}_p$ with $p \in S$, and by construction $\widetilde{\chi}_p \equiv 1$ on $\mathbb{Z}_p$ for each $p \in S$ (since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$). For $x \in \mathbb{Q}^{(p)} \subseteq R_S$ we have

$$\widetilde{\chi}(\delta(x)) = \prod_{q \in S} \widetilde{\chi}_q(x) = \widetilde{\chi}_\infty(x)\widetilde{\chi}_p(x) = 1$$

since $\chi \in \delta(R_S)^\perp$, so

$$\widetilde{\chi}_p(x) = \mathrm{e}^{2\pi \mathrm{i} x}.$$

Now let $\chi^* \in \widehat{\mathbb{Q}_{\mathbb{A},S}}$ be an arbitrary character. Then by continuity we have $\chi_p^* \in \mathbb{Z}_p^\perp$ for all but finitely many $p \in S$, and by Weil [203, Sect. II.5] there is some $a_p \in \mathbb{Q}_p$ for which $\chi_p^*(x) = \widetilde{\chi}_p(a_p x)$ for all $x \in \mathbb{Q}_p$ and $p \in S$. Moreover, since $\chi_p^* \in \mathbb{Z}_p^\perp$ for all but finitely many $p \in S$, the vector $(a_p)_{p \in S}$ is itself an element of $\mathbb{Q}_{\mathbb{A},S}$, so this gives a surjective homomorphism

$$a = (a_p)_{p \in S} \longmapsto \widetilde{\chi}_a = \widetilde{\chi}(a \cdot) = \prod_{p \in S} \widetilde{\chi}_p(a_p \cdot) \tag{C.3}$$

from $\mathbb{Q}_{\mathbb{A},S}$ to $\widehat{\mathbb{Q}_{\mathbb{A},S}}$. The map is clearly injective on each quasi-factor so is injective; it is also continuous since it is continuous on $k_{\mathbb{A}}(P)$ for each finite set P of places with $\mathsf{P}_\infty(k) \subseteq P \subseteq S$.

Now if $a = \delta(a)$ for some $a \in R_S$ then $\widetilde{\chi}(a \cdot) \in \delta(R_S)^\perp$, so the map in (C.3) takes $\delta(R_S)$ into $\delta(R_S)^\perp$. We claim that in fact

$$\{\widetilde{\chi}_{\delta(a)} \mid a \in R_S\} = \delta(R_S)^{\perp}. \tag{C.4}$$

If $\widetilde{\chi}_b \in \delta(R_S)^{\perp}$ then by Theorem C.10 we may write $b = c + d$ with $c \in \delta(R_S)$ and $d \in [-\frac{1}{2}, \frac{1}{2}] \times \prod_{p \in S \smallsetminus \{\infty\}} \mathbb{Z}_p$. Since $\widetilde{\chi}_b = \widetilde{\chi}_c \widetilde{\chi}_d$, it follows that $\widetilde{\chi}_d \equiv 1$ on the compact subgroup $\{0\} \times \prod_{p \in S \smallsetminus \{\infty\}} \mathbb{Z}_p$, so $\widetilde{\chi}_d \equiv 1$ and hence $d = 0$. This shows (C.4).

All that remains is to check that the maps constructed are indeed continuous, and this may be seen from general considerations as follows. The map $a \mapsto \widetilde{\chi}_a$ from the discrete group $\delta(R_S)$ induces a continuous map from the compact set $\mathbb{Q}_{\mathbb{A},S}/\delta(R_S) \to \widehat{\mathbb{Q}_{\mathbb{A},S}}/\delta(R_S)^{\perp}$, so $\widehat{\mathbb{Q}_{\mathbb{A},S}}/\delta(R_S)^{\perp}$ is the continuous image of a compact set and is therefore compact. The group $\delta(R_S)^{\perp}$ is the character group of the compact group $\mathbb{Q}_{\mathbb{A},S}/\delta(R_S)$, so is discrete. Finally, $\mathbb{Q}_{\mathbb{A},S}/\delta(R_S)$ is the quotient of a LCA group by a discrete subgroup, so is Hausdorff. Thus the induced map from $\widehat{R_S}$ to $\mathbb{Q}_{\mathbb{A},S}/\delta(R_S)$ is a continuous bijection from a compact set to a Hausdorff space, so is a homeomorphism. Since $\delta(R_S)$ and $\delta(R_S)^{\perp}$ are discrete subgroups, it follows that the map from $\mathbb{Q}_{\mathbb{A},S}$ to $\widehat{\mathbb{Q}_{\mathbb{A},S}}$ is also a homeomorphism. □

C.4 Automorphisms of Solenoids

With the machinery developed in Section 6.3 and the adelic language developed above, it is relatively straightforward to extend Theorem 6.12 to automorphisms of solenoids(56), which is the main step in computing the entropy of any group automorphism.

Definition C.13. *A* solenoid *is a finite-dimensional, connected, compact, abelian group. Equivalently, its dual group is a finite torsion-free rank, torsion-free, discrete abelian group, and thus is a subgroup of $\mathbb{Q}^r$ for some $r \geqslant 1$.*

A formula for the topological entropy of an automorphism of a solenoid was found by Yuzvinskiĭ [209]; in this section we use adeles to give a simple proof of Yuzvinskiĭ's formula(57). In particular, this approach shows that the topological entropy of an automorphism of a solenoid is made up of geometrical contributions in exactly the same way as is the case for toral automorphisms.

In order to illustrate the simple geometrical ideas, we outline in an example the issues that arise. One way to compute the entropy of $T : x \mapsto 2x$ on $\mathbb{T}$ using the methods developed earlier in this chapter is as follows. The map T lifts to the map $\widetilde{T} : x \mapsto 2x$ on $\mathbb{R}$, via the locally isometric covering map $x \mapsto x$ modulo 1 from $\mathbb{R}$ to $\mathbb{T}$ so we can apply Proposition 6.7. The topological entropy of this linear map can be easily computed using the volume growth of Bowen balls,

$$\begin{aligned}
h_{\mathrm{top}}(T) &= h_{\mathrm{top}}(\widetilde{T}) && \text{(by Proposition 6.7)}\\
&= k_{\mathsf{d}}(m_{\mathbb{R}}, \widetilde{T}) && \text{(by Theorem 6.9)}\\
&= \lim_{\varepsilon\searrow 0}\limsup_{n\to\infty} -\frac{1}{n}\log m_{\mathbb{R}}\left(\bigcap_{k=0}^{n-1}\widetilde{T}^{-k}(-\varepsilon,\varepsilon)\right)\\
&= \lim_{\varepsilon\searrow 0}\limsup_{n\to\infty} -\frac{1}{n}\log(2^{-n+2}\varepsilon) = \log 2. && \text{(C.5)}
\end{aligned}$$

In this section we will emulate this calculation for solenoids.

Example C.14. Let $T : X \to X$ be the automorphism dual to $x \mapsto \frac{3}{2}x$ on $\mathbb{Z}[\frac{1}{6}]$. We can replace the local isometry $\mathbb{R} \to \mathbb{T}$ with a local isometry

$$\mathbb{R}\times\mathbb{Q}_2\times\mathbb{Q}_3 \to \mathbb{R}\times\mathbb{Q}_2\times\mathbb{Q}_3/\Delta(\mathbb{Z}[\tfrac{1}{6}]) \cong X,$$

where $\Delta(r) = (r, r, r)$ is the diagonal embedding. This allows us to lift the map T to the map $\widetilde{T} : (x, y, z) \mapsto (\frac{3}{2}x, \frac{3}{2}y, \frac{3}{2}z)$ on $\mathbb{R}\times\mathbb{Q}_2\times\mathbb{Q}_3$. We may choose a metric on $\mathbb{R}\times\mathbb{Q}_2\times\mathbb{Q}_3$ so that a metric ball of radius $\frac{1}{m}$ is a set of the form

$$(-\tfrac{1}{m}, \tfrac{1}{m}) \times 2^m\mathbb{Z}_2 \times 3^m\mathbb{Z}_3$$

with volume $\frac{2}{m} \times 2^{-m} \times 3^{-m}$ with respect to Haar measure

$$m_{\mathbb{R}\times\mathbb{Q}_2\times\mathbb{Q}_3} = m_{\mathbb{R}} \times m_{\mathbb{Q}_2} \times m_{\mathbb{Q}_3}.$$

The Bowen ball is easily computed using the ultrametric property of the metric on $\mathbb{Q}_2$ and $\mathbb{Q}_3$,

$$\bigcap_{k=0}^{n-1}\widetilde{T}^{-k}\big((\tfrac{1}{m},\tfrac{1}{m})\times 2^m\mathbb{Z}_2\times 3^m\mathbb{Z}_3\big) = \big(-(\tfrac{2}{3})^{n-1}\tfrac{1}{m}, (\tfrac{2}{3})^{n-1}\tfrac{1}{m}\big)\times 2^{m+k}\mathbb{Z}_2\times 3^m\mathbb{Z}_3,$$

since multiplication by $\frac{2}{3}$ shrinks distances in $\mathbb{Z}_2$ and expands distances in $\mathbb{Z}_3$. Following the steps in (C.5) gives

$$\begin{aligned}
h_{\mathrm{top}}(T) &= \lim_{m\to\infty}\limsup_{n\to\infty} -\frac{1}{n}\log\left(2\cdot\left(\tfrac{2}{3}\right)^{n-1}\tfrac{1}{m}\cdot 2^{-m-(n-1)}\cdot 3^m\right)\\
&= \log\frac{3}{2} + \log 2 = \log 3.
\end{aligned}$$

Returning to the general case, let T be an automorphism of an r-dimensional solenoid X, so the dual automorphism $\widehat{T} : \widehat{X} \to \widehat{X}$ extends to an automorphism of $\mathbb{Q}^r$, which may be described by an element of $\mathrm{GL}_r(\mathbb{Q})$, which we denote by A and write $T = T_A$. In the toral case $X = \mathbb{T}^r$, the matrix A lies in $\mathrm{GL}_r(\mathbb{Z})$, and as shown in Theorem 6.12 we have

$$h_{\mathrm{top}}(T_A) = \sum_{|\lambda|>1}\log|\lambda|,$$

where the sum is taken over the set of eigenvalues of A, with multiplicities. Moreover, as seen in the proof of Theorem 6.12 (and, in a different setting, in Section 1.6) an eigenvalue that dilates distance by some factor $\rho > 1$ contributes $\log \rho$ to the entropy.

Returning to the case of an automorphism T_A of a solenoid corresponding to a matrix $A \in \mathrm{GL}_r(\mathbb{Q})$, write $\chi_A(t) = \det(A - tI_r) \in \mathbb{Q}[t]$ for the characteristic polynomial and let $s = s(A) \geqslant 1$ denote the least common multiple of the denominators of the coefficients of χ_A. Yuzvinskiĭ's formula states that

$$h(A) = \log s + \sum_{|\lambda|>1} \log |\lambda|, \tag{C.6}$$

where the sum is taken over the set of eigenvalues of A, with multiplicities. The second term in (C.6) accords exactly with our geometrical view of entropy, but the first term $\log s$ does not. It turns out that the two terms are on the same footing: both are sums of contributions $\log \rho$ corresponding to eigenvalues that dilate by a factor ρ, and we will prove in Theorems C.17 and C.18 that

$$h(A) = \sum_{p \leqslant \infty} \sum_{|\lambda_{j,p}|_p > 1} \log |\lambda_j|_p,$$

where $\{\lambda_{j,p}\}$ denotes the set of eigenvalues of A in a finite extension of $\mathbb{Q}_p$.

The first step is to show that we can reduce to the case of the specific solenoid $\widehat{\mathbb{Q}}^r$ without changing the entropy. Let $T : X \to X$ be an automorphism of a solenoid with $\widehat{X} \leqslant \mathbb{Q}^r$ (with r chosen to be minimal with this property); the automorphism $\widehat{T} : \widehat{X} \to \widehat{X}$ extends to an automorphism of $\mathbb{Q}^d$. If we write Σ for the solenoid $\widehat{\mathbb{Q}}^r$ and $T_{\mathbb{Q}}$ for the automorphism of Σ dual to this automorphism of $\mathbb{Q}^r$, then the injective map $\widehat{X} \to \mathbb{Q}^r$ dualizes to give a commutative diagram

$$\begin{array}{ccc} \Sigma & \xrightarrow{T_{\mathbb{Q}}} & \Sigma \\ \downarrow & & \downarrow \\ X & \xrightarrow[T]{} & X \end{array}$$

realizing T as a topological factor of $T_{\mathbb{Q}}$.

The system defined by a full solenoid $\mathbb{Q}^r$ is an inverse limit in a natural way. The behavior of topological entropy under inverse limits is worth recording separately, so we formulate this in a more general setting.

Lemma C.15. *Let $((X_n, T_n))_{n\geqslant 1}$ be a sequence of compact metric spaces X_n with continuous maps $T_n : X_n \to X_n$, with continuous factor maps*

$$\pi_{n+1,n} : (X_{n+1}, T_{n+1}) \to (X_n, T_n)$$

for $n \geqslant 1$. Then if

$$(X, T) = \varprojlim (X_n, T_n, \pi_{n+1,n})$$

denotes the inverse limit system,

$$h_{\mathrm{top}}(X,T)=\sup_{n\geqslant 1} h_{\mathrm{top}}(X_n,T_n).$$

PROOF. For $m>n$ write $\pi_{m,n}$ for the induced factor map from (X_m,T_m) to (X_n,T_n), and write $\pi_n:(X,T)\to(X_n,T_n)$ for the factor map from the inverse limit to (X_n,T_n). Then $h_{\mathrm{top}}(T)\geqslant h_{\mathrm{top}}(T_n)$ for all $n\geqslant 1$ by Exercise 5.2.2, so

$$h_{\mathrm{top}}(X,T)\geqslant\sup_{n\geqslant 1} h_{\mathrm{top}}(X_n,T_n).$$

By definition of the topology on the inverse limit, the collection

$$\left\{\pi_n^{-1}(U_n)\mid U_n\subseteq X_n \text{ open },n\geqslant 1\right\}$$

is a basis for the topology on X, so any open cover $\mathscr{U}$ of X is refined by some open cover $\mathscr{V}$ comprising elements of the form $\pi_n^{-1}(U_n)$ for various n. By compactness, it follows that there is a choice of $N\geqslant 1$ and open sets $U_1,\dots,U_N$ so that

$$\mathscr{V}=\{\pi_n^{-1}(U_n)\mid 1\leqslant n\leqslant N\}.$$

For $m>N$ and $n\leqslant N$ we have $\pi_n^{-1}(U_n)=\pi_m^{-1}\left(\pi_{m,n}^{-1}(U_n)\right)$ for each $n\leqslant N$, so $\mathscr{V}=\{\pi_m^{-1}(V_i)\mid 1\leqslant i\leqslant N\}$ for open sets V_i in X_m. It follows that $\pi_m^{-1}\pi_m\mathscr{V}=\mathscr{V}$, and that $\pi_m\mathscr{V}$ is an open cover of X_m. Thus

$$h_{\mathrm{top}}(T,\mathscr{U})\leqslant h_{\mathrm{top}}(T,\mathscr{V})=h_{\mathrm{top}}(T_m,\pi_m\mathscr{V})\leqslant h_{\mathrm{top}}(T_m)\leqslant\sup_{n\geqslant 1}h_{\mathrm{top}}(T_n).$$

Thus $h_{\mathrm{top}}(T)\leqslant\sup_{n\geqslant 1}h_{\mathrm{top}}(T_n)$, completing the proof. □

Lemma C.16. *Let $T:X\to X$ be an automorphism of a solenoid. Then*

$$h_{\mathrm{top}}(T)=h_{\mathrm{top}}(T_{\mathbb{Q}}).$$

PROOF. Recall that $T:X\to X$ is an automorphism, and $\widehat{X}\leqslant\mathbb{Q}^r$. For any $n\geqslant 1$ the subgroup $\frac{1}{n!}\widehat{X}$ is a $\widehat{T}$-invariant subgroup of $\mathbb{Q}^r$, this defines an increasing sequence of subgroups whose union is all of $\mathbb{Q}^r$. Thus $\mathbb{Q}^r$ is the direct limit

$$\widehat{X}\xrightarrow{\phi_1}\tfrac{1}{2!}\widehat{X}\xrightarrow{\phi_2}\cdots$$

where $\phi_n:\frac{1}{n!}\widehat{X}\to\frac{1}{(n+1)!}\widehat{X}$ is the map defined by $\phi_n(x)=x$. Writing Σ_n for the dual of $\frac{1}{n!}\widehat{X}$, this means that Σ is the inverse limit

$$\Sigma_1\xleftarrow{\widehat{\phi_1}}\Sigma_2\xleftarrow{\widehat{\phi_2}}\cdots.$$

Equivalently,

$$\Sigma = \{(x_n) \in \Sigma_1 \times \Sigma_2 \times \cdots \mid x_n = \widehat{\phi_n} x_{n+1} \text{ for all } n \geqslant 1\}$$

with the topology induced from the product topology. On the other hand, as an abstract group each $\frac{1}{n!}\widehat{X}$ is isomorphic to $\widehat{X}$ (since they are all torsion-free groups), so $h_{\mathrm{top}}(T_n) = h_{\mathrm{top}}(T)$ for all $n \geqslant 1$. Then Lemma C.15 gives

$$h_{\mathrm{top}}(T_{\mathbb{Q}}) = \lim_{n\to\infty} h_{\mathrm{top}}(T, \Sigma_n) = h_{\mathrm{top}}(T).$$

□

Thus we may assume that T is the map $T_A : \Sigma \to \Sigma$ where $\Sigma = \widehat{\mathbb{Q}}^r$ and $A \in \mathrm{GL}_r(\mathbb{Q})$. The matrix A acts as an automorphism of the r-dimensional solenoid Σ, and also as a uniformly continuous linear map on $\mathbb{Q}_p^r$ for $p \leqslant \infty$ (the infinite place giving $\mathbb{Q}_\infty^r = \mathbb{R}^r$). Write d_p for the maximum metric on $\mathbb{Q}_p^r$ for each $p \leqslant \infty$, and d for the metric

$$\mathsf{d}\left((x_p)_{p\in P}, (y_p)_{p\in P}\right) = \max_{p\in P}\{d_p(x_p, y_p)\}$$

on the various sets of the form $\mathbb{A}_{\mathbb{Q}}(P)^r$ arising in the proof. The entropy of T_A, dual to the action of A on the vector space $\mathbb{Q}^r$ over the global field $\mathbb{Q}$, decomposes into a sum of local contributions corresponding to the places of $\mathbb{Q}$.

Theorem C.17. $h_{\mathrm{top}}(T_A) = \sum_{p\leqslant\infty} h_{\mathsf{d}_p}(A \text{ } on \text{ } \mathbb{Q}_p^r)$.

Proof. The adele ring $\mathbb{A}_{\mathbb{Q}}^r$ contains $\mathbb{Q}^r$ as a discrete subring via the map δ (by Theorem C.10), so we may view it as a $\mathbb{Q}$-vector space and thus extend the action of A on $\mathbb{Q}^r$ to a uniformly continuous map on $\mathbb{A}_{\mathbb{Q}}^r$ by defining

$$(Ax)_p = A(x_p)$$

for $x \in \mathbb{Q}_p^r$. Under this action the embedded copy of $\mathbb{Q}^r$ is invariant, so by Theorem C.12, the induced action of A on $\mathbb{A}_{\mathbb{Q}}^r/\delta(\mathbb{Q}^r)$ is isomorphic to the action of $T_{\mathbb{Q}}$ on Σ. The quotient map

$$\mathbb{A}_{\mathbb{Q}}^r \to \mathbb{A}_{\mathbb{Q}}^r/\delta(\mathbb{Q}^r)$$

is a local isometry (since $\delta(\mathbb{Q}^r)$ is a discrete subgroup), so

$$h(T_{\mathbb{Q}}) = h(A \text{ on } \mathbb{A}_{\mathbb{Q}}^r)$$

by Proposition 6.7.

Just as in Section 6.3, we use Haar measure to compute entropy via the decay of volume of Bowen balls.

Both A and A^{-1} have entries in $\mathbb{Z}_p$ for all but finitely many p; let P_A be the set of primes p for which some entry of A or of A^{-1} is not in $\mathbb{Z}_p$, together with ∞. Thus $A \in \mathrm{GL}_r(\mathbb{Z}_p)$ for any $p \notin P$, so

$$\mathbb{A}_{\mathbb{Q}}(P)^r = \prod_{p\in P} \mathbb{Q}_p^r \times \prod_{p\notin P} \mathbb{Z}_p$$

(as in Section C.2) is an A-invariant neighborhood of the identity. By Theorem 6.9, it follows that

$$h_{\mathsf{d}}(A \text{ on } \mathbb{A}_{\mathbb{Q}}^r) = h_{\mathsf{d}}(A \text{ on } \mathbb{A}_{\mathbb{Q}}(P)^r).$$

Now the second factor in $\mathbb{A}_{\mathbb{Q}}(P)^r = \prod_{p\in P} \mathbb{Q}_p^r \times \prod_{p\notin P} \mathbb{Z}_p^r$ is compact, so

$$h_{\mathsf{d}}\left(A \text{ on } \mathbb{A}_{\mathbb{Q}}(P)^r\right) = h_{\mathsf{d}}\left(A \text{ on } \prod_{p\in P} \mathbb{Q}_p^r\right) + h_{\mathsf{d}}\left(A \text{ on } \prod_{p\notin P} \mathbb{Z}_p^r\right). \tag{C.7}$$

by Lemma 6.6. If F is any finite set of primes in the complement Q of P then, for any $m \geqslant 1$,

$$\prod_{p\in F} p^m \mathbb{Z}_p^r \times \prod_{p\in Q\smallsetminus F} \mathbb{Z}_p^r$$

is an A-invariant neighborhood of the identity since $A \in \mathrm{GL}_r(\mathbb{Z}_p)$ for $p \notin P$. These neighborhoods form a basis for the topology, so $h_{\mathsf{d}}(A \text{ on } \prod_{p\notin P} \mathbb{Z}_p^r) = 0$. We deduce that

$$h_{\mathsf{d}}\left(A \text{ on } \mathbb{A}_{\mathbb{Q}}(P)^r\right) = h_{\mathsf{d}}\left(A \text{ on } \prod_{p\in P} \mathbb{Q}_p^r\right).$$

Now the Haar measure $m_{\mathbb{Q}_p}$ on $\mathbb{Q}_p$, normalized to have $m_{\mathbb{Q}_p}(\mathbb{Z}_p) = 1$ for all $p < \infty$, and the Haar measure

$$m_{\mathbb{A}_{\mathbb{Q}}} = \prod_{p\leqslant\infty} m_{\mathbb{Q}_p}$$

are A-homogeneous, and we will show in the proof of Theorem C.18 that

$$-\frac{1}{n}\log m_{\mathbb{Q}_p}^r\left(\bigcap_{k=0}^{n-1} A^{-k}(B^r)\right) \longrightarrow h(A \text{ on } \mathbb{Q}_p^r) \tag{C.8}$$

as $n \to \infty$ for $p < \infty$ (in particular, showing the convergence); for the real case $p = \infty$ this is shown in the proof of Theorem 6.11. By Lemma 6.10, it follows that

$$h_{\mathsf{d}}\left(A \text{ on } \mathbb{A}_{\mathbb{Q}}(P)^r\right) = \sum_{p\in P} h_{\mathsf{d}_p}(A \text{ on } \mathbb{Q}_p^r).$$

For $p \notin P$, the fact that $p^m\mathbb{Z}_p$ forms a basis of A-invariant open neighborhoods of the identity shows that $h_{\mathsf{d}_p}(A \text{ on } \mathbb{Q}_p^r) = 0$, so this completes the proof of Theorem C.17 by (C.7). $\square$

We now turn to the calculation of the local entropies appearing in Theorem C.17. For $p = \infty$ we have done this already, so it is sufficient to consider the case of a finite prime.

Theorem C.18. *Assume that $p < \infty$. If A has eigenvalues $\lambda_1, \dots, \lambda_r$ in a finite extension of $\mathbb{Q}_p$, then*

$$h_{\mathsf{d}_p}(A \text{ on } \mathbb{Q}_p^r) = \sum_{|\lambda_j|_p > 1} \log |\lambda_j|_p,$$

where eigenvalues are counted with multiplicity and $|\cdot|_p$ denotes the unique extension of the p-adic norm to the splitting field of the characteristic polynomial of A. Moreover, there is convergence in (C.8).

The proof below follows exactly the lines of the case $p = \infty$ in Theorem 6.11, but the ultrametric inequality in $\mathbb{Q}_p$ makes the analysis considerably easier. The entire argument is visible in Figure 6.1 on p. 164.

Outline Proof of Theorem C.18. Let K be a finite extension of $\mathbb{Q}_p$ containing all zeros of the characteristic polynomial χ_A of A, and set

$$d = [K : \mathbb{Q}_p].$$

Then $\mathbb{Q}_p^r \otimes_{\mathbb{Q}_p} K \cong K^r$, and A extends to the map $A \otimes 1_K$ acting on K^r. Since K is a d-dimensional vector space over $\mathbb{Q}_p$, and $A \in \mathrm{GL}_r(\mathbb{Q}) \leqslant \mathrm{GL}_r(\mathbb{Q}_p)$, the map $A \otimes 1_K$ is isomorphic to the direct sum of d copies of the map A acting on $\mathbb{Q}_p^r$. Thus $h_{\mathsf{d}_p}(A \otimes 1_K \text{ on } K^r) = d h_{\mathsf{d}_p}(A \text{ on } \mathbb{Q}_p^r)$.

Since K contains the eigenvalues of $A \otimes 1_K$, we can proceed just as in the final part of the proof of Theorem 6.11 and put $A \otimes 1_K$ into its Jordan form

$$A \otimes 1_K \cong \begin{bmatrix} J(\lambda_1, d_1) & & & \\ & J(\lambda_2, d_2) & & \\ & & \ddots & \\ & & & J(\lambda_k, d_k) \end{bmatrix}$$

where $J(\lambda_i, d_i)$ denotes the Jordan block of size d_i corresponding to λ_i. Once convergence is established for the decay of Bowen balls in each block, the entropy sums over the blocks so it is sufficient to prove the result for one block $J = J(\lambda, m)$ acting on K^m equipped with the maximum norm. We may assume without loss that $|\lambda|_p > 1$, since otherwise both sides are 0, and the calculation used in the case of $\mathbb{R}$ or $\mathbb{C}$ shows that there is convergence and the formula required. Finally,

$$\begin{aligned} h_{\mathsf{d}_p}(A \text{ on } \mathbb{Q}_p^r) &= \frac{1}{r} h(A \otimes 1_K \text{ on } K^r) \\ &= \frac{1}{r} \sum_{i=1}^{k} h(J(\lambda_i, d_i) \text{ on } K^{d_i}) \\ &= \frac{1}{r} \sum_{i=1}^{k} r d_i \log^+ |\lambda_i|_p = \sum_{|\lambda_j|_p > 1} \log^+ |\lambda_j|_p. \end{aligned}$$

□

Thus the entropy of an automorphism of a solenoid is a sum over local contributions. The way in which these contributions fit together is illustrated by some examples.

Example C.19. If $A \in \mathrm{GL}_d(\mathbb{Z})$, then for any $p < \infty$ we have $|\lambda_{i,p}|_p = 1$ for all of the eigenvalues $\lambda_{i,p}$ of A in an extension of $\mathbb{Q}_p$[(58)]. Thus all the entropy of the automorphism of Σ^d induced by A comes from the infinite place, with all the finite contributions being zero.

Example C.20. An opposite extreme to Example C.19 is given by an example used by Lind [118] to show that a general exponential rate of recurrence phenomena for ergodic group automorphisms may be driven entirely by p-adic hyperbolicity. Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & \frac{6}{5} \end{bmatrix},$$

with characteristic polynomial $\chi_A(t) = t^2 - \frac{6}{5}t + 1$. The complex eigenvalues $\frac{3}{5} \pm \frac{4}{5}\mathrm{i}$ of A have modulus 1, so the entropy contribution from the infinite place is zero. The 5-adic eigenvalues λ_1, λ_2 satisfy $|\lambda_1\lambda_2|_5 = |1|_5 = 1$ and $|\lambda_1 + \lambda_2|_5 = |\frac{6}{5}|_5 = 5$, so $|\lambda_1|_5 = 5$ and $|\lambda_2|_5 = \frac{1}{5}$. For any finite $p \neq 5$ the p-adic eigenvalues λ_1, λ_2 satisfy $|\lambda_1\lambda_2|_p = 1$ and

$$|\lambda_1 + \lambda_2|_p = |\tfrac{6}{5}|_p \begin{cases} < 1 & \text{if } p \in \{2,3\}; \\ = 1 & \text{if not.} \end{cases}$$

It follows that $|\lambda_1|_p = |\lambda_2|_p = 1$, since if one of $|\lambda_1|_p$ or $|\lambda_2|_p$ exceeds 1, then

$$|\lambda_1 + \lambda_2|_p = |\lambda_1 + \lambda_1^{-1}|_p > 1.$$

Thus

$$h_{\mathrm{top}}(A \text{ on } \Sigma^2) = h_{\mathrm{top}}(A \text{ on } \mathbb{Q}_5^2) = \log 5,$$

and the only positive contribution comes from the 5-adic place.

Example C.21. In general we expect there to be a mixture of infinite and finite contributions to entropy, and this may already be seen in the case of a one-dimensional solenoid X. Here an automorphism is defined by a matrix $[\frac{a}{b}]$ in $\mathrm{GL}_1(\mathbb{Q}) = \mathbb{Q}^\times$, written as a rational $\frac{a}{b}$ in lowest terms, and Theorems C.17 and C.18 show that

$$h_{\mathrm{top}}([\tfrac{a}{b}] \text{ on } X) = \textstyle\sum_{p \leqslant \infty} \log^+ |\tfrac{a}{b}|_p = \log \max\{|a|, |b|\}, \tag{C.9}$$

with contributions only coming from the infinite place and those $p < \infty$ dividing b. This recovers a formula due to Abramov [3]. For example,

$$h_{\mathrm{top}}([\tfrac{3}{2}] \text{ on } \Sigma) = \log^+ |\tfrac{3}{2}| + \log^+ |\tfrac{3}{2}|_2 + \log^+ |\tfrac{3}{2}|_3 = \log \tfrac{3}{2} + \log 2 + 0 = \log 3.$$

On the other hand, Lemma 5.19 shows that $h_{\mathrm{top}}([\frac{2}{3}]$ on $\Sigma) = \log 3$ also, and this arises entirely from the 3-adic contribution:

$$h_{\mathrm{top}}([\tfrac{2}{3}] \text{ on } \Sigma) = \log^+ |\tfrac{2}{3}| + \log^+ |\tfrac{2}{3}|_2 + \log^+ |\tfrac{2}{3}|_3 = 0 + 0 + \log 3 = \log 3.$$

Finally, we show how Theorems C.17 and C.18 together show Yuzvinskiĭ's formula in (C.6). As before, we let $A \in \mathrm{GL}_d(\mathbb{Q})$ be a rational matrix with eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{C}$, and let s be the least common multiple of the denominators of the coefficients of the characteristic polynomial $\chi_A(t)$.

Theorem C.22 (Yuzvinskiĭ). *Let $A \in \mathrm{GL}_d(\mathbb{Q})$ act as an automorphism of a solenoid X with $\widehat{X} \subseteq \mathbb{Q}^d$. Then*

$$h(A \text{ on } X) = \log s + \sum_{|\lambda_j|>1} \log |\lambda_j|$$

where the sum is taken over the eigenvalues of A with multiplicities.

Proof. Let $\chi_A(t) = t^d + a_1 t^{d-1} + \dots + a_d$. If $|s|_p = p^{-e}$, then

$$p^e = \max\{|a_1|_p, \dots, |a_d|_p, 1\}.$$

We claim that

$$h_{\mathrm{top}}(A \text{ on } \mathbb{Q}_p^d) = \log p^e. \tag{C.10}$$

Theorem C.18 then shows that $\log s$ the sum over finite primes p of the p-adic contributions to the topological entropy, proving Theorem C.22. All that remains is to show (C.10). The characteristic polynomial factorizes as

$$\chi_A(t) = \prod_{j=1}^{d} (t - \lambda_j)$$

over a finite extension of $\mathbb{Q}_p$, and we may arrange the eigenvalues so that

$$|\lambda_1|_p \geqslant |\lambda_2|_p \geqslant \dots \geqslant |\lambda_m|_p > 1 \geqslant |\lambda_{m+1}|_p \geqslant \dots \geqslant |\lambda_d|_p.$$

If $|\lambda_j|_p \leqslant 1$ for all j, then $e = 0$ and $h(A \text{ on } \mathbb{Q}_p^d) = 0$ also. Thus we may suppose that $|\lambda_1|_p > 1$. Using the ultrametric inequality, we have

$$\begin{aligned} |a_m|_p &= \Big| \sum_{i_1 < \dots < i_m} \lambda_{i_1} \dots \lambda_{i_m} \Big|_p \\ &= |\lambda_1 \dots \lambda_m + \text{ terms smaller in } p\text{-adic norm}|_p \\ &= |\lambda_1 \dots \lambda_m|_p, \end{aligned}$$

and

$$|a_j|_p \leqslant |a_m|_p$$

similarly. Thus

$$p^e = \max_{1 \leqslant j \leqslant d} \{|a_j|_p\} = \prod_{|\lambda_j^{(p)}|_p > 1} |\lambda_j^{(p)}|_p = h_{\mathrm{top}}(A \text{ on } \mathbb{Q}_p^d),$$

completing the proof. □

Notes to Appendix C

(55) (Page 274) An elegant treatment of the usual adeles may be found in the monograph of Weil [203], and a more accessible version in the book of Ramakrishnan and Valenza [167]. The more flexible version we need here is closer to that used in Tate's thesis [193].

(56) (Page 279) A simple example of a solenoid is a torus; in [53, Exercise 2.1.9] the solenoid dual to $\mathbb{Z}[\frac{1}{2}]$ is constructed. The results in this section formally subsume Theorem 6.11, but the proofs are not a generalization of the proof of Theorem 6.11; indeed we will use that theorem to prove the more general case. To see how diverse solenoids are, notice that for any subset S of the set of rational primes, the ring of S-integers R_S (see Section C.2) is a subgroup of $\mathbb{Q}$, and if $S \neq S'$ then R_S and $R_{S'}$ are not isomorphic. This shows there are uncountably many different one-dimensional solenoids; an easy calculation confirms that these may be used to find uncountably many topologically distinct algebraic dynamical systems with the same topological entropy [201].

(57) (Page 279) We follow Lind and Ward [121], [200] closely here. The possibility of computing entropy for solenoidal automorphisms using p-adic entropy contributions was suggested by Lind in 1980, and ultimately goes back to a suggestion of Furstenberg.

(58) (Page 286) This is most easily seen using the Newton polygon of the characteristic polynomial of A (see Koblitz [105, Sect. IV.3] for example); the proof of Theorem C.22 also shows this in a more general setting.