Chapter 5
Compact Groups

In this chapter we will obtain fundamental results concerning all unitary representations of compact groups. We will assume implicitly that the group considered is metric as well and simply refer to it as *the compact group* G. As we will see, the description of general unitary representations in terms of irreducible representations is easier from the analytic point of view than in the abelian case, due to complete reducibility of unitary representations into irreducible representations. However, from an algebraic point of view the discussions here are harder, since the irreducible representations can be of any finite dimension instead of just one-dimensional. For convenience, we will assume throughout the chapter that the Haar measure m on G is normalized to satisfy $m(G) = 1$. We recall that any compact group is unimodular (see Exercise 1.14(a) and its hint on p. 473).

We begin the chapter, however, by discussing two fundamental constructions for unitary representations that are particularly important for compact groups.

5.1 The Contragredient Representation

For the first construction, recall from the Fréchet–Riesz representation theorem that any Hilbert space $\mathcal{H}$ is isometrically isomorphic to its dual $\mathcal{H}'$. However, this canonical isomorphism sending $w \in \mathcal{H}$ to the map $w' \colon v \mapsto \langle v, w\rangle$ in $\mathcal{H}'$ is semi-linear over $\mathbb{C}$, and so in particular is not an isomorphism of complex Hilbert spaces. Transporting a unitary representation π via this semi-linear isomorphism to the dual $\mathcal{H}_\pi'$ may therefore create a representation not unitarily isomorphic to π.

Definition 5.1 (Contragredient). For a Hilbert space $\mathcal{H}$ the dual $\mathcal{H}'$ may be endowed with the inner product

$$\left\langle v', w'\right\rangle_{\mathcal{H}'} = \overline{\langle v, w\rangle_{\mathcal{H}}} = \langle w, v\rangle_{\mathcal{H}} \tag{5.1}$$

for any $v, w \in \mathcal{H}$ and this inner product induces the operator norm on $\mathcal{H}'_\pi$. For a unitary representation π of G on $\mathcal{H}_\pi$ the *contragredient representation* $\overline{\pi}$ is defined as the unitary representation on $\mathcal{H}_{\overline{\pi}} = \mathcal{H}'_\pi$ induced by the dual operators. That is,

$$\overline{\pi}_g(v') = v' \circ \pi_g^{-1} \in \mathcal{H}_{\overline{\pi}} = \mathcal{H}'_\pi$$

for $v' \in \mathcal{H}'_\pi$ and $g \in G$.

Let us make a few comments that should help the reader to become familiar with this construction, which may be confusing at first sight.

We verify that $\overline{\pi}$ is as claimed a unitary representation. First, it is easy to see that $\mathcal{H}'_\pi$ is again a (complex) Hilbert space. For example, by (5.1),

$$\langle v', w' \rangle_{\mathcal{H}'} = \langle w, v \rangle_{\mathcal{H}}$$

depends linearly on $v' \in \mathcal{H}'_\pi$ and semi-linearly on $w' \in \mathcal{H}'_\pi$ by semi-linearity of the isomorphism.

For the representation $\overline{\pi}$ we first note that for $g \in G$ and $v' \in \mathcal{H}'$ the linear functional $\overline{\pi}_g(v') = v' \circ \pi_g^{-1}$ sends any $w \in \mathcal{H}$ to

$$v'(\pi_g^{-1}(w)) = \left\langle \pi_g^{-1} w, v \right\rangle_{\mathcal{H}_\pi} = \langle w, \pi_g v \rangle_{\mathcal{H}_\pi} = (\pi_g v)'(w),$$

which shows that $\overline{\pi}_g(v') = (\pi_g v)'$, in other words the diagram

$$\begin{array}{ccc} \mathcal{H}_\pi \ni v & \xrightarrow{\pi_g} & \pi_g v \in \mathcal{H}_\pi \\ \downarrow & & \downarrow \\ \mathcal{H}'_\pi \ni v' & \xrightarrow{\overline{\pi}_g} & \overline{\pi}_g v' \in \mathcal{H}'_\pi \end{array}$$

commutes. From this, we see that

$$\begin{aligned} \overline{\pi}_g(\alpha_1 v'_1 + \alpha_2 v'_2) &= \overline{\pi}_g((\overline{\alpha_1} v_1 + \overline{\alpha_2} v_2)') \\ &= (\pi_g(\overline{\alpha_1} v_1 + \overline{\alpha_2} v_2))' \\ &= (\overline{\alpha_1} \pi_g v_1 + \overline{\alpha_2} \pi_g v_2)' = \alpha_1 \overline{\pi}_g v'_1 + \alpha_2 \overline{\pi}_g v'_2 \end{aligned}$$

for any $\alpha_1, \alpha_2 \in \mathbb{C}$ and $v'_1, v'_2 \in \mathcal{H}'_\pi$. Similarly, we have

$$\|\overline{\pi}_g v'\|_{\mathcal{H}'_\pi} = \|(\pi_g v)'\|_{\mathcal{H}'_\pi} = \|\pi_g v\|_{\mathcal{H}_\pi} = \|v\|_{\mathcal{H}_\pi} = \|v'\|_{\mathcal{H}'_\pi}$$

for all $v \in \mathcal{H}$ and $g \in G$, which shows that π_g is unitary for all $g \in G$. The homomorphism property and the continuity requirement follow in the same way from the respective properties of π.

Next we calculate the matrix coefficient

$$\varphi^{\overline{\pi}}_{v',w'}(g) = \left\langle \overline{\pi}_g v', w' \right\rangle_{\mathcal{H}'_\pi} = \left\langle (\pi_g v)', w' \right\rangle_{\mathcal{H}'_\pi}$$
$$= \left\langle w, \pi_g v \right\rangle_{\mathcal{H}_\pi} = \overline{\left\langle \pi_g v, w \right\rangle_{\mathcal{H}_\pi}} = \overline{\varphi^{\pi}_{v,w}(g)} \tag{5.2}$$

of $v', w' \in \mathcal{H}'_\pi$ at $g \in G$, and obtain the conjugate of the matrix coefficient of $v, w \in \mathcal{H}_\pi$. In particular, if the operator π_g is described by a matrix with respect to some orthonormal basis of $\mathcal{H}$, then $\overline{\pi}_g$ is described by the complex conjugate of the same matrix with respect to the dual basis of $\mathcal{H}'$.

In the special case of a one-dimensional unitary representation defined by a unitary character χ on G, this shows that the contragredient is the unitary representation defined by the complex conjugate character $\overline{\chi}$ on G. If χ does not have purely real values then χ is not isomorphic to $\overline{\chi}$ as a unitary representation.

We remark that $\overline{\pi}$ is irreducible if and only if π is, since there is a semi-linear equivariant isomorphism $v \mapsto v'$ between π and $\overline{\pi}$.

Exercise 5.2. (a) Show that the contragredient of the left-regular representation is isomorphic to the left-regular representation.
(b) Give a sufficient criterion for a general unitary representation π to be isomorphic to its contragredient $\overline{\pi}$, that in particular applies to (a).

Exercise 5.3. Let G be abelian as in Chapter 2, and let π be a unitary representation of G. Apply the spectral theorem (Corollary 2.12, or the more refined Theorem 2.65) to π. Describe the contragredient of π in terms of the data arising in the spectral theorem.

Exercise 5.4. (a) Calculate the contragredient representation of all elements of $\widehat{G}$ for the group $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ as in Section 3.3.1.
(b) Repeat this for the affine '$ax + b$' group in Section 3.3.2.
(c) Repeat this for the Heisenberg group in Section 3.3.4.

5.1.1 Continuity

We now prove continuity of the 'contragredient map' with respect to the Fell topology.

Proposition 5.5 (Continuity of contragredient). *The contragredient map defined by*

$$\mathscr{U}(G) \ni \pi \longmapsto \overline{\pi} \in \mathscr{U}(G)$$

is continuous with respect to the Fell topology on $\mathscr{U}(G)$.

Proof. We note that for any $\phi \in \mathcal{P}^1(G)$, compact $K \subseteq G$, $\varepsilon > 0$, unitary representation $\pi \in \mathscr{U}(G)$, and unit vector $v \in \mathcal{H}_\pi$ we have

$$\varphi^{v'}_{\overline{\pi}} = \overline{\varphi^{v}_{\pi}} \in CO(\phi, K, \varepsilon)$$

if and only if $\varphi^v_\pi \in CO(\overline{\phi}, K, \varepsilon)$, which implies that the image of $\mathscr{F}\!\mathscr{O}_{\mathrm{diag}}(\phi, K, \varepsilon)$ is precisely $\mathscr{F}\!\mathscr{O}_{\mathrm{diag}}(\overline{\phi}, K, \varepsilon)$. □

Exercise 5.6. Prove Proposition 5.5 using the principal Fell open sets $\mathscr{F}\!\mathscr{O}(f,\alpha)$ for all values $\alpha \in \mathbb{R}$ and $f \in L^1(G)$.

5.2 The Tensor Product Representation

The second construction we wish to present is the definition of the tensor product of two representations, which will include the definition in Lemma 1.26 as a very special case.

5.2.1 Basic Construction and Properties

In the following, we let $\mathcal{V}$ and $\mathcal{W}$ denote two separable Hilbert spaces.

Proposition 5.7 (Tensor product). *There exists a Hilbert space $\mathcal{V} \otimes \mathcal{W}$ together with a bilinear map $\mathcal{V} \times \mathcal{W} \ni (v,w) \mapsto v \otimes w \in \mathcal{V} \otimes \mathcal{W}$ such that for all $v, v_1, v_2 \in \mathcal{V}$ and $w, w_1, w_2 \in \mathcal{W}$ we have*

(1) $\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{\mathcal{V}\otimes\mathcal{W}} = \langle v_1, v_2 \rangle_{\mathcal{V}} \langle w_1, w_2 \rangle_{\mathcal{W}}$,
(2) $\|v \otimes w\|_{\mathcal{V}\otimes\mathcal{W}} = \|v\|_{\mathcal{V}} \|w\|_{\mathcal{W}}$, *and*
(3) $\{v \otimes w \mid v \in \mathcal{V}, w \in \mathcal{W}\}$ *spans a dense subspace of* $\mathcal{V} \otimes \mathcal{W}$.

Moreover, the subspace spanned by $\{v \otimes w \mid v \in \mathcal{V}, w \in \mathcal{W}\}$ is isomorphic to the tensor product in the sense of linear algebra $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$, the map

$$\mathcal{V} \times \mathcal{W} \ni (v,w) \longmapsto v \otimes w$$

is continuous, and the tensor product $\mathcal{V} \otimes \mathcal{W}$ is separable.

Recall that the tensor product in the sense of linear algebra, written as $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$, of the complex vector spaces $\mathcal{V}$ and $\mathcal{W}$ is defined as the universal vector space together with the bilinear map

$$\mathcal{V} \times \mathcal{W} \ni (v,w) \longmapsto v \otimes w \in \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$$

with the property that for any other bilinear map

$$B\colon \mathcal{V} \times \mathcal{W} \ni (v,w) \longmapsto B(v,w) \in \mathcal{Z}$$

with values in another complex vector space $\mathcal{Z}$ there is a unique linear map

$$L\colon \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W} \longrightarrow \mathcal{Z}$$

such that $B(v,w) = L(v \otimes w)$ for all $(v,w) \in \mathcal{V} \times \mathcal{W}$. Moreover, we recall that $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ can be constructed by taking the formal linear hull of all pure tensors $v \otimes w$ and taking the quotient by the subspace generated by the necessary relations to enforce bilinearity of the map

$$\mathcal{V} \times \mathcal{W} \ni (v, w) \longmapsto v \otimes w \in \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W};$$

we refer to Hungerford [34, Sec. IV.5] for the details and the general setting of tensor products of modules. We will use this construction to give a canonical definition of the inner product on $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$, and define $\mathcal{V} \otimes \mathcal{W}$ as its completion. After the following abstract and coordinate-free construction, we will discuss a more concrete viewpoint in Corollary 5.10.

Proof of Proposition 5.7. We want to define $\mathcal{V} \otimes \mathcal{W}$ as the completion of the tensor product in the sense of linear algebra $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ with respect to some inner product, which we now construct.

Fix some $v_1 \in \mathcal{V}$ and $w_1 \in \mathcal{W}$. Then the map

$$\mathcal{V} \times \mathcal{W} \ni (v, w) \longmapsto \langle v, v_1 \rangle \langle w, w_1 \rangle \in \mathbb{C}$$

is bilinear, and so by the universal property of the tensor product there exists a unique linear functional $L_{(v_1,w_1)} \colon \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W} \ni t \mapsto L_{(v_1,w_1)}(t) \in \mathbb{C}$ with

$$L_{(v_1,w_1)}(v \otimes w) = \langle v, v_1 \rangle \langle w, w_1 \rangle \tag{5.3}$$

for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$.

We now take the conjugate and note that the expression $\overline{L_{(v_1,w_1)}(t)}$ depends bilinearly on (v_1, w_1) for any fixed $t \in \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$. Indeed, for a pure tensor $t = v \otimes w$ this follows from (5.3), and the general case follows by taking sums of pure tensors. In other words, $\mathcal{V} \times \mathcal{W} \ni (v_1, w_1) \mapsto \overline{L_{(v_1,w_1)}}$ is bilinear and has values in the complex vector space

$$\mathcal{Z} = \{S \colon \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W} \to \mathbb{C} \mid S \text{ is semi-linear}\}.$$

By the universal property of the tensor product, it follows that there exists a linear map $I_p \colon \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W} \to \mathcal{Z}$ with $I_p(v_1 \otimes w_1) = \overline{L_{(v_1,w_1)}}$ and so, by (5.3),

$$I_p(v_1 \otimes w_1)(v_2 \otimes w_2) = \overline{L_{(v_1,w_1)}(v_2, w_2)} = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle$$

for all $v_1, v_2 \in \mathcal{V}$ and $w_1, w_2 \in \mathcal{W}$.

We claim that

$$\langle t_1, t_2 \rangle_{\otimes} = I_p(t_1)(t_2)$$

defines an inner product for tensors $t_1, t_2 \in \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$, which will allow us to define $\mathcal{V} \otimes \mathcal{W}$ as its completion. From the construction (and in particular, from the definition of $\mathcal{Z}$), it is clear that $I_p(t_1)(t_2)$ depends linearly on $t_1 \in \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ and semi-linearly on $t_2 \in \mathcal{V} \otimes \mathcal{W}$. For the conjugate symmetry we note that for pure tensors $v_1 \otimes w_1, v_2 \otimes w_2 \in \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ we have

$$\overline{I_p(v_1 \otimes w_1)(v_2 \otimes w_2)} = \langle v_2, v_1 \rangle \langle w_2, w_1 \rangle = I_p(v_2 \otimes w_2)(v_1 \otimes w_1).$$

By sesqui-linearity, this extends to the identity

$$\overline{\langle t_1, t_2 \rangle_\otimes} = \overline{I_p(t_1)(t_2)} = I_p(t_2)(t_1) = \langle t_2, t_1 \rangle_\otimes$$

for all tensors $t_1, t_2 \in \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W} = \langle v \otimes w \mid v \in \mathcal{V}, w \in \mathcal{W} \rangle_{\mathbb{C}}$.

For definiteness, let $t \in \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ be non-zero and write t as a sum

$$t = \sum_\ell v_\ell \otimes w_\ell$$

for some $v_1, \ldots, v_L \in \mathcal{V}$, and $w_1, \ldots, w_L \in \mathcal{W}$. Now choose an orthonormal basis $(e_1, \ldots, e_J)$ of $\langle v_\ell \mid \ell = 1, \ldots, L \rangle_{\mathbb{C}} \subseteq \mathcal{V}$ and an orthonormal basis $(f_1, \ldots, f_K)$ of $\langle w_\ell \mid \ell = 1, \ldots, L \rangle_{\mathbb{C}} \subseteq \mathcal{W}$ for some $J, K \in \mathbb{N}$. Expressing each v_ℓ (resp. w_ℓ) in terms of this orthonormal basis, putting these into the expression for t, and expanding bilinearly, we obtain

$$t = \sum_{j,k} t_{j,k} e_j \otimes f_k$$

with $t_{j,k} \in \mathbb{C}$ for $1 \leqslant j \leqslant J$ and $1 \leqslant k \leqslant K$. If all of these coefficients vanish, then clearly $t = 0$. If not, we use sesqui-linearity and the construction of $\langle \cdot, \cdot \rangle_\otimes$ to obtain

$$\begin{aligned} \langle t, t \rangle_\otimes &= \sum_{\substack{j_1, k_1 \\ j_2, k_2}} t_{j_1, k_1} \overline{t_{j_2, k_2}} \left\langle e_{j_1} \otimes f_{k_1}, e_{j_2} \otimes f_{k_2} \right\rangle_\otimes \\ &= \sum_{\substack{j_1, k_1 \\ j_2, k_2}} t_{j_1, k_1} \overline{t_{j_2, k_2}} \left\langle e_{j_1}, e_{j_2} \right\rangle \left\langle f_{k_1}, f_{k_2} \right\rangle = \sum_{j,k} |t_{j,k}|^2 > 0. \end{aligned}$$

Having shown that $\langle \cdot, \cdot \rangle_\otimes$ is an inner product on $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$, we define $\mathcal{V} \otimes \mathcal{W}$ as the completion of $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ with respect to the norm $\|\cdot\|_\otimes$ induced by $\langle \cdot, \cdot \rangle_\otimes$. This shows the existence.

To see that the properties of $\mathcal{V} \otimes \mathcal{W}$ imply that $\langle v \otimes w \mid v \in \mathcal{V}, w \in \mathcal{W} \rangle_{\mathbb{C}}$ is isomorphic to $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$, note that the universal property of $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ applied to the bilinear map $\otimes \colon \mathcal{V} \times \mathcal{W} \to \mathcal{V} \otimes \mathcal{W}$ provides a linear map

$$L \colon \mathcal{V} \otimes_{\mathrm{la}} \mathcal{W} \to \langle v \otimes w \mid v \in \mathcal{V}, w \in \mathcal{W} \rangle_{\mathbb{C}}$$

with $L(v \otimes w) = v \otimes w$, which is an isometric bijection when $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ is given the inner product $\langle \cdot, \cdot \rangle_\otimes$ constructed above. (This also implies uniqueness; see Exercise 5.8.)

It remains to show continuity of the tensor product map, since this will then imply that $\{v \otimes w \mid v \in \mathcal{V}, w \in \mathcal{W}\}$ is separable, and hence its closed linear hull $\mathcal{V} \otimes \mathcal{W}$ is also separable. So suppose that $v_n \to v$ in $\mathcal{V}$ and $w_n \to w$ in $\mathcal{W}$ as $n \to \infty$. Then

$$\begin{aligned}\|v_n \otimes w_n - v \otimes w\|_{\otimes} &\leqslant \|v_n \otimes w_n - v \otimes w_n\|_{\otimes} + \|v \otimes w_n - v \otimes w\|_{\otimes}\\ &= \|v_n - v\|\|w_n\| + \|v\|\|w_n - w\| \longrightarrow 0\end{aligned}$$

as $n \to \infty$, as required. □

Exercise 5.8. Show that the properties (1) to (3) in Proposition 5.7 uniquely determine $\mathcal{V} \otimes \mathcal{W}$ up to isomorphism.

Exercise 5.9. (a) Let $v_1, v_2 \in \mathcal{V}$ be linearly independent and $w_1, w_2 \in \mathcal{W}$ be linearly independent. Show that an element of the form $v_1 \otimes w_1 + v_2 \otimes w_2$ is not of the form $v \otimes w$ for any $v \in \mathcal{V}$ and $w \in \mathcal{W}$.
(b) Show that $\mathcal{V} \otimes \mathcal{W}'$ is canonically isomorphic to the space $\mathrm{HS}(\mathcal{W}, \mathcal{V})$ of Hilbert–Schmidt operators from $\mathcal{W}$ to $\mathcal{V}$ (see [21, Ex. 6.53]).
(c) Show that if both $\mathcal{V}$ and $\mathcal{W}$ are infinite dimensional, then $\mathcal{V} \otimes_{\mathrm{la}} \mathcal{W}$ is a proper subspace of $\mathcal{V} \otimes \mathcal{W}$.

Corollary 5.10 (Tensor products of L^2-spaces). *Let μ and ν be locally finite measures on X respectively on Y. Then the tensor product*

$$L^2_\mu(X) \otimes L^2_\nu(Y)$$

is canonically isomorphic to $L^2_{\mu\times\nu}(X \times Y)$. Under this isomorphism the tensor product $f_X \otimes f_Y$ of $f_X \in L^2_\mu(X)$ and $f_Y \in L^2_\nu(Y)$ corresponds to the function $f_X \otimes f_Y(x, y) = f_X(x) f_Y(y)$ for $(x, y) \in X \times Y$. Moreover, for closed subspaces $\mathcal{V} \subseteq L^2_\mu(X)$ and $\mathcal{W} \subseteq L^2_\nu(Y)$ the tensor product $\mathcal{V} \otimes \mathcal{W}$ is canonically isomorphic to the closed subspace of $L^2_{\mu\times\nu}(X \times Y)$ generated by elements $f_X \otimes f_Y$ for $f_X \in \mathcal{V}$ and $f_Y \in \mathcal{W}$.

In particular, given bases $\big(e_j \mid j \in J\big)$ and $(f_k \mid k \in K)$ of separable Hilbert spaces $\mathcal{V}$ and $\mathcal{W}$ respectively, with $J, K \subseteq \mathbb{N}$, we obtain an isomorphism

$$\mathcal{V} \otimes \mathcal{W} \cong \ell^2(J) \otimes \ell^2(K) = \ell^2(J \times K).$$

In other words, the family $(e_j \otimes f_k \mid j \in J, k \in K)$ forms an orthonormal basis of $\mathcal{V} \otimes \mathcal{W}$.

Proof. Let $B\colon L^2_\mu(X) \times L^2_\nu(Y) \to L^2_{\mu\times\nu}(X \times Y)$ be defined by

$$B(f_X, f_Y)(x, y) = f_X(x) f_Y(y)$$

for $f_X \in L^2_\mu(X)$, $f_Y \in L^2_\nu(Y)$ and $(x, y) \in X \times Y$. By Fubini's theorem we have $B(f_X, f_Y) \in L^2_{\mu\times\nu}(X \times Y)$ and $\|B(f_X, f_Y)\|_2 = \|f_X\|_2 \|f_Y\|_2$. Since B is bilinear, it induces a linear map

$$\imath\colon L^2_\mu(X) \otimes_{\mathrm{la}} L^2_\nu(Y) \longrightarrow L^2_{\mu\times\nu}(X \times Y)$$

with $\imath(f_X \otimes f_Y) = B(f_X, f_Y)$ for all $f_X \in L^2_\mu(X)$ and $f_Y \in L^2_\nu(Y)$. Moreover, by Proposition 5.7 and Fubini's theorem again, we have

$$\begin{aligned}\left\langle f_X \otimes f_Y, \widetilde{f_X} \otimes \widetilde{f_Y} \right\rangle_\otimes &= \left\langle f_X, \widetilde{f_X} \right\rangle \left\langle f_Y, \widetilde{f_Y} \right\rangle \\ &= \int_X f_X \overline{\widetilde{f_X}} \,\mathrm{d}\mu \int_Y f_Y \overline{\widetilde{f_Y}} \,\mathrm{d}\nu \\ &= \int_{X\times Y} f_X(x) f_Y(y) \overline{\widetilde{f_X}(x) \widetilde{f_Y}(y)} \,\mathrm{d}\mu(x) \,\mathrm{d}\nu(y) \\ &= \left\langle \imath(f_X \otimes f_Y), \imath(\widetilde{f_X} \otimes \widetilde{f_Y}) \right\rangle_{L^2(X\times Y)}\end{aligned}$$

for all $f_X, \widetilde{f_X} \in L^2_\mu(X)$ and $f_Y, \widetilde{f_Y} \in L^2_\nu(Y)$. For any tensor

$$t \in L^2_\mu(X) \otimes_{\mathrm{la}} L^2_\nu(Y)$$

we write $t = \sum_\ell v_\ell \otimes w_\ell$ and use the above to obtain

$$\begin{aligned}\langle t, t\rangle_\otimes &= \sum_{\ell_1,\ell_2} \left\langle v_{\ell_1} \otimes w_{\ell_1}, v_{\ell_2} \otimes w_{\ell_2} \right\rangle_\otimes \\ &= \sum_{\ell_1,\ell_2} \left\langle \imath(v_{\ell_1} \otimes w_{\ell_1}), \imath(v_{\ell_2} \otimes w_{\ell_2}) \right\rangle_{L^2_{\mu\times\nu}(X\times Y)} = \langle \imath(t), \imath(t)\rangle_{L^2_{\mu\times\nu}(X\times Y)}.\end{aligned}$$

Thus $\imath$ is an isometry and it extends from $L^2_\mu(X) \otimes_{\mathrm{la}} L^2_\nu(Y)$ to its completion $L^2_\mu(X) \otimes L^2_\nu(Y)$. Since the image of the unique extension is complete, and contains all characteristic functions of the form $\mathbb{1}_{B_X \times B_Y}$ for finite measure sets $B_X \subseteq X$ and $B_Y \subseteq Y$, it follows that $\imath$ is an isomorphism.

The argument above also applies to closed subspaces $\mathcal{V}$ and $\mathcal{W}$ of $L^2_\mu(X)$ and $L^2_\nu(Y)$ respectively, and defines an isomorphism between $\mathcal{V} \otimes \mathcal{W}$ and a closed subspace of $L^2_{\mu\times\nu}(X \times Y)$.

Recalling that $\ell^2(J)$ for a finite or countably infinite index set J is equal to $L^2(J)$ with respect to the counting measure, the final claim follows from the above. □

With the exception of not being canonical, the final claim in Corollary 5.10 gives a convenient way of thinking about the tensor product of Hilbert spaces. Fix a basis $(e_j \mid j \in J)$ of $\mathcal{V}$ and a basis $(f_k \mid k \in K)$ of $\mathcal{W}$ for some finite or countable index sets J and K. Then let us write $e_j \otimes f_k$ for the basis vector in $\ell^2(J \times K)$ corresponding to the index $(j, k) \in J \times K$. This gives the identification

$$\mathcal{V} \otimes \mathcal{W} \cong \ell^2(J) \otimes \ell^2(K) = \ell^2(J \times K),$$

and for $v = \sum_j \alpha_j e_j$ and $w = \sum_k \beta_k f_k$ the tensor product map

$$v \otimes w = \sum_{j,k} \alpha_j \beta_k e_j \otimes f_k. \tag{5.4}$$

Exercise 5.11. Show that for convergent series $v = \sum_{m=1}^\infty v_m$ in $\mathcal{V}$ and $w = \sum_{n=1}^\infty w_n$ in $\mathcal{W}$, we have that $v \otimes w = \sum_{m=1}^\infty \sum_{n=1}^\infty v_m \otimes w_n$ converges also.

Corollary 5.12 (Tensor operators). *Let $A\colon \mathcal{V}\to\mathcal{V}$ and $B\colon \mathcal{W}\to\mathcal{W}$ be bounded operators. Then there exists a uniquely determined bounded operator*

$$A\otimes B\colon \mathcal{V}\otimes\mathcal{W}\to\mathcal{V}\otimes\mathcal{W}$$

with

$$A\otimes B(v\otimes w)=Av\otimes Bw \tag{5.5}$$

for $v\in\mathcal{V}$ and $w\in\mathcal{W}$. Moreover,

- $\|A\otimes B\|_{\mathrm{op}}=\|A\|_{\mathrm{op}}\|B\|_{\mathrm{op}}$*;*
- $(A\otimes B)^*=A^*\otimes B^*$*;*
- *if A and B are self-adjoint, then $A\otimes B$ is self-adjoint;*
- *if A and B are unitary, then $A\otimes B$ is unitary; and*
- $(A\otimes B)(A'\otimes B')=(AA')\otimes(BB')$

for bounded operators $A'\colon\mathcal{V}\to\mathcal{V}$ and $B'\colon\mathcal{W}\to\mathcal{W}$.

Proof. Let (f_k) be a basis of $\mathcal{W}$, where we implicitly let k run over all $k\in\mathbb{N}$ with $k\leqslant\dim\mathcal{W}$. We define

$$A\otimes I\colon\mathcal{V}\otimes\mathcal{W}\longrightarrow\mathcal{V}\otimes\mathcal{W}$$

by using the isomorphism[†]

$$\mathcal{V}^{\dim\mathcal{W}}\ni(v_k)_k\longmapsto\sum_k v_k\otimes f_k\in\bigoplus_k\mathcal{V}\otimes f_k=\mathcal{V}\otimes\mathcal{W} \tag{5.6}$$

and applying A in each component. More formally, we use the isomorphism in (5.6) and its inverse to make the definition

$$(A\otimes I)\bigl(\sum_k v_k\otimes f_k\bigr)=\sum_k(Av_k)\otimes f_k$$

for any sequence $(v_k)\in\mathcal{V}^{\dim\mathcal{W}}$ with $\sum_k\|v_k\|^2<\infty$. We claim that this indeed defines a bounded operator satisfying $\|A\otimes I\|_{\mathrm{op}}=\|A\|_{\mathrm{op}}$. In fact, for any $t=\sum_k v_k\otimes f_k\in\mathcal{V}\otimes\mathcal{W}$ we have

$$\begin{aligned}\|(A\otimes I)t\|_\otimes^2&=\Bigl\|\sum_k(Av_k)\otimes f_k\Bigr\|_\otimes^2\\&=\sum_k\|(Av_k)\otimes f_k\|_\otimes^2\\&=\sum_k\|Av_k\|^2\leqslant\|A\|_{\mathrm{op}}^2\sum_k\|v_k\|^2=\|A\|_{\mathrm{op}}^2\|t\|^2,\end{aligned}$$

[†] To see that this is indeed an isomorphism, choose a basis of $\mathcal{V}$ and use the last statement of Corollary 5.10.

which gives $\|A \otimes I\|_{\mathrm{op}} \leqslant \|A\|_{\mathrm{op}}$. Together with

$$\sup_{v \in \mathcal{V}, \|v\| \leqslant 1} \|(A \otimes I)(v \otimes f_k)\|_{\otimes} = \sup_{v \in \mathcal{V}, \|v\| \leqslant 1} \|Av\| = \|A\|_{\mathrm{op}}$$

for any k, we in fact obtain $\|A \otimes I\|_{\mathrm{op}} = \|A\|_{\mathrm{op}}$.

By continuity of $A \otimes I$ and bilinearity and continuity of the tensor product, we also have

$$A \otimes I(v \otimes w) = \sum_k \beta_k \underbrace{(A \otimes I)(v \otimes f_k)}_{(Av \otimes f_k)} = Av \otimes w \tag{5.7}$$

for $v \in \mathcal{V}$ and $w = \sum_k \beta_k f_k \in \mathcal{W}$.

Given another bounded operator $B \colon \mathcal{W} \to \mathcal{W}$ we define $I \otimes B$ similarly, also satisfying $\|I \otimes B\| = \|B\|$ and the relation

$$(I \otimes B)(v \otimes w) = v \otimes Bw$$

for all $v \in \mathcal{V}$ and $w \in \mathcal{W}$. Using (5.7) and the latter identity we obtain

$$(A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I),$$

which suggests the definition

$$A \otimes B = (A \otimes I)(I \otimes B),$$

satisfying (5.5). The remaining properties we leave as an exercise. □

Essential Exercise 5.13. Prove the remaining properties of $A \otimes B$ claimed in Corollary 5.12.

Proposition 5.14 (Outer tensor product). *Given unitary representations π of G and ρ of H, there exists a unitary representation $\pi \otimes \rho$ of $G \times H$, called the* outer *or* Kronecker tensor product representation, *on the tensor product $\mathcal{H}_\pi \otimes \mathcal{H}_\rho$ with the property that*

$$\big\langle (\pi \otimes \rho)_{(g,h)}(v_1 \otimes w_1), v_2 \otimes w_2 \big\rangle = \big\langle \pi_g v_1, v_2 \big\rangle \left\langle \rho_h w_1, w_2 \right\rangle \tag{5.8}$$

for all $g \in G$, $h \in H$, $v_1, v_2 \in \mathcal{H}_\pi$, and $w_1, w_2 \in \mathcal{H}_\rho$.

Proof. We define $(\pi \otimes \rho)_{(g,h)} = \pi_g \otimes \rho_h$ for $g \in G$ and $h \in H$, which satisfies (5.8) due to the definition of $\pi_g \otimes \rho_h$ in (5.5) and the definition of the tensor product in Proposition 5.7. Moreover, Corollary 5.12 also implies that $(\pi \otimes \rho)_{(g,h)}$ is unitary for all $(g, h) \in G \times H$ and that $\pi \otimes \rho$ is a homomorphism. It remains to prove continuity, where we will apply Lemma 1.9 for $D = \{v \otimes w \mid v \in \mathcal{V}, w \in \mathcal{W}\}$. So assume $v \in \mathcal{V}$ and $w \in \mathcal{W}$. Then $G \ni g \mapsto \pi_g v \in \mathcal{V}$ and $H \ni h \mapsto \rho_h w \in \mathcal{W}$ are continuous by assumption, and

$$G \times H \ni (g,h) \longmapsto (\pi_g \otimes \rho_h)(v \otimes w) = \pi_g v \otimes \rho_h w$$

is therefore continuous by Proposition 5.7. This shows the continuity of the representation. □

Exercise 5.15 (Tensor product of cyclic representations). Let π be a cyclic representation of G with generator $v_0 \in \mathcal{H}_\pi$ and let ρ be a cyclic representation of H with generator $w_0 \in \mathcal{H}_\rho$. Show that $v_0 \otimes w_0 \in \mathcal{H}_\pi \otimes \mathcal{H}_\rho$ is a generator for $\pi \otimes \rho$.

In the case of $G = H$ we define the *inner tensor representation* $\pi \otimes \rho$ of G by

$$(\pi \otimes \rho)_g = \pi_g \otimes \rho_g \tag{5.9}$$

for all $g \in G$. We will always make clear whether we are dealing with an inner or an outer tensor product. This concludes the material required for the discussion of unitary representations of compact groups.

Exercise 5.16. Let $K \subseteq G$ be a compact subset. Show that the linear hull of all diagonal and non-diagonal matrix coefficients of all irreducible unitary representations of G restricted to K spans a dense subspace of $C(K)$.

Exercise 5.17. Let G and H be abelian groups as in Chapter 2. Let π be a unitary representation of G, and let ρ be a unitary representation of H.
(a) Describe the spectral measure of $v \otimes w$ for $v \in \mathcal{H}_\pi$ and $w \in \mathcal{H}_\rho$ with respect to the outer tensor product representation $\pi \otimes \rho$ of $G \times H$ on $\mathcal{H}_\pi \otimes \mathcal{H}_\rho$ in terms of the spectral measures of v and w.
(b) Assume that $G = H$ and repeat the above for the inner tensor product representation.

Exercise 5.18. (a) Let $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the isometry group of the plane as in Section 3.3.1. Let $\pi, \rho \in \widehat{G}$ be irreducible unitary representations. Describe when the inner tensor product representation $\pi \otimes \rho$ is again irreducible, and describe the support of $\pi \otimes \rho$.
(b) Repeat the above for the '$ax + b$' group in Section 3.3.2.
(c) Repeat the above for the Heisenberg group in Section 3.3.4.

5.2.2 Irreducibility of Outer Tensor Products

Proposition 5.19 (Irreducibility of tensor products). *Given irreducible unitary representations π of G and ρ of H, the outer tensor product representation $\pi \otimes \rho$ of $G \times H$ on $\mathcal{H}_\pi \otimes \mathcal{H}_\rho$ is again irreducible.*

The following lemma, which is based on Schur's lemma, will be essential for the proof.

Lemma 5.20. *Let π be an irreducible unitary representation of G and $\mathcal{W}$ a Hilbert space. Suppose that $T \in \mathrm{B}(\mathcal{H}_\pi \otimes \mathcal{W})$ is equivariant for the representation $\pi \otimes I$ of G defined by $(\pi \otimes I)_g = \pi_g \otimes I$ for $g \in G$. Then $T = I \otimes B$ for some $B \in \mathrm{B}(\mathcal{W})$.*

Proof. We fix a basis (f_k) of $\mathcal{W}$, and will again use the unitary isomorphism

$$\mathcal{H}_\pi \otimes \mathcal{W} = \bigoplus_k \mathcal{H}_\pi \otimes f_k \cong \mathcal{H}_\pi^{\dim \mathcal{W}}. \tag{5.10}$$

By the properties of tensor operators in Corollary 5.12, this isomorphism is also equivariant between $\pi \otimes I$ and $\pi^{\dim \mathcal{W}}$ on $\mathcal{H}_\pi^{\dim \mathcal{W}}$. For an index $k \in \mathbb{N}$ with $k \leqslant \dim \mathcal{W}$, we write

$$\imath_k \colon \mathcal{H}_\pi \ni v \longmapsto v \otimes f_k \in \mathcal{H}_\pi \otimes \mathcal{W}$$

for the equivariant embedding into the kth subspace $\mathcal{H}_\pi \otimes f_k \subseteq \mathcal{H}_\pi \otimes \mathcal{W}$, and

$$P_k \colon \mathcal{H}_\pi \otimes \mathcal{W} \ni t = \sum_\ell v_\ell \otimes f_\ell \longmapsto v_k$$

for the equivariant projection.

For $T \in \mathrm{B}(\mathcal{H}_\pi \otimes \mathcal{W})$ as in the lemma and indices $k, \ell \leqslant \dim \mathcal{W}$ we then have an equivariant operator

$$T_{k,\ell} = P_k \circ T \circ \imath_\ell \colon \mathcal{H}_\pi \longrightarrow \mathcal{H}_\pi,$$

which by Schur's lemma (Theorem 1.27) must equal $b_{k,\ell} I$ for some $b_{k,\ell} \in \mathbb{C}$, where I denotes the identity on $\mathcal{H}_\pi$. For a unit vector $v \in \mathcal{H}_\pi$ and the basis vector f_ℓ, this shows that

$$\begin{aligned} T(v \otimes f_\ell) &= \sum_k \underbrace{P_k\bigl(T(v \otimes f_\ell)\bigr)}_{=T_{k,\ell}(v)} \otimes f_k & (5.11)\\ &= \sum_k v \otimes (b_{k,\ell} f_k) \\ &= v \otimes \sum_k b_{k,\ell} f_k. & (5.12) \end{aligned}$$

In fact the sum in (5.11) over $k \in \mathbb{N}$ converges due to the unitary isomorphism in (5.10), and (5.12) follows since

$$\mathcal{W} \ni w \longmapsto v \otimes w \in v \otimes \mathcal{W} \tag{5.13}$$

is an isometry.

Said differently, the above shows that $T(v \otimes \mathcal{W}) \subseteq v \otimes \mathcal{W}$ for any unit vector $v \in \mathcal{H}_\pi$. Using the isometry in (5.13) and its inverse, it follows that there exists a bounded operator $B_v \in \mathrm{B}(\mathcal{W})$ with

$$T(v \otimes w) = v \otimes B_v w \tag{5.14}$$

for all $w \in \mathcal{W}$.

Let $e_1, e_2, \dots \in \mathcal{H}_\pi$ be an orthonormal basis, let $j, k \leqslant \dim \mathcal{H}_\pi$ with $j \neq k$ be two indices, and define $v = \frac{1}{\sqrt{2}}(e_j + e_k)$. Then for $w \in \mathcal{W}$ we have

$$\begin{aligned} e_j \otimes B_{e_j} w + e_k \otimes B_{e_k} w &= T(e_j \otimes w + e_k \otimes w) \\ &= T\Big(\underbrace{\tfrac{1}{\sqrt{2}}(e_j + e_k)}_{v} \otimes \sqrt{2} w\Big) \\ &= \tfrac{1}{\sqrt{2}}(e_j + e_k) \otimes B_v(\sqrt{2} w) \\ &= e_j \otimes B_v(w) + e_k \otimes B_v(w), \end{aligned}$$

which implies that $B = B_{e_j} = B_{e_k}$ is independent of the basis vector. Using the fact that the linear hull of the subspaces $e_j \otimes \mathcal{W}$ for $j \leqslant \dim \mathcal{H}_\pi$ is dense and T is bounded, we obtain $T = I \otimes B$. □

With this lemma, we are ready to prove the proposition.

PROOF OF PROPOSITION 5.19. Suppose that $T \in \mathrm{B}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho)$ is equivariant for the unitary representation $\pi \otimes \rho$ of $G \times H$. Restricting the unitary representation to G, we may apply Lemma 5.20 and obtain $T = I \otimes B$ for a bounded operator B on $\mathcal{H}_\rho$. Similarly, we obtain $T = A \otimes I$ for a bounded operator A on $\mathcal{H}_\pi$. We claim that this implies $T = sI$ for some $s \in \mathbb{C}$. Applying this to the equivariant projection operator $A = P_{\mathcal{V}}$ corresponding to a closed $\pi \otimes \rho$-invariant subspace $\mathcal{V} \subseteq \mathcal{H}_\pi \otimes \mathcal{H}_\rho$, it follows that either $\mathcal{V} = \{0\}$ or $\mathcal{V} = \mathcal{H}_\pi \otimes \mathcal{H}_\rho$ and so $\pi \otimes \rho$ is irreducible.

To see the claim, suppose first that $v \in \mathcal{H}_\pi$ and $w \in \mathcal{H}_\rho$ are unit vectors and extend these to orthonormal bases (e_j) of $\mathcal{H}_\pi$ with $e_1 = v$ and (f_k) of $\mathcal{H}_\rho$ with $f_1 = w$. By the last claim in Corollary 5.10 the vectors $e_j \otimes f_k$ for $(j, k) \in J \times K$ form an orthonormal basis of $\mathcal{H}_\pi \otimes \mathcal{H}_\rho$. Let $(j, k) \neq (1, 1)$. If $j \neq 1$ then we use $T = I \otimes B$ to obtain

$$\langle T(v \otimes w), e_j \otimes f_k \rangle = \langle v \otimes Bw, e_j \otimes f_k \rangle = \underbrace{\langle v, e_j \rangle}_{=0} \langle Bw, f_k \rangle = 0.$$

Similarly, if $k \neq 1$ then we use $T = A \otimes I$ to obtain that $T(v \otimes w)$ is orthogonal to $e_j \otimes f_k$. As this holds for any $(j, k) \neq (1, 1)$ it follows that

$$T(v \otimes w) = s_{v,w} v \otimes w$$

for some $s_{v,w} \in \mathbb{C}$ depending on the unit vectors $v \in \mathcal{H}_\pi$ and $w \in \mathcal{H}_\rho$. Let us now fix an orthonormal basis $e_1, e_2, \dots$ of $\mathcal{H}_\pi$ and an orthonormal basis $f_1, f_2, \dots$ of $\mathcal{H}_\rho$. By the above, there exists for every pair (j, k) of indices some $s_{j,k} \in \mathbb{C}$ with $T(e_j \otimes f_k) = s_{j,k} e_j \otimes f_k$. Using also $v = \frac{1}{\sqrt{2}}(e_j + e_{j'})$ for a second index $j' \leqslant \dim \mathcal{H}_\pi$, we obtain

$$s_{j,k} e_j \otimes f_k + s_{j',k} e_{j'} \otimes f_k = \sqrt{2} T\Big(\underbrace{\tfrac{1}{\sqrt{2}}(e_j + e_{j'})}_{v} \otimes f_k\Big) = s_{v,e_k}(e_j + e_{j'}) \otimes f_k$$

which implies that $s_{j,k} = s_{j',k} = s_k$ is independent of the first index. The argument for the second index is similar. Hence $T(e_j \otimes f_k) = se_j \otimes f_k$ for a fixed $s \in \mathbb{C}$ and any pair (j,k) of indices. Since the vectors $e_j \otimes f_k$ for all pairs (j,k) form a basis, it follows that $T = sI$, as claimed. □

The converse to Proposition 5.19 does not hold for all pairs of groups, but does hold if at least one of the two groups is somewhat reasonable (the groups with unreasonable unitary dual considered in Chapter 3 are also unreasonable in this sense).

Proposition 5.21 (Partial characterization for products). *Let τ be an irreducible unitary representation of $G\times H$. Suppose that there exists a unitary representation $\pi \in \widehat{G}$ and $K \in \mathbb{N}\cup\{\infty\}$ such that for the restriction of τ to G we have $\mathcal{H}_\tau \cong \mathcal{H}_\pi^K$. Then there exists an irreducible representation $\rho \in \widehat{H}$ so that τ is isomorphic to the outer tensor product representation $\pi \otimes \rho$.*

Proof. Let $\mathcal{W}$ be a K-dimensional Hilbert space so that

$$\mathcal{H}_\tau \cong \mathcal{H}_\pi^K \cong \mathcal{H}_\pi \otimes \mathcal{W}$$

as in (5.10). We may therefore assume that $\mathcal{H}_\tau = \mathcal{H}_\pi \otimes \mathcal{W}$, and that τ restricted to G is equal to $\pi \otimes I$. Note that τ_h for $h \in H$ is equivariant for $\tau|_G = \pi \otimes I$. By Lemma 5.20 this implies for $h \in H$ that $\tau_h = I \otimes \rho_h$ for an operator $\rho_h \in \mathrm{B}(\mathcal{W})$. Since $\tau|_H$ is a unitary representation and $\mathcal{W} \cong v\otimes\mathcal{W}$ for any unit vector $v \in \mathcal{H}_\pi$, it follows that ρ is a unitary representation of H on $\mathcal{H}_\rho = \mathcal{W}$. Thus τ is the outer tensor product $\pi \otimes \rho$ on $\mathcal{H}_\tau = \mathcal{H}_\pi \otimes \mathcal{H}_\rho$. Irreducibility of τ now also implies that ρ must be irreducible. □

To summarize, Proposition 5.19 shows that the outer tensor product of irreducible unitary representations of G and H give rise to irreducible unitary representations of $G\times H$. Conversely, by Proposition 5.21 irreducible unitary representations of $G \times H$ arise in this way if the restriction to G is easily described with *one* irreducible representation of G. It may feel like the latter should always hold, since different irreducible representations should give rise to invariant subspaces, but this line of argument only works if it is somehow possible to separate different elements of $\widehat{G}$ by, for example, equivariant operators.

Example 5.22. Let $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the group of isometries considered in Section 3.3.1, and let H simply satisfy our standing assumptions. We will show that any irreducible unitary representation τ of $G \times H$ is isomorphic to the outer tensor product $\pi \otimes \rho$ for some $\pi \in \widehat{G}$ and $\rho \in \widehat{H}$.

So let $\tau \in \widehat{G \times H}$. To prove the assumption of Proposition 5.21, we apply the arguments to Proposition 3.13. Indeed, if the restriction of τ to $\mathbb{R}^2 \lhd G$ is trivial, then τ induces an irreducible unitary representation of $\mathrm{SO}_2(\mathbb{R}) \times H$ and $\tau = \chi_n \otimes \rho$ for a character $\chi_n \in \widehat{\mathrm{SO}_2(\mathbb{R})}$ and a representation $\rho \in \widehat{H}$, by Example 1.33.

So suppose the restriction of τ to $\mathbb{R}^2$ is non-trivial. We again define the radius function $R\colon \mathbb{R}^2 \to [0,\infty)$, which is invariant under the action of $\mathrm{SO}_2(\mathbb{R})$ on $\mathbb{R}^2$. As in the proof of Proposition 3.13, this shows that the operator $\tau_{\mathrm{FC}}(R)$ is equivariant for $\tau|_G$. By the properties of the measurable functional calculus (Proposition 2.55(5) applied to τ_h for $h \in H$), $\tau_{\mathrm{FC}}(R)$ is also equivariant under $\tau|_H$. Hence Schur's lemma (Theorem 1.27) applies again, and we see that $\tau_{\mathrm{FC}}(R) = rI$ for some $r > 0$.

Let $\mathcal{W}_n < \mathcal{H}_\tau$ denote the eigenspace for $\tau|_{\mathrm{SO}_2(\mathbb{R})}$ and a character χ_n for some $n \in \mathbb{Z}$. Also let $D\colon \mathbb{R}^2 \to \mathbb{S}^1 \subseteq \mathbb{C}$ denote the direction function as in the proof of Proposition 2.55. Then $\tau_{\mathrm{FC}}(D)$ is a unitary operator sending $\mathcal{W}_n$ onto $\mathcal{W}_{n-1}$ for all $n \in \mathbb{Z}$. Therefore $K = \dim \mathcal{W}_n$ is independent of $n \in \mathbb{Z}$. Given a unit vector $v \in \mathcal{W}_0$, we define $\mathcal{V} = \langle \tau_{\mathrm{FC}}(D^n)v \mid n \in \mathbb{Z}\rangle$ and apply the proof of Proposition 2.55 to see that τ restricted to G and $\mathcal{V}$ is isomorphic to the representation π_r. Hence τ restricted to G is isomorphic to π_r^K, and Proposition 5.21 shows that τ is isomorphic to $\pi_r \otimes \rho$ for some $\rho \in \widehat{H}$.

Exercise 5.23. Repeat Example 5.22 for any metabelian group G satisfying the assumptions in Section 3.5.

5.2.3 Compatibility and Continuity of Tensor Products

We now establish compatibility with weak containment, and continuity with respect to the Fell topology, for the tensor product construction.

Proposition 5.24 (Compatibility and continuity). *Let π_1 and π_2 be unitary representations of G and let ρ_1 and ρ_2 be unitary representations of H. If $\pi_1 \prec \pi_2$ and $\rho_1 \prec \rho_2$ then the outer tensor products also satisfy*

$$\pi_1 \otimes \rho_1 \prec \pi_2 \otimes \rho_2.$$

Moreover, the tensor product construction

$$\mathscr{U}(G) \times \mathscr{U}(H) \ni (\pi,\rho) \longmapsto \pi \otimes \rho \in \mathscr{U}(G \times H)$$

is continuous. If $G = H$ then both claims also hold for the inner tensor products.

Proof. Suppose first that π_1 and ρ_1 are cyclic with generators $v \in \mathcal{H}_{\pi_1}$ and $w \in \mathcal{H}_{\rho_1}$ respectively. Then $\pi_1 \prec \pi_2$ and $\rho_1 \prec \rho_2$ imply that, for any compact subsets $K \subseteq G$ and $L \subseteq H$ and any $\varepsilon > 0$, there exist $\phi^{\pi_2} \in \mathcal{P}^1_{\pi_2}$ and $\phi^{\rho_2} \in \mathcal{P}^1_{\rho_2}$ with $\|\varphi^{\pi_1}_v - \phi^{\pi_2}\|_{K,\infty} < \varepsilon$ and $\|\varphi^{\rho_1}_w - \phi^{\rho_2}\|_{L,\infty} < \varepsilon$. We recall that $\varphi^{\pi_1\otimes\rho_1}_{v\otimes w} = \varphi^{\pi_1}_v \otimes \varphi^{\rho_1}_w$ (that is, $\varphi^{\pi_1\otimes\rho_1}_{v\otimes w}(g,h) = \varphi^{\pi_1}_v(g)\varphi^{\rho_1}_w(h)$ for all $(g,h) \in G \times H$), which extends by linearity to $\phi = \phi^{\pi_2} \otimes \phi^{\rho_2} \in \mathcal{P}^1_{\pi_2\otimes\rho_2}$. Multiplying the approximation claims above with $\|\varphi^{\rho_1}_w\|_{L,\infty}$ and $\|\phi^{\pi_2}\|_{K,\infty}$ respectively gives

$$\begin{aligned}\big\|\varphi_{v\otimes w}^{\pi_1\otimes\rho_1}-\phi\big\|_{K\times L,\infty}&\leqslant\big\|\varphi_v^{\pi_1}-\phi^{\pi_2}\big\|_{K,\infty}\|\varphi_w^{\rho_1}\|_{L,\infty}+\big\|\phi^{\pi_2}\big\|_{K,\infty}\big\|\varphi_w^{\rho_1}-\phi^{\rho_2}\big\|_{L,\infty}\\&<2\varepsilon.\end{aligned}$$

We also note that $v\otimes w\in\mathcal{H}_{\pi_1}\otimes\mathcal{H}_{\rho_1}$ is a generator for $\pi_1\otimes\rho_1$ by Exercise 5.15. As the compact subsets and $\varepsilon>0$ were arbitrary, we obtain from Corollary 4.32 that $\pi_1\otimes\rho_1\prec\pi_2\otimes\rho_2$.

If now π_1 (resp. ρ_1) is not cyclic, we write $\pi_1=\bigoplus_{m=1}^{\infty}\pi_{1,m}$ (and, respectively, $\rho_1=\bigoplus_{n=1}^{\infty}\rho_{1,n}$) as a direct sum of cyclic representations. This gives

$$\pi_1\otimes\rho_1=\bigoplus_{m,n=1}^{\infty}\pi_{1,m}\otimes\rho_{1,n}\prec\big(\pi_2\otimes\rho_2\big)^{\infty}\prec\pi_2\otimes\rho_2,$$

and hence the first part of the proposition.

The proof of the second part is similar. Suppose that (π_n) is a sequence in $\mathscr{U}(G)$ converging to $\pi\in\mathscr{U}(G)$, and (ρ_n) is a sequence in $\mathscr{U}(H)$ converging to $\rho\in\mathscr{U}(H)$. We claim that $\pi\otimes\rho\prec\bigoplus_{n=1}^{\infty}\pi_n\otimes\rho_n$.

For the proof of the claim, we first assume that $v\in\mathcal{H}_\pi$ and $w\in\mathcal{H}_\rho$ are generators so that $v\otimes w$ is a generator for $\pi\otimes\rho$ by Exercise 5.15. By Proposition 4.49 we can find, for compact subsets $K\subseteq G$ and $L\subseteq H$, $\varepsilon>0$ and every large enough $n\in\mathbb{N}$, positive-definite functions $\phi^{\pi_n}\in\mathcal{P}^1_{\pi_n}$ and $\phi^{\rho_n}\in\mathcal{P}^1_{\rho_n}$ with $\|\varphi_v^{\pi}-\phi^{\pi_n}\|_{K,\infty}<\varepsilon$ and $\|\varphi_w^{\rho}-\phi^{\rho_n}\|_{L,\infty}<\varepsilon$. As in the first part of the proof, this implies that

$$\|\varphi_{v\otimes w}^{\pi\otimes\rho}-\phi^{\pi_n}\otimes\phi^{\rho_n}\|_{K\times L,\infty}<2\varepsilon.$$

Together with Corollary 4.32, we see that $\pi\otimes\rho\prec\bigoplus_{n=1}^{\infty}\pi_n\otimes\rho_n$ as claimed.

As the claim holds for any sequences (π_n) and (ρ_n) converging to π and ρ respectively, we can also apply it to subsequences. By Corollary 4.42, this shows that $\pi_n\otimes\rho_n\to\pi\otimes\rho$ as $n\to\infty$. If π (resp. ρ) is not cyclic, we can apply the above to all its cyclic subspaces, and conclude the argument as in the first part of the proof.

By restricting the uniform approximation in the weak containment statement or the convergence statement to compact subsets of

$$\varDelta_G=\{(g,g)\mid g\in G\}\subseteq G\times G,$$

we obtain the two statements also for the inner tensor product representations. □

Exercise 5.25. Let G have property (T) and let $\pi\in\widehat{G}$ be a finite-dimensional irreducible representation. Show that for any unitary representation ρ of G we have that $\pi\prec\rho$ implies $\pi<\rho$.

5.3 Structure of Unitary Representations

As mentioned in the introduction to the chapter, compactness of G has important consequences for its unitary representations.

Theorem 5.26 (Finite dimension). *Let π be an irreducible unitary representation of the compact group G. Then $\mathcal{H}_\pi$ has finite dimension.*

For non-compact groups this can fail quite badly. Indeed, Exercise 1.79 shows for the group $\mathrm{SL}_2(\mathbb{R})$ that there are no non-trivial finite-dimensional unitary representations.

Theorem 5.27 (Decomposability[(7)]). *Let ρ be a unitary representation of the compact group G. Then $\mathcal{H}_\rho$ is a direct sum of mutually orthogonal irreducible subspaces. More precisely, for every $[\pi] \in \widehat{G}$ define the linear hull $\mathcal{H}_\rho^{[\pi]}$ of all subspaces $\mathcal{V} \subseteq \mathcal{H}_\rho$ such that $\rho|_{\mathcal{V}} \cong \pi$. Then these subspaces are closed, mutually orthogonal,*

$$\mathcal{H}_\rho = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{H}_\rho^{[\pi]},$$

and ρ restricted to $\mathcal{H}_\rho^{[\pi]}$ is isomorphic to $\mathrm{mult}(\pi,\rho) \in \mathbb{N}_0 \cup \{\infty\}$ *many copies of π. Here the* multiplicity $\mathrm{mult}(\pi,\rho)$ *is uniquely determined by ρ, but the isomorphism between $\mathcal{H}_\rho^{[\pi]}$ and $\mathcal{H}_\pi^{\mathrm{mult}(\pi,\rho)}$ is not.*

We have already seen in the paragraph after Exercise 2.3 (see also Theorem 2.15 and Proposition 2.54) that the regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ does not even contain one irreducible subspace, also showing how Theorem 5.27 is a special property of compact groups.

To lighten the notation, we will sometimes write π as an abbreviation for $[\pi]$, as for example in the notation $\mathrm{mult}(\pi,\rho)$ for $[\pi] \in \widehat{G}$ and an arbitrary unitary representation ρ of G.

5.3.1 Equivariant Maps

For the proof of Theorems 5.26 and 5.27 we will need to construct equivariant maps using the next two lemmas.

Lemma 5.28 (Existence of equivariant operator). *Let π and ρ be unitary representations of the compact group G, and let $A \in \mathrm{B}(\mathcal{H}_\pi, \mathcal{H}_\rho)$ be a bounded operator. Then*

$$\widetilde{A}v = \int_G \rho_g A \pi_{g^{-1}} v \,\mathrm{d}m(g) \tag{5.15}$$

for $v \in \mathcal{H}_\pi$ defines another bounded operator from $\mathcal{H}_\pi$ to $\mathcal{H}_\rho$ that is equivariant.

PROOF. The integral in (5.15) is to be understood weakly, as in our definition of convolution operators. More precisely, for $v \in \mathcal{H}_\pi$ we define $\widetilde{A}v \in \mathcal{H}_\rho$ using the Fréchet–Riesz representation theorem and the formula

$$\langle \widetilde{A}v, w\rangle = \int_G \langle \rho_g A \pi_{g^{-1}} v, w\rangle \,\mathrm{d}m(g)$$

for every $w \in \mathcal{H}_\rho$, where the function $g \mapsto \langle \rho_g A \pi_{g^{-1}} v, w\rangle = \langle A\pi_{g^{-1}} v, \rho_{g^{-1}} w\rangle$ is continuous (by continuity of the unitary representation and the inner product) and satisfies $|\langle \rho_g A \pi_{g^{-1}} v, w\rangle| \leqslant \|A\|\|v\|\|w\|$. Hence $\widetilde{A}$ is a well-defined linear operator with $\|\widetilde{A}\| \leqslant \|A\|$ (see also the argument in Section 1.4.3).

Equivariance of $\widetilde{A}$ now follows quickly from the properties of Haar measure. Indeed,

$$\widetilde{A}\pi_{g_0} v = \int_G \rho_g A \pi_{g^{-1} g_0} v \,\mathrm{d}m(g) = \int_G \rho_{g_0} \rho_k A \pi_{k^{-1}} v \,\mathrm{d}m(k) = \rho_{g_0} \widetilde{A} v$$

for $g_0 \in G$ by the substitution $k = g_0^{-1} g$. □

Lemma 5.29 (Existence of compact equivariant operator). *Let π be a unitary representation of the compact group G. For a given unit vector $u \in \mathcal{H}_\pi$ we define the operator T by*

$$Tv = \int_G \langle v, \pi_g u\rangle \pi_g u \,\mathrm{d}m(g)$$

for all $v \in \mathcal{H}_\pi$ Then T is positive, self-adjoint, equivariant, non-trivial, and compact. Moreover, $u \in (\ker T)^\perp$.

PROOF. The equivariance of T follows from Lemma 5.28, since for $Av = \langle v, u\rangle u$ and $g \in G$ we have

$$\pi_g A \pi_{g^{-1}} v = \langle \pi_{g^{-1}} v, u\rangle \pi_g u = \langle v, \pi_g u\rangle \pi_g v$$

for all $v \in \mathcal{H}_\pi$. Hence $\widetilde{A} = T$.

To prove positivity, let $v \in \mathcal{H}_\pi$ and calculate

$$\langle Tv, v\rangle = \int_G \underbrace{\langle v, \pi_g u\rangle\langle \pi_g u, v\rangle}_{=|\langle v, \pi_g u\rangle|^2} \,\mathrm{d}m(g) \geqslant 0.$$

Positivity also implies that T is self-adjoint, which can also be checked directly.

If now $v \in \ker T$, then $\langle Tv, v\rangle = 0$ and so by the argument above and the continuity of the map $g \mapsto \langle v, \pi_g u\rangle$ it follows that $\langle v, \pi_g u\rangle = 0$ for all $g \in G$. In particular, $u \perp \ker T$ and so $T \neq 0$.

It remains to prove that T is compact, which we will do by approximating T uniformly by operators with finite-dimensional range. For this, note first that $g \mapsto \pi_g u$ is uniformly continuous on G by compactness. Hence, given some $\varepsilon > 0$ there exists a finite measurable partition $\mathcal{P} = \{B_1, \dots, B_n\}$ of G with $\max\{\operatorname{diam} B_j \mid j = 1, \dots, n\}$ small enough to ensure that $g, g_j \in B_j$ implies that $\|\pi_g u - \pi_{g_j} u\| < \varepsilon$. Hence

$$\begin{aligned}\|\langle v, \pi_g u\rangle \pi_g u - \langle v, \pi_{g_j} u\rangle \pi_{g_j} u\| \\ \leqslant |\langle v, \pi_g u\rangle| \|\pi_g u - \pi_{g_j} u\| + |\langle v, \pi_g u - \pi_{g_j} u\rangle| \|\pi_{g_j} u\| \leqslant 2\varepsilon \|v\|\end{aligned}$$

for all $v \in \mathcal{H}_\pi$, since u is a unit vector. Using this partition and the sample points $g_j \in B_j$ for $j = 1, \dots, n$ we define the Riemann sum approximation

$$T_{\mathcal{P}} \colon \mathcal{H}_\pi \ni v \longmapsto T_{\mathcal{P}} v = \sum_{j=1}^{n} \langle v, \pi_{g_j} u\rangle m(B_j) \pi_{g_j} u$$

with values in $\langle \pi_{g_1} u, \dots, \pi_{g_n} u\rangle$. This defines a bounded operator on $\mathcal{H}_\pi$ with

$$\begin{aligned}\|T - T_{\mathcal{P}}\| &= \sup_{\|v\| \leqslant 1} \|Tv - T_{\mathcal{P}} v\| \\ &= \sup_{\|v\| \leqslant 1} \left\| \sum_{j=1}^{n} \int_{B_j} \big(\langle v, \pi_g u\rangle \pi_g u - \langle v, \pi_{g_j} u\rangle \pi_{g_j} u\big) \,\mathrm{d}m(g) \right\| \leqslant 2\varepsilon.\end{aligned}$$

As $T_{\mathcal{P}}$ has finite-dimensional range and $\varepsilon > 0$ was arbitrary, the lemma follows from [21, Lemma 6.7]. □

5.3.2 Proof of Theorems

Proof of Theorem 5.26. Suppose that π is an irreducible unitary representation and that $u \in \mathcal{H}_\pi$ is a unit vector. Applying Lemma 5.29 we find a non-trivial compact equivariant operator $T \in \mathrm{B}(\mathcal{H}_\pi)$. By Schur's lemma (Theorem 1.27), irreducibility implies that $T = \lambda I$ for some $\lambda \in \mathbb{C}$. However, as $T \neq 0$ is a compact operator, we see that $\dim \mathcal{H}_\pi < \infty$ (see, for example, the discussion after [21, Def. 6.2]). □

Proof of Theorem 5.27: Decomposability. Let ρ be a unitary representation of the compact group G and let $u \in \mathcal{H}_\rho$ be a unit vector. Apply Lemma 5.29 again to obtain an equivariant compact operator T so that $u \in (\ker T)^\perp$. Since T is compact and self-adjoint, there exists a (possibly finite) sequence (λ_n) of non-zero eigenvalues of T such that

$$\mathcal{H}_\rho = \bigoplus_{n\geqslant 1} \mathcal{W}_n \oplus \ker T,$$

where $\mathcal{W}_n = \{v \in \mathcal{H}_\rho \mid Tv = \lambda_n v\}$ is the finite-dimensional eigenspace of T with eigenvalue λ_n (see [21, Th. 6.27]). As T is equivariant it follows that each subspace $\mathcal{W}_n$ is ρ-invariant. Indeed $g \in G$ and $v \in \mathcal{W}_n$ implies that

$$T\rho_g v = \rho_g T v = \lambda_n \rho_g v,$$

and so $\rho_g v \in \mathcal{W}_n$. Using induction on the dimension and the fact that every invariant subspace has an invariant complement (see Exercise 1.21) it follows that every finite-dimensional representation, and in particular each of the subspaces $\mathcal{W}_n$ is a finite direct sum of irreducible subspaces. It follows that $\bigoplus_{n\geqslant 1} \mathcal{W}_n$ is at most a countable direct sum of irreducible subspaces. Recall that $u \in \bigoplus_{n\geqslant 1} \mathcal{W}_n$.

To deduce the theorem from this, let $v_1, v_2, \dots$ be an orthonormal basis of $\mathcal{H}_\rho$. Apply the argument above, first to $u = v_1$ to obtain a direct sum $\mathcal{W}$ of irreducible subspaces that contains v_1. Choose the next integer $n \geqslant 2$ with $v_n \notin \mathcal{W}$, project this vector to $\mathcal{W}^\perp$, and define the operator T as above but for the restriction of π to $\mathcal{W}^\perp$ and the normalized projection of v_n to $\mathcal{W}^\perp$. This produces a further collection of mutually orthogonal irreducible subspaces in $\mathcal{W}^\perp$ such that $v_1, \dots, v_n$ belongs to the direct sum $\widetilde{W}$ of $\mathcal{W}$ and these new irreducible subspaces.

Repeating the argument inductively produces mutually orthogonal irreducible subspaces whose direct sum contains all basis vectors. □

To complete the proof of Theorem 5.27 it remains to discuss the subspaces $\mathcal{H}_\rho^{[\pi]}$ and the multiplicity $\operatorname{mult}(\pi, \rho)$ for all $[\pi] \in \widehat{G}$. For this, let us first summarize what we have obtained thus far in a convenient notation. Given the representation ρ we have found (finite-dimensional) irreducible subspaces $\mathcal{V}_1, \mathcal{V}_2, \dots < \mathcal{H}_\rho$ such that $\mathcal{H}_\rho = \bigoplus_{n\geqslant 1} \mathcal{V}_n$. For any $[\pi] \in \widehat{G}$ we denote those subspaces $\mathcal{V}_n$ with $\rho|_{\mathcal{V}_n} \cong \pi$ by $\mathcal{V}_\ell^\pi$ for $\ell = 1, \dots, m([\pi]) = m(\pi)$, where $m(\pi) \in \mathbb{N} \cup \{\infty\}$ is the total number of such subspaces. In this notation, we have shown that

$$\mathcal{H}_\rho = \bigoplus_{[\pi]\in\widehat{G}} \underbrace{\bigoplus_{\ell=1}^{m(\pi)} \mathcal{V}_\ell^\pi}_{=\mathcal{W}^\pi} = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{W}^\pi. \tag{5.16}$$

The following lemma will be useful for the second part of the proof of Theorem 5.27, but is also of independent interest (see also Exercise 5.31).

Lemma 5.30 (Bound on multiplicity). *If ρ is a cyclic representation of the compact group G and*

$$\mathcal{H}_\rho = \bigoplus_{[\pi]\in\widehat{G}} \bigoplus_{\ell=1}^{m(\pi)} \mathcal{V}_\ell^\pi,$$

then $m(\pi) \leqslant \dim \mathcal{H}_\pi$ *for every* $[\pi] \in \widehat{G}$.

Proof. Fix some $[\pi] \in \widehat{G}$. Projecting the generator to $\bigoplus_{\ell=1}^{m(\pi)} \mathcal{V}_\ell^\pi$ it follows that the latter is also cyclic, and we may suppose that

$$\mathcal{H}_\rho = \bigoplus_{\ell=1}^{m(\pi)} \mathcal{V}_\ell^\pi.$$

Let $d = \dim \mathcal{H}_\pi$ and suppose that $m = m(\pi) > d$. Projecting the generator to the orthogonal sum of the first $d+1$ subspaces, we may assume that m is $d+1$ and that $\mathcal{H}_\rho = \mathcal{H}_\pi^{d+1}$ is cyclic with generator $v = (v_1, \dots, v_{d+1})$. Permuting the indices if necessary, it follows that $v_{d+1} = \alpha_1 v_1 + \cdots + \alpha_d v_d$ for some $\alpha_1, \dots, \alpha_d \in \mathbb{C}$. Now define the proper subspace

$$\mathcal{W} = \big\{(w_1, \dots, w_d, \alpha_1 w_1 + \cdots + \alpha_d w_d) \mid w_1, \dots, w_d \in \mathcal{H}_\pi\big\} < \mathcal{H}_\pi^{d+1}$$

and notice that it is invariant under $\rho = \bigoplus_{\ell=1}^{d+1} \pi$ and contains the vector v that generates $\mathcal{H}_\pi^{d+1}$. This contradiction proves the lemma. □

Proof of Theorem 5.27 continued: multiplicities. Let ρ be unitary representation of G. As already done in the theorem, we define for every irreducible representation π the subspace $\mathcal{H}_\rho^{[\pi]}$ as the linear hull over those subspaces $\mathcal{V} \subseteq \mathcal{H}_\rho$ such that $\rho|_{\mathcal{V}} \cong \pi$. Furthermore, let $\mathcal{V}_\ell^\pi$ for $\ell = 1, \dots, m(\pi)$ and $\mathcal{W}^\pi = \bigoplus_{\ell=1}^{m(\pi)} \mathcal{V}_\ell^\pi$ for all $\pi \in \widehat{G}$ be as in (5.16) so that the space $\mathcal{H}_\rho$ is the orthogonal direct sum of the subspaces $\mathcal{W}^\pi$ for $\pi \in \widehat{G}$. We wish to show that $\mathcal{H}_\rho^{[\pi_0]} = \mathcal{W}^{\pi_0}$ for every $[\pi_0] \in \widehat{G}$.

Suppose for a moment that π_1 and π_2 are inequivalent irreducible representations of G and $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{H}_\rho$ are invariant subspaces such that $\rho|_{\mathcal{V}_j}$ is isomorphic to π_j for $j = 1, 2$. Using invariance of $\mathcal{V}_2$ we see first that the orthogonal projection $P\colon \mathcal{H}_\rho \to \mathcal{V}_2$ is equivariant. Composing the projection with the equivariant isometries $\mathcal{H}_{\pi_1} \to \mathcal{V}_1$ and $\mathcal{V}_2 \to \mathcal{H}_{\pi_2}$ gives now an equivariant map $\mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$. By Schur's lemma (Theorem 1.27) it follows that $P|_{\mathcal{V}_1} = 0$ and so $\mathcal{V}_1 \perp \mathcal{V}_2$.

We fix some $[\pi_0] \in \widehat{G}$ and return to our discussion of the subspaces $\mathcal{W}^{\pi_0}$ and $\mathcal{H}_\rho^{[\pi_0]}$. Let $\mathcal{V}_0 \subseteq \mathcal{H}_\rho$ be an irreducible subspace with $\rho|_{\mathcal{V}_0} \cong \pi_0$, let π be an irreducible representation with $\pi \neq [\pi_0]$, and let $\mathcal{V}_\ell^\pi \subseteq \mathcal{W}^\pi$ be one of the subspaces in the definition of $\mathcal{W}^\pi$. Then the argument above shows $\mathcal{V}_0 \perp \mathcal{V}_\ell^\pi$. Varying the subspaces we obtain $\mathcal{H}_\rho^{[\pi_0]} \perp \mathcal{W}^\pi$ and varying $[\pi] \in \widehat{G} \smallsetminus \{[\pi_0]\}$, we obtain $\mathcal{H}_\rho^{[\pi_0]} \subseteq \mathcal{W}^{\pi_0}$, since $\mathcal{H}_\rho$ equals the orthogonal direct sum $\bigoplus_{[\pi]\in\widehat{G}} \mathcal{W}^\pi$.

For the converse inclusion, let $v \in \mathcal{W}^{\pi_0} \smallsetminus \{0\}$ and apply the first part of the theorem to the cyclic representation $\langle v \rangle_\rho \subseteq \mathcal{W}^{\pi_0}$ generated by v. This shows that $\langle v \rangle_\rho = \bigoplus_{k=1}^{K} \mathcal{V}_k'$ for some irreducible subspaces $\mathcal{V}_k' \subseteq \langle v \rangle_\rho$ and K in $\mathbb{N} \cup \{\infty\}$. By the above we also have $\mathcal{H}_\rho^{[\pi]} \subseteq \mathcal{W}^\pi$ for $[\pi] \in \widehat{G} \smallsetminus \{[\pi_0]\}$, which implies that $\rho|_{\mathcal{V}_k'} \cong \pi_0$ for all k. By Lemma 5.30 we now see that $K \leqslant \dim \mathcal{H}_\pi$ and hence that $\langle v \rangle_\rho$ is a finite sum of irreducible subspaces, which implies that $v \in \mathcal{H}_\rho^{[\pi_0]}$. As $v \in \mathcal{W}^\pi \smallsetminus \{0\}$ was arbitrary, we see that $\mathcal{W}^{\pi_0} = \mathcal{H}_\rho^{[\pi_0]}$, and in particular that $\mathcal{H}_\rho^{[\pi_0]}$ is closed.

As $\mathcal{H}_\rho^{[\pi_0]}$ is canonically defined, it follows that

$$\dim \mathcal{H}_\rho^{[\pi_0]} = m(\pi_0) \dim \mathcal{H}_{\pi_0}$$

is independent of the choices that were made to arrive at the splitting of $\mathcal{H}$ into irreducible subspaces $\mathcal{V}_1, \mathcal{V}_2, \ldots$. As $\dim \mathcal{H}_{\pi_0} < \infty$ by Theorem 5.26, this implies that

$$\mathrm{mult}(\pi_0, \rho) = m(\pi_0) = \frac{\dim \mathcal{H}_\rho^{[\pi_0]}}{\dim \mathcal{H}_{\pi_0}}$$

is well-defined.

Finally, note that the isomorphism U_π between $\mathcal{H}_\rho^\pi$ and $\mathcal{H}_\pi^{\mathrm{mult}(\pi,\rho)}$ can be changed in many ways. For instance, U_π can be composed with a permutation of the subspaces if $\mathrm{mult}(\pi, \rho) > 1$ or multiplied by a scalar of absolute value one whenever $\mathcal{H}_\rho^\pi \neq \{0\}$. □

Exercise 5.31. (a) Let π be a finite-dimensional irreducible unitary representation of G. Show that the unitary representation on $\mathcal{H}_\pi^{\dim \mathcal{H}_\pi}$ is cyclic.
(b) Let $I \subseteq \mathbb{N}$ and $[\pi_j] \in \widehat{G}$ for $j \in I$ such that $j \neq k$ in I implies that $\pi_j \not\cong \pi_k$ (equivalently, $[\pi_j] \neq [\pi_k]$). Show that $\bigoplus_{j \in I} \mathcal{H}_\pi^{\dim \mathcal{H}_\pi}$ is cyclic.

Exercise 5.32. Let $G = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{T}$, with group operation defined by

$$(a_1, x_1) \cdot (a_2, x_2) = (a_1 + a_2, x_1 + (-1)^{a_1} x_2).$$

Describe the unitary dual $\widehat{G}$.

5.3.3 Containment

The description of unitary representations in Theorem 5.27 allows us to describe containment quite clearly.

Corollary 5.33 (Characterization of containment). *Let ρ_1 and ρ_2 be unitary representations of the compact group G. Then $\rho_1 < \rho_2$ if and only if for every $[\pi] \in \widehat{G}$ we have $\mathrm{mult}(\pi, \rho_1) \leqslant \mathrm{mult}(\pi, \rho_2)$.*

PROOF. Suppose that $\rho_1 < \rho_2$. Then we may assume that $\mathcal{H}_{\rho_1} \subseteq \mathcal{H}_{\rho_2}$ and $\rho_1 = \rho_2|_{\mathcal{H}_{\rho_1}}$. Applying the decomposition theorem (Theorem 5.27) to $\mathcal{H}_{\rho_1}$ and $\mathcal{H}_{\rho_1}^{\perp} \subseteq \mathcal{H}_{\rho_2}$, it follows that

$$\operatorname{mult}(\pi, \rho_1) \leqslant \operatorname{mult}(\pi, \rho_1) + \operatorname{mult}(\pi, \rho_2|_{\mathcal{H}_{\rho_1}^{\perp}}) = \operatorname{mult}(\pi, \rho_2)$$

for every $[\pi] \in \widehat{G}$.

For the converse, we suppose that $\operatorname{mult}(\pi, \rho_1) \leqslant \operatorname{mult}(\pi, \rho_2)$ for every π in $\widehat{G}$, and obtain from the decomposition theorem for ρ_1 and ρ_2 that

$$\mathcal{H}_{\rho_1} \cong \bigoplus_{[\pi]\in\widehat{G}} \mathcal{H}_{\pi}^{\operatorname{mult}(\pi,\rho_1)} \subseteq \bigoplus_{[\pi]\in\widehat{G}} \mathcal{H}_{\pi}^{\operatorname{mult}(\pi,\rho_2)} \cong \mathcal{H}_{\rho_2}$$

as required. □

5.3.4 Irreducible Representations of Products

In this section we let G denote a compact metric group and H a locally compact, σ-compact metric group. We wish to relate the irreducible representations in $\widehat{G}$ and $\widehat{H}$ to the irreducible representations of $G \times H$ giving the converse to Proposition 5.14.

Proposition 5.34 (The unitary dual of compact products). *Let G be a compact metric group and H a locally compact, σ-compact metric group. Then the irreducible representations of the direct product $G \times H$ are precisely of the form $\pi \otimes \rho$ for $[\pi] \in \widehat{G}$ and $[\rho] \in \widehat{H}$.*

PROOF. The irreducibility of $\pi \otimes \rho$ for $[\pi] \in \widehat{G}$ and $[\rho] \in \widehat{H}$ holds more generally, and was established in Proposition 5.19.

For the converse, we suppose that τ is an irreducible unitary representation of $G \times H$. We restrict τ to the compact group G and apply Theorem 5.27. It follows that

$$\mathcal{H}_\tau = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{H}_\tau^{[\pi]},$$

where $\mathcal{H}_\tau^{[\pi]}$ is the linear hull of all irreducible subspaces $\mathcal{V} < \mathcal{H}_\tau$ isomorphic to $\mathcal{H}_\pi$. Since the direct factors of $G \times H$ commute, it follows for $h \in H$ and a closed subspace $\mathcal{V} < \mathcal{H}_\tau$ that $\tau_h \mathcal{V} < \mathcal{H}_\tau$ is isomorphic to $\mathcal{V}$ with respect to the restriction of τ to G. Therefore $\tau_h \mathcal{H}_\tau^{[\pi]} < \mathcal{H}_\tau^{[\pi]}$ for $[\pi] \in \widehat{G}$ and $h \in H$. By irreducibility of $\mathcal{H}_\tau$ we must have $\mathcal{H}_\tau = \mathcal{H}_\tau^{[\pi]}$ for some $[\pi] \in \widehat{G}$. Let

$$K = \operatorname{mult}(\pi, \tau|_G) \in \mathbb{N} \cup \{\infty\}$$

so we may assume that $\mathcal{H}_\tau \cong \mathcal{H}_\pi^K$ by Theorem 5.27. Applying Proposition 5.21 it follows that there exists an irreducible unitary representation $\rho \in \widehat{H}$ on a K-dimensional Hilbert space so that $\tau = \pi \otimes \rho$. □

5.3.5 The Unitary Assumption

In all of our discussions up to this point (and in many of the following ones) we always start by assuming that the representation in question is unitary. In the context of compact groups this assumption can be weakened significantly.

Proposition 5.35 (Averaging the inner product). *Let $\mathcal{H}$ be a Hilbert space and suppose that π is a continuous representation of the compact group G on $\mathcal{H}$ (so properties* (1) *and* (3) *of Definition* 1.1 *hold, but we do not assume* (2)*). Then*

$$\langle v, w\rangle_\pi = \int \langle \pi_g v, \pi_g w\rangle_{\mathcal{H}} \,\mathrm{d}m(g)$$

for $v, w \in \mathcal{H}$ defines an inner product on $\mathcal{H}$ with the property that the induced norm is equivalent to the norm induced by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and π is a unitary representation with respect to $\langle\cdot,\cdot\rangle_\pi$. In particular, this applies to every continuous finite-dimensional representation.

Proof. Using the continuity of $G \ni g \mapsto \pi_g v$ for $v \in \mathcal{H}$, it is easy to see that $\langle\cdot,\cdot\rangle_\pi$ is well-defined and satisfies the axioms of an inner product. Moreover, also by compactness of G and the continuity requirement of the representation we have that $\{\pi_g v \mid g \in G\} \subseteq \mathcal{H}$ is bounded for every $v \in \mathcal{H}$. As $\mathcal{H}$ is a Hilbert space, the Banach–Steinhaus theorem on uniform boundedness (see [21, Th. 4.1]) applies and shows that $M = \sup\{\|\pi_g\| \mid g \in G\}$ is finite. Notice that $\|\pi_g v\|_{\mathcal{H}} \leqslant M\|v\|_{\mathcal{H}}$, which may be applied to g^{-1} and its action on the vector $w = \pi_g v$ to see that

$$\tfrac{1}{M}\|v\|_{\mathcal{H}} \leqslant \|\pi_g v\|_{\mathcal{H}} \leqslant M\|v\|_{\mathcal{H}}$$

for all $v \in \mathcal{H}$ and $g \in G$.

Let $\|\cdot\|_\pi$ denote the norm induced by $\langle\cdot,\cdot\rangle_\pi$. By integration, we now obtain

$$\tfrac{1}{M^2}\|v\|_{\mathcal{H}}^2 \leqslant \|v\|_\pi^2 = \int \|\pi_g v\|_{\mathcal{H}}^2 \,\mathrm{d}m(g) \leqslant M^2\|v\|_{\mathcal{H}}^2$$

for all v, which shows that $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_\pi$ are equivalent norms.

Finally, note that

$$\|\pi_g v\|_\pi^2 = \int \|\pi_{hg} v\|_{\mathcal{H}}^2 \,\mathrm{d}m(h) = \int \|\pi_k v\|_{\mathcal{H}}^2 \,\mathrm{d}m(k) = \|v\|_\pi^2$$

for all $v \in \mathcal{H}$, showing that π_g is unitary for all $g \in G$ with respect to the inner product $\langle\cdot,\cdot\rangle_\pi$ on $\mathcal{H}_\pi$. As $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_\pi$ are equivalent norms, we see that $\mathcal{H}_\pi = \mathcal{H}$ is still a Hilbert space when equipped with the latter norm and that π still satisfies the continuity requirement, and so gives a unitary representation. □

5.4 The Regular Representation*

Having obtained the complete description of any unitary representation of the compact group G, we now move on to the regular representation λ. Here we will, in particular, also obtain complete knowledge of the multiplicities $\operatorname{mult}(\pi,\lambda)$ of a given irreducible unitary representation π of G. As a first step towards that goal, we study the matrix coefficients of unitary representations (which are continuous functions on G, and in particular are elements of $L^2(G)$ by compactness).

5.4.1 Schur Orthogonality

Definition 5.36. For a unitary representation π of the compact group G we define $\mathcal{M}_\pi = \langle \varphi_{u,v} \mid u,v \in \mathcal{H}_\pi \rangle$ to be the linear hull of all matrix coefficient for vectors in $\mathcal{H}_\pi$.

It is clear that $\mathcal{M}_\pi$ only depends on π up to unitary equivalence and so we may also define $\mathcal{M}_{[\pi]} = \mathcal{M}_\pi$ for any $[\pi] \in \widehat{G}$.

Proposition 5.37 (Dimension bound). *Let π be a unitary representation of the compact group G. Then $\mathcal{M}_\pi$ is invariant under the left- and right-regular representations, is a two-sided ideal in $L^1(G)$ with respect to convolution, and satisfies*

$$\dim \mathcal{M}_\pi \leqslant (\dim \mathcal{H}_\pi)^2.$$

Proof. To see the invariance under the regular representation let $u,v \in \mathcal{H}_\pi$ and $g,h \in G$. Then

$$\lambda_g(\varphi_{u,v})(h) = \varphi_{u,v}(g^{-1}h) = \left\langle \pi_{g^{-1}h}u, v \right\rangle = \left\langle \pi_h u, \pi_g v \right\rangle = \varphi_{u,\pi_g v}(h)$$

and

$$\rho_g(\varphi_{u,v})(h) = \varphi_{u,v}(hg) = \left\langle \pi_{hg}u, v \right\rangle = \left\langle \pi_h \pi_g u, v \right\rangle = \varphi_{\pi_g u,v}(h)$$

show that $\lambda_g \mathcal{M}_\pi \subseteq \mathcal{M}_\pi$ and $\rho_g \mathcal{M}_\pi \subseteq \mathcal{M}_\pi$. For the claim regarding convolution let $f \in L^1(G)$ to see that

$$\begin{aligned} f * \varphi_{u,v}(g) &= \int f(h)\varphi_{u,v}(h^{-1}g)\,\mathrm{d}m(h) \\ &= \int f(h)\left\langle \pi_{h^{-1}g}u, v\right\rangle\,\mathrm{d}m(h) \\ &= \int f(h)\left\langle \pi_g u, \pi_h v\right\rangle\,\mathrm{d}m(h) \\ &= \left\langle \pi_g u, \pi_*(\overline{f})v\right\rangle = \varphi_{u,\pi_*(\overline{f})v}(g) \end{aligned}$$

by definition of the convolution operator $\pi_*(\overline{f})$.

Similarly, we define $\widetilde{f}(k) = f(k^{-1})$ for $k \in G$ and obtain

$$\begin{aligned} \varphi_{u,v} * f(g) &= \int \varphi_{u,v}(h) f(h^{-1}g)\,\mathrm{d}m(h) \\ &= \int \varphi_{u,v}(gk) f(k^{-1})\,\mathrm{d}m(k) \\ &= \int \left\langle \pi_g\pi_k u, v\right\rangle \widetilde{f}(k)\,\mathrm{d}m(k) \\ &= \left\langle \pi_g\pi_*(\widetilde{f})u, v\right\rangle = \varphi_{\pi_*(\widetilde{f})u,v}(g) \end{aligned}$$

by using the substitution $h = gk$, and again the definition of the convolution operator $\pi_*(\widetilde{f})$. It follows that $L^1(G) * \mathcal{M}_\pi \cup \mathcal{M}_\pi * L^1(G) \subseteq \mathcal{M}_\pi$.

Finally, if $w_1, w_2, \dots, w_d \in \mathcal{H}_\pi$ is an orthonormal basis, then sesquilinearity shows that $\mathcal{M}_\pi$ is spanned by φ_{w_i,w_j} for $i, j = 1, \dots, d$. However, if $\mathcal{H}_\pi$ is infinite-dimensional, then there is nothing to prove. □

Theorem 5.38 (Schur orthogonality relations). *Let π and ρ be irreducible unitary representations of the compact group G and consider the subspaces $\mathcal{M}_\pi$ and $\mathcal{M}_\rho$ inside $L^2(G)$. If π and ρ are not unitarily equivalent, then $\mathcal{M}_\pi \perp \mathcal{M}_\rho$. Moreover, if $w_1, \dots, w_{d_\pi}$ is an orthonormal basis of $\mathcal{H}_\pi$ then $\sqrt{d_\pi}\pi_{i,j}$ is an orthonormal basis of $\mathcal{M}_\pi$, where $d_\pi = \dim \mathcal{H}_\pi$ and $\pi_{i,j} = \varphi^\pi_{w_i,w_j}$ for $i, j = 1, \dots, d_\pi$. In particular, $\dim \mathcal{M}_\pi = d_\pi^2$.*

Proof. We will apply Lemma 5.28 for various choices of A. We fix some vector $u_0 \in \mathcal{H}_\pi$, some $v_0 \in \mathcal{H}_\rho$ and consider the map A defined by

$$Au = \langle u, u_0\rangle\, v_0$$

for $u \in \mathcal{H}_\pi$, which gives rise to

$$\widetilde{A}u = \int \left\langle \pi_{g^{-1}}u, u_0\right\rangle \rho_g v_0\,\mathrm{d}m(g).$$

By definition we have

$$\left\langle \widetilde{A}u, v \right\rangle = \int \underbrace{\langle u, \pi_g u_0 \rangle}_{\overline{\varphi^{\pi}_{u_0,u}}} \underbrace{\langle \rho_g v_0, v \rangle}_{\varphi^{\rho}_{v_0,v}} \,\mathrm{d}m(g) = \langle \varphi^{\rho}_{v_0,v}, \varphi^{\pi}_{u_0,u} \rangle$$

for all $u \in \mathcal{H}_\pi$ and $v \in \mathcal{H}_\rho$, where we write $\varphi^{\pi}_{u_0,u}$ for the matrix coefficient of $u_0, u \in \mathcal{H}_\pi$ defined by π and $\varphi^{\rho}_{v_0,v}$ for the matrix coefficient of $v_0, v \in \mathcal{H}_\rho$ defined by ρ.

Suppose now that π and ρ are not unitarily equivalent. Then $\widetilde{A} = 0$ by Schur's lemma (Theorem 1.27) and so the argument above shows that

$$\varphi^{\pi}_{u_0,u} \perp \varphi^{\rho}_{v_0,v}$$

for all $u_0, u \in \mathcal{H}_\pi$ and $v_0, v \in \mathcal{H}_\rho$, or equivalently that $\mathcal{M}_\pi \perp \mathcal{M}_\rho$.

Suppose now that $w_1, \dots, w_{d_\pi}$ is an orthonormal basis of $\mathcal{H}_\pi$, let $\rho = \pi$ in the above discussion, and set $u_0 = w_k$ and $v_0 = w_j$ so that the linear map A has trace $\mathrm{tr}(A) = \delta_{k,j}$. Since $d_\pi = \dim \mathcal{H}_\pi < \infty$ we can obtain $\widetilde{A}$ also via the (matrix valued, Riemann-) integral

$$\widetilde{A} = \int \pi_g A \pi_g^{-1} \,\mathrm{d}m(g).$$

Taking the trace we see that

$$\mathrm{tr}\, \widetilde{A} = \int \mathrm{tr}(\pi_g A \pi_g^{-1}) \,\mathrm{d}m(g) = \mathrm{tr}(A) = \delta_{k,j}.$$

By Schur's lemma (Theorem 1.27), $\widetilde{A} = \lambda I$ on $\mathcal{H}_\pi$ for some $\lambda \in \mathbb{C}$, which gives $\mathrm{tr}\, \widetilde{A} = \lambda d_\pi = \delta_{k,j}$. Hence we see that

$$\widetilde{A} = \tfrac{1}{d_\pi} \delta_{k,j} I.$$

We now apply $\widetilde{A}$ (which we defined using the basis vectors $u_0 = w_k$, respectively $v_0 = w_j$) to $u = w_\ell$ and take the inner product with $v = w_i$ to obtain

$$\tfrac{1}{d_\pi} \delta_{k,j} \delta_{\ell,i} = \left\langle \tfrac{1}{d_\pi} \delta_{k,j} w_\ell, w_i \right\rangle = \left\langle \widetilde{A}u, v \right\rangle = \langle \varphi^{\pi}_{v_0,v}, \varphi^{\pi}_{u_0,u} \rangle = \langle \pi_{i,j}, \pi_{k,\ell} \rangle,$$

where $\pi_{i,j} = \varphi^{\pi}_{w_i,w_j}$ and $\pi_{k,\ell} = \varphi^{\pi}_{w_k,w_\ell}$. Multiplying by d_π gives

$$\left\langle \sqrt{d_\pi}\pi_{i,j}, \sqrt{d_\pi}\pi_{k,\ell} \right\rangle = \delta_{k,i}\delta_{\ell,j}.$$

As in the proof of Proposition 5.37 sesqui-linearity of the matrix coefficients shows that $\mathcal{M}_\pi$ is generated by the vectors $\pi_{i,j}$ for $i, j = 1, \dots, d_\pi$. Since these are orthogonal, it follows that $\dim \mathcal{M}_\pi = d_\pi^2$ and the theorem follows. □

5.4.2 A Dense Algebra

Recall that in the study of compact abelian groups a trigonometric polynomial is a finite linear combination of characters of the group, and that these form a dense sub-algebra of the space of continuous functions on the group with respect to the supremum norm. In the context of compact groups the following is the appropriate generalization, which as we will see will have the same properties.

Definition 5.39 (Matrix coefficient algebra). We define the *matrix coefficient algebra* of the compact group G to be the linear hull

$$\mathcal{M}(G) = \langle \mathcal{M}_\pi \mid [\pi] \in \widehat{G} \rangle.$$

The purpose of this section is to develop the main properties of $\mathcal{M}(G)$ as summarised in the following theorem.

Theorem 5.40 (Dense algebra). *For the compact group G the subspace*

$$\mathcal{M}(G) \subseteq C(G)$$

is a dense sub-algebra with respect to pointwise multiplication.

Our first step is to justify the nomenclature 'algebra' for $\mathcal{M}(G)$.

Lemma 5.41 (Algebra). *For the compact group G we have*

$$\mathcal{M}(G) = \langle \mathcal{M}_\rho \mid \rho \text{ is a unitary representation of } G \text{ with } \dim \mathcal{H}_\rho < \infty \rangle,$$

and that $\mathcal{M}(G)$ is a sub-algebra of $C(G)$, where $\mathcal{M}_\rho$ is defined as in Definition 5.36 *for any unitary representation ρ of G.*

Proof. Let us write $\mathcal{F}$ for the linear span of all matrix coefficients of all finite-dimensional unitary representations of G. We first prove that $\mathcal{M}(G) = \mathcal{F}$ as claimed in the lemma. By Theorem 5.26 we have $\mathcal{M}_\pi \subseteq \mathcal{F}$ for all $\pi \in \widehat{G}$ and so $\mathcal{M}(G) \subseteq \mathcal{F}$. Suppose now that ρ is a finite-dimensional unitary representation of G and let $\mathcal{H}_\rho = \bigoplus_{n=1}^N \mathcal{V}_n$ be the decomposition of $\mathcal{H}_\rho$ into irreducible representations. Given $u = \sum_{n=1}^N u_n$ and $v = \sum_{n=1}^N v_n$ with $u_n, v_n \in \mathcal{V}_n$ for $n = 1, \ldots, N$ we have

$$\varphi^\rho_{u,v} = \sum_{n=1}^N \varphi^\rho_{u_n,v_n} \in \langle \mathcal{M}_{\rho|_{\mathcal{V}_n}} \mid n = 1, \ldots, N \rangle \subseteq \mathcal{M}(G).$$

Since $u, v \in \mathcal{H}_\rho$ and ρ were arbitrary as in the definition of $\mathcal{F}$ we obtain the opposite inclusion $\mathcal{F} \subseteq \mathcal{M}(G)$.

Suppose now that $\varphi^{\rho_1}_{u_1,v_1}$ and $\varphi^{\rho_2}_{u_2,v_2}$ are matrix coefficients for finite-dimensional unitary representations ρ_1 and ρ_2 respectively as in the definition of $\mathcal{F}$. Recalling the construction of the inner tensor product representation $\rho_1 \otimes \rho_2$ from Section 5.1 and especially Proposition 5.14 (which in the case at hand is easier as we are currently only dealing with finite-dimensional representations) we see that the product

$$\varphi^{\rho_1}_{u_1,v_1}(g)\varphi^{\rho_2}_{u_2,v_2}(g) = \langle (\rho_1 \otimes \rho_2)_g u_1 \otimes u_2, v_1 \otimes v_2 \rangle = \varphi^{\rho_1 \otimes \rho_2}_{u_1 \otimes u_2, v_1 \otimes v_2}(g)$$

is again a matrix coefficient for a finite-dimensional representation $\rho_1 \otimes \rho_2$ on $\mathcal{H}_{\rho_1} \otimes \mathcal{H}_{\rho_2}$ for the vectors $u_1 \otimes u_2$ and $v_1 \otimes v_2$. This implies that $\mathcal{M}(G) = \mathcal{F}$ is a sub-algebra of $C(G)$. □

Proof of Theorem 5.40. We are going to apply the Stone–Weierstrass theorem to $\mathcal{M}(G)$. By Lemma 5.41 we know that $\mathcal{M}(G)$ is a sub-algebra of $C(G)$. Using the trivial representation we also see that the constant function $\mathbb{1}$ belongs to $\mathcal{M}(G)$. Moreover, given a matrix coefficient $\varphi^{\rho}_{u,v}$ of a finite-dimensional unitary representation ρ and vectors $u, v \in \mathcal{H}_\rho$, the formula (5.2) shows that $\overline{\varphi^{\rho}_{u,v}}$ is also a matrix coefficient of a finite-dimensional unitary representation, namely the contragredient $\overline{\rho}$. Therefore, $\mathcal{M}(G)$ is closed under conjugation.

Finally, recall from the Gelfand–Raikov theorem (Corollary 1.75) that for any $g_1 \neq g_2$ in G there exists a unitary representation $\pi \in \widehat{G}$ with $\pi_{g_1} \neq \pi_{g_2}$. Hence there exist $u, v \in \mathcal{H}_\pi$ with $\varphi^{\pi}_{u,v}(g_1) \neq \varphi^{\pi}_{u,v}(g_2)$. Hence $\mathcal{M}(G)$ also separates points. By the Stone–Weierstrass theorem, it follows that $\mathcal{M}(G)$ is dense in $C(G)$. □

5.4.3 The Peter–Weyl Theorem

The material of Section 5.3 and our preparations in Sections 5.4.1 and 5.4.2 leads naturally to the following complete description of the regular representation of the compact group.[(8)]

Theorem 5.42 (Peter–Weyl). *The regular representation of the compact group G is isomorphic to the direct sum of irreducible representations where each $\pi \in \widehat{G}$ appears with multiplicity $d_\pi = \dim \mathcal{H}_\pi$. More precisely, we have the decomposition*

$$L^2(G) = \bigoplus_{[\pi] \in \widehat{G}} \mathcal{M}_\pi,$$

and the subspace $\mathcal{M}_\pi$ is invariant for every $[\pi] \in \widehat{G}$. In fact the right-regular representation ρ restricted to $\mathcal{M}_\pi$ is isomorphic to d_π copies of π, and the left-regular representation λ restricted to $\mathcal{M}_\pi$ is isomorphic to d_π copies of the contragredient $\overline{\pi}$ of π.

As we will see, the proof consists of reviewing what we have obtained so far.

Proof of Theorem 5.42. By Theorem 5.38 we know that $\mathcal{M}_{\pi_1} \perp \mathcal{M}_{\pi_2}$ if $\pi_1, \pi_2 \in \widehat{G}$ have $\pi_1 \not\sim \pi_2$. Therefore the direct sum $\bigoplus_{[\pi]\in\widehat{G}} \mathcal{M}_\pi$ is an orthogonal decomposition of a subspace of $L^2(G)$. Moreover, by Theorem 5.40 the linear hull $\mathcal{M} = \langle \mathcal{M}_\pi \mid [\pi] \in \widehat{G}\rangle$ is dense in $C(G)$ and so also in $L^2(G)$ in the L^2 norm, which implies that $L^2(G) = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{M}_\pi$.

Let us first consider the right-regular representation ρ. Fix some $\pi \in \widehat{G}$, some orthonormal basis $w_1, \ldots, w_{d_\pi}$ of $\mathcal{H}_\pi$, where $d_\pi = \dim \mathcal{H}_\pi$, and finally some index $\ell \in \{1, \ldots, d_\pi\}$. Then the map U_ℓ defined by $U_\ell v = \sqrt{d_\pi}\varphi_{v,w_\ell}$ for $v \in \mathcal{H}_\pi$ satisfies

$$\rho_g(U_\ell v)(h) = \sqrt{d_\pi}\varphi_{v,w_\ell}(hg) = \sqrt{d_\pi}\langle \pi_h \pi_g v, w_\ell\rangle = U_\ell(\pi_g v)(h)$$

for all $g, h \in G$ and has $\|U_\ell w_k\| = \|\sqrt{d_\pi}\pi_{k,\ell}\| = 1$ for any $k \in \{1, \ldots, d_\pi\}$ by Schur's orthogonality relations in Theorem 5.38. By Schur's lemma (Theorem 1.27)

$$U_\ell \colon \mathcal{H}_\pi \longrightarrow \operatorname{Im} U_\ell \subseteq \mathcal{M}_\pi$$

is an isometric isomorphism. By Schur's orthogonality relations we also have $\operatorname{Im} U_{\ell_1} \perp \operatorname{Im} U_{\ell_2}$ if $\ell_1 \neq \ell_2$ in $\{1, \ldots, d_\pi\}$ and hence the right-regular representation restricted to $\mathcal{M}_\pi$ is isomorphic to d_π copies of π.

For the left-regular representation we again fix some $k \in \{1, \ldots, d_\pi\}$ and define B_k on $\mathcal{H}_\pi$ by $B_k v = \sqrt{d_\pi}\varphi_{w_k,v}$ satisfying

$$\lambda_g(B_k v)(h) = \sqrt{d_\pi}\varphi_{w_k,v}(g^{-1}h) = \sqrt{d_\pi}\langle \pi_h w_k, \pi_g v\rangle = B_k(\pi_g v)(h)$$

for all $g, h \in G$. However, the map B_k is semi-linear, which we can correct by using the same notation as in Section 5.1 and considering instead the linear map $B'_k \colon \mathcal{H}'_\pi \ni v' \mapsto B_k v = \sqrt{d_\pi}\varphi_{w_k,v}$. Now we have

$$\lambda_g B'_k v' = \lambda_g B_k v = B_k \pi_g v = B'_k \overline{\pi_g} v'$$

for all $g \in G$ and $v' \in \mathcal{H}'_\pi$, which shows that B'_k is an isomorphism between the contragredient representation $\overline{\pi}$ and the restriction of the left-regular representation to $\operatorname{Im} B'_k$. As in the case of the right-regular representation the multiplicity is d_π. □

Exercise 5.43. Consider the left-right representation γ of $G \times G$ on $L^2(G)$ defined by $\gamma_{(g_1,g_2)}(f)(h) = f(g_1^{-1} h g_2)$ for all $f \in \mathcal{H}_\gamma = L^2(G)$ and $g_1, g_2, h \in G$. Show that

$$L^2(G) = \bigoplus_{[\pi]\in\widehat{G}} \mathcal{M}_\pi$$

is precisely the decomposition into inequivalent irreducible representations for the unitary representation α of $G \times G$ in $\mathcal{H}_\gamma = L^2(G)$, each appearing with multiplicity one.

Exercise 5.44. Let π be a cyclic unitary representation of a compact group. Show that π is contained in the regular representation, that is $\pi < \lambda$.

Exercise 5.45. What additional consequences about $(d_\pi \mid [\pi] \in \widehat{G})$ can be derived from the Peter–Weyl theorem (Theorem 5.42) if G is a finite group? (See also Exercise 5.60.)

5.4.4 Weak Containment and Discreteness of the Dual

For the compact group G the notion of weak containment $\rho_1 \prec \rho_2$ for unitary representations ρ_1 and ρ_2 from Section 4.1 has the following special properties.

Corollary 5.46 (Characterization of weak containment). *Let ρ_1 and ρ_2 be unitary representations of the compact group G. Then the following are equivalent:*

(1) $\rho_1 \prec \rho_2$;
(2) $\pi < \rho_1$ *implies that* $\pi < \rho_2$ *for any* $\pi \in \widehat{G}$; *and*
(3) $\rho_1 < \rho_2^\infty$.

Moreover, for $\pi \in \widehat{G}$ and a unitary representation ρ of G, we have $\pi \prec \rho$ if and only if $\pi < \rho$. In particular, the Fell topology on $\widehat{G}$ is discrete.

Proof. We begin by proving the final claim in the corollary (which also follows from Exercises 5.25 and 4.18). Suppose that π is in $\widehat{G}$ and ρ is a unitary representation of G. Clearly $\pi < \rho$ implies $\pi \prec \rho$, so suppose for the converse that π is not contained in ρ. By the decomposition theorem (Theorem 5.27) this gives

$$\mathcal{H}_\rho = \bigoplus_{\ell \in I} \mathcal{V}_\ell$$

for a finite or countable index set I and closed subspaces $\mathcal{V}_\ell$ for $\ell \in I$ with $\rho|_{\mathcal{V}_\ell} \cong \pi_\ell \in \widehat{G}\smallsetminus\{\pi\}$. Let $v \in \mathcal{H}_\pi$ be a unit vector and let φ_v^π be its principal matrix coefficient. For any $w \in \mathcal{H}_{\pi_0}$ with $\pi_0 \in \widehat{G}\smallsetminus\{\pi\}$ we obtain from Schur orthogonality (Theorem 5.38) that $\varphi_w^{\pi_0} \perp \varphi_v^\pi$, where it does not matter whether w is normalized to unit length or not.

If now $w = \sum_{\ell \in I} w_\ell \in \mathcal{H}_\rho$ with $w_\ell \in \mathcal{V}_\ell$, then

$$\varphi_w^\rho(g) = \langle \rho_g w, w \rangle = \sum_{\ell \in I} \langle \rho_g w_\ell, w_\ell \rangle = \sum_{\ell \in I} \varphi_{w_\ell}^\rho(g)$$

for all $g \in G$. Since $\|\varphi_{w_\ell}^\rho\| = \|w_\ell\|^2$ this series converges uniformly. Together with the above, this implies that $\varphi_w^\rho \perp \varphi_v^\pi$. Clearly this extends to finite sums as in the definition of weak containment (Definition 4.1), and shows that π is not weakly contained in ρ. The discreteness of $\widehat{G}$ in the Fell topology now follows from Corollary 4.42: If $\pi \in \widehat{G}$ and $\pi_n \in \widehat{G} \smallsetminus \{\pi\}$ for all $n \in \mathbb{N}$, then π is not weakly contained in $\bigoplus_{n=1}^{\infty} \pi_n$ and hence (π_n) cannot converge to π.

Suppose now that ρ_1 and ρ_2 are unitary representations of G and $\rho_1 \prec \rho_2$ as in (1). If now $\pi < \rho_1$, then by transitivity of weak containment (see Exercise 4.4 and its hint on p. 480) we have $\pi \prec \rho_2$, which by the above also implies that $\pi < \rho_2$ as required.

If now ρ_1 and ρ_2 satisfy (2), then

$$\operatorname{mult}(\pi, \rho_1) \leqslant \infty \cdot \operatorname{mult}(\pi, \rho_2) = \operatorname{mult}(\pi, \rho_2^{\infty}),$$

and (3) follows from the characterization of containment in Corollary 5.33.

Now suppose that $\rho_1 < \rho_2^{\infty}$ as in (3). Any $v \in \mathcal{H}_{\rho_1}$ then corresponds under the assumed unitary equivariant isomorphism to some $w = (w_n)_n$ lying in $\bigoplus_{n=1}^{\infty} \mathcal{H}_{\rho_2}$. This implies that $\varphi_v^{\rho_1} = \varphi_w^{\rho_2^{\infty}} = \sum_{n=1}^{\infty} \varphi_{w_n}^{\rho_2}$, and the series converges uniformly. As $v \in \mathcal{H}_\rho$ was arbitrary, this implies that $\rho_1 \prec \rho_2$ by definition of weak containment. □

5.5 The Space of Conjugacy Classes*

Recall that for two elements g_1, g_2 of a group G we say that g_1 is *conjugate* to g_2 if there exists some $k \in G$ with $g_2 = k g_1 k^{-1}$, equivalently if g_1 and g_2 are on the same G-orbit under the action of G on itself by inner automorphisms. The equivalence classes (or G-orbits) are called the *conjugacy classes* and we write $G^\sharp$ for the space of conjugacy classes. We also write

$$\begin{aligned} p\colon G &\longrightarrow G^\sharp \\ g &\longmapsto [g] = \{kgk^{-1} \mid k \in G\} \end{aligned}$$

for the canonical projection onto the space of conjugacy classes.

If G is a topological group we also use the map p to define the quotient topology on $G^\sharp$. In general, this map and the resulting topological space may not be well-behaved. For example, $G^\sharp$ may not be Hausdorff.

Exercise 5.47. Show that the images under p of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are distinct but do not have disjoint neighbourhoods in the space $\mathrm{SL}_2(\mathbb{R})^\sharp$. Show that p is not a closed map, specifically that the image of $\left\{ \begin{pmatrix} a & \\ & 1/a \end{pmatrix} \mid a > 0 \right\}$ is not closed.

However, if G is compact the situation is much better[(9)] and we can use the results of this chapter to give an introduction to harmonic analysis on $G^\sharp$. Moreover, as we will see this will also lead to a better understanding of the unitary representations of G.

5.5.1 The Topological Space

Lemma 5.48 (Topology on $G^\sharp$). *For the compact metric group G the quotient topology on $G^\sharp$ is compact and metrizable. In fact the metric $\mathsf{d}_\sharp$ on $G^\sharp$ can be defined by*

$$\mathsf{d}_\sharp\big([g_1],[g_2]\big) = \inf_{\substack{h_1\in[g_1],\\ h_2\in[g_2]}} \mathsf{d}(h_1,h_2)$$

for $[g_1],[g_2]\in G^\sharp$.

Proof. Since G is compact, the equivalence classes $[g]$ for $g\in G$ are (as images under the continuous map $G\ni k\mapsto kgk^{-1}$) also compact. It follows that the distance between $[g_1]$ and $[g_2]$ is actually a minimum and so is positive unless $[g_1]=[g_2]$. From this it is easy to see that $\mathsf{d}_\sharp$ defines a metric on $G^\sharp$, and it remains to show that the topology induced by $\mathsf{d}_\sharp$ coincides with the quotient topology.

Given $g\in G$ and $\varepsilon>0$ it is also clear that

$$\bigcup_{\substack{k_1\in G\\ k_2\in G}} k_1 B_\varepsilon\big(k_2 g k_2^{-1}\big)k_1^{-1}\subseteq G$$

is open and equal to

$$\{h\in G\mid \mathsf{d}_\sharp([h],[g])<\varepsilon\} = p^{-1}B_\varepsilon^{G^\sharp}([g]).$$

It follows that every metric open set is open in the quotient topology. For the converse, suppose that $O^\sharp\subseteq G^\sharp$ is an open neighbourhood of $[g]\in G^\sharp$ so that $O=p^{-1}(O^\sharp)$ is an open set containing $[g]$ and invariant under conjugation by elements of G. It follows that $[g]$ and $G\smallsetminus O$ are disjoint compact sets so that their distance

$$\varepsilon = \min_{h\in[g],k\in G\smallsetminus O} \mathsf{d}(h,k) = d_\sharp([g],G\smallsetminus O)$$

is positive. We claim that $B_\varepsilon([g])\subseteq O^\sharp$, which implies that $O^\sharp$ is also a neighbourhood of $[g]$ with respect to the metric $\mathsf{d}_\sharp$. For the proof of the claim suppose that

$$\mathsf{d}_\sharp([h],[g]) = \mathsf{d}(k_1hk_1^{-1},k_2gk_2^{-1})<\varepsilon$$

for some $h,k_1,k_2\in G$. Since $k_2gk_2^{-1}\in[g]$ it follows from the definition of ε that $k_1hk_1^{-1}\in O$ or equivalently $[h]\in O^\sharp$ as claimed.

Since $G^\sharp=p(G)$ is a continuous image of the compact group G, $G^\sharp$ is also compact and the lemma follows. □

5.5.2 The Centre of the Convolution Algebra

Many groups (and, in particular, many semi-simple groups) have very small or trivial centre; see, for example, the discussion of $\mathrm{SU}_2(\mathbb{R})$ in the next chapter. This and Schur's lemma (Theorem 1.27) makes the following result important. For this and the following, we endow $G^\sharp$ with the push-forward of the Haar measure m on G, which we again denote by m. Moreover, we identify a function on G that is invariant under conjugation by all $g \in G$ with a function on $G^\sharp$ and obtain in this way the inclusions $C(G^\sharp) \subseteq C(G)$, $L^1(G^\sharp) \subseteq L^1(G)$, and $L^2(G^\sharp) \subseteq L^2(G)$.

Proposition 5.49 (The centre of the convolution algebra). *The centre of $L^1(G)$ for the compact group G is given by $L^1(G^\sharp)$. That is, $f_c \in L^1(G)$ satisfies $f_c * \psi = \psi * f_c$ for all $\psi \in L^1(G)$ if and only if $f_c \in L^1(G^\sharp)$.*

Proof. We suppose first that $f_c \in L^1(G^\sharp)$, so that $f_c(ghg^{-1}) = f_c(h)$ for all $g, h \in G$. For $\psi \in L^1(G^\sharp)$ we then have

$$f_c * \psi(g) = \int f_c(g\ell^{-1})\psi(\ell)\,\mathrm{d}m(\ell) = \int f_c(\ell^{-1}g)\psi(\ell)\,\mathrm{d}m(\ell) = \psi * f_c(g)$$

for almost every $g \in G$ by the definition of convolution in (1.12), as required.

For the converse, we suppose now that $f_c \in L^1(G)$ satisfies $f_c * \psi = \psi * f_c$ for all $\psi \in L^1(G)$. We wish to apply this to $\psi = \lambda_g \psi_n$ for some $g \in G$ and an approximate identity (ψ_n) as in Proposition 1.42. In fact, using the compactness of G we see that conjugation is uniformly continuous and so we may choose a decreasing sequence (B_n) of neighbourhoods of $e \in G$ that are invariant under conjugation and satisfy $\bigcap_{n\geqslant 1} B_n = \{e\}$. We set

$$\psi_n = \frac{1}{m(B_n)} \mathbb{1}_{B_n} \in L^1(G^\sharp)$$

for all $n \geqslant 1$. Also recall (1.14) and Exercise 1.46, which we can express as saying that

$$\delta_g * f = \lambda_g f$$

and

$$f * \delta_g = \rho_{g^{-1}} f$$

for all $f \in L^1(G)$ and $g \in G$. For $f = \psi_n$ and $g, h \in G$ this gives

$$\bigl(\delta_g * \psi_n\bigr)(h) = \psi_n(g^{-1}h) = \psi_n(hg^{-1}) = \bigl(\psi_n * \delta_g\bigr)(h).$$

By the assumption on f_c and associativity of convolution in $M(G)$, this gives

$$(f_c * \delta_g) * \psi_n = f_c * (\underbrace{\delta_g * \psi_n}_{=\psi}) = (\underbrace{\psi_n * \delta_g}_{=\psi}) * f_c = \psi_n * (\delta_g * f_c).$$

Letting $n \to \infty$ and applying Proposition 1.42, we obtain $f_c * \delta_g = \delta_g * f_c$ and so

$$f_c(hg^{-1}) = \rho_{g^{-1}}(f_c)(h) = \lambda_g(f_c)(h) = f_c(g^{-1}h)$$

for every $g \in G$ and almost every $h \in G$, or equivalently

$$f_c(ghg^{-1}) = f_c(h) \tag{5.17}$$

for every $g \in G$ and almost every $h \in G$.

Strictly speaking, the null set excluded in this statement may depend on $g \in G$, but one can replace f_c by an equivalent function so that (5.17) then holds for all $g, h \in G$. We refer to [20, Prop. 8.3] or Exercise 5.50. This proves the converse. □

Exercise 5.50. Show that (5.17) for every $g \in G$ and almost every $h \in G$ implies that there exists a null set $N \subseteq G$ such that $h \in G \smallsetminus N$ implies (5.17) for almost every $g \in G$, and conclude that

$$f_c(h) = \int_G f_c(ghg^{-1}) \, \mathrm{d}m(g)$$

for almost every $h \in G$.

Exercise 5.51. Generalize Proposition 5.49 to the measure algebra by showing that the centre of $M(G)$ is given by

$$M(G^\sharp) = \{\mu \in M(G) \mid \mu \text{ is invariant under conjugation}\}.$$

For unitary representations of G we obtain the following consequence.

Proposition 5.52 (Unitary representations and the centre of $L^1(G)$). *Let π be a unitary representation of the compact group G. Then the convolution operators $\pi_*(f_c)$ for $f_c \in L^1(G^\sharp)$ are equivariant. If $\pi \in \widehat{G}$, then $\pi_*(f_c)$ is a multiple of the identity on $\mathcal{H}_\pi$.*

Proof. Let π be a unitary representation of the compact group G, $g \in G$, and $f_c \in L^1(G^\sharp)$. Then

$$\begin{aligned}\big\langle \pi_*(f_c)\pi_g u, v \big\rangle &= \int f_c(h) \langle \pi_{hg} u, v \rangle \, \mathrm{d}m(h) \\ &= \int f_c(kg^{-1}) \langle \pi_k u, v \rangle \, \mathrm{d}m(k) \\ &= \int f_c(g^{-1}k) \langle \pi_k u, v \rangle \, \mathrm{d}m(k) \\ &= \int f_c(\ell) \langle \pi_{g\ell} u, v \rangle \, \mathrm{d}m(\ell) = \big\langle \pi_g \pi_*(f_c) u, v \big\rangle\end{aligned}$$

for all $u, v \in \mathcal{H}_\pi$, which implies that

$$\pi_*(f_c)\pi_g = \pi_g \pi_*(f_c)$$

for all $g \in G$ as required. For $\pi \in \widehat{G}$ the final conclusion follows from Schur's lemma (Theorem 1.27). $\square$

The following 'projection' from $L^1(G)$ to $L^1(G^\sharp)$ will be useful.

Lemma 5.53 (Averaging projection). *Given the compact group G we let $A\colon L^1(G) \to L^1(G^\sharp)$ be the* averaging projection *defined by*

$$A(f)(g) = \int_G f(hgh^{-1})\,\mathrm{d}m(h)$$

for $g \in G$ and $f \in L^1(G)$. Then $A(f) = f$ for all $f \in L^1(G^\sharp)$, and $A(f)$ lies in $C(G^\sharp)$ for all f in $C(G)$. Moreover, for a finite-dimensional representation π of G we also have

$$\pi_*(A(f)) = \int_G \pi_h \pi_*(f) \pi_h^{-1}\,\mathrm{d}m(h)$$

for all $f \in L^1(G)$.

Proof. Let $f \in L^1(G)$. Since $g \mapsto f(kgk^{-1})$ is integrable for all $k \in G$ and $m(G) = 1$, Fubini's theorem implies that the integral defining $A(f)(g)$ exists for almost every $g \in G$ and that $\|A(f)\|_1 \leqslant \|f\|_1$. By right-invariance of m, we also have $A(f) \in L^1(G^\sharp)$. If $f \in L^1(G^\sharp)$ we have

$$f(kgk^{-1}) = f(g)$$

for $g, k \in G$, and so $A(f) = f$ as $m(G) = 1$. The continuity of $A(f)$ for f in $C(G)$ follows from dominated convergence.

Suppose now that π is a finite-dimensional representation of G and u, v are in $\mathcal{H}_\pi$. Then

$$\begin{aligned}
\big\langle \pi_*\big(A(f)\big)u, v\big\rangle &= \int A(f)(g)\langle \pi_g u, v\rangle\,\mathrm{d}m(g) \\
&= \iint f(hgh^{-1})\langle \pi_g u, v\rangle\,\mathrm{d}m(h)\,\mathrm{d}m(g) \\
&= \iint f(k)\langle \pi_{h^{-1}kh} u, v\rangle\,\mathrm{d}m(k)\,\mathrm{d}m(h) \\
&= \iint f(k)\langle \pi_\ell \pi_k \pi_\ell^{-1} u, v\rangle\,\mathrm{d}m(k)\,\mathrm{d}m(\ell) \\
&= \int \langle \pi_\ell \pi_*(f) \pi_\ell^{-1} u, v\rangle\,\mathrm{d}m(\ell) = \left\langle \int \pi_\ell \pi_*(f)\pi_\ell^{-1}\,\mathrm{d}m(\ell) u, v \right\rangle,
\end{aligned}$$

which gives the lemma as $u, v \in \mathcal{H}_\pi$ were arbitrary. □

5.5.3 Characters

In the context of non-abelian groups the following notion of characters is important. We note however that these are not multiplicative characters (except for some exceptional cases) as we have considered since Chapter 1, and particularly in Chapter 2.

Definition 5.54 (Characters). Given a finite-dimensional representation ρ of a group G the *character of the representation* ρ is the complex-valued function χ_ρ defined by $\chi_\rho(g) = \operatorname{tr}(\rho_g)$ for $g \in G$.

We note that for a finite-dimensional representation ρ of a group G and $g, k \in G$ we have

$$\chi_\rho(kgk^{-1}) = \operatorname{tr}(\rho_k\rho_g\rho_k^{-1}) = \operatorname{tr}(\rho_g) = \chi_\rho(g),$$

which shows that the character of a representation can be considered as a function on $G^\sharp$. Hence, if ρ is a continuous finite-dimensional representation of a topological group, then χ_ρ is a continuous function on $G^\sharp$. Moreover, the characters of two isomorphic finite-dimensional representations are equal. In particular, if G is compact, then for every $[\pi] \in \widehat{G}$ we have a well-defined continuous character $\chi_{[\pi]} = \chi_\pi \in C(G^\sharp)$.

The characters appear quickly when the averaging projection is applied.

Proposition 5.55 (Averaging). *Let π be an irreducible representation of the compact group G, and let $w_1, \ldots, w_{d_\pi} \in \mathcal{H}_\pi$ be an orthonormal basis of $\mathcal{H}_\pi$. Then we have*

$$A(\pi_{m,n}) = \tfrac{1}{d_\pi}\delta_{m,n}\chi_{[\pi]}$$

for all $m, n \in \{1, \ldots, d_\pi\}$.

Proof. Let $m, n \in \{1, \ldots, d_\pi\}$. By the Schur orthogonality relations (Theorem 5.38) we have for all $k, \ell \in \{1, \ldots, d_\pi\}$ that

$$\begin{aligned} \big\langle \pi_*\big(d_\pi\overline{\pi_{m,n}}\big)w_k, w_\ell\big\rangle_{\mathcal{H}_\pi} &= \int d_\pi\overline{\pi_{m,n}(g)}\,\underbrace{\big\langle \pi_g w_k, w_\ell\big\rangle_{\mathcal{H}_\pi}}_{\pi_{k,\ell}(g)}\,\mathrm{d}m(g) \\ &= d_\pi\big\langle \pi_{k,\ell}, \pi_{m,n}\big\rangle_{L^2(G)} = \delta_{km}\delta_{\ell n}. \end{aligned} \tag{5.18}$$

In other words, $\pi_*(d_\pi\overline{\pi_{m,n}})$ is the elementary linear map that sends w_m to w_n and all other basis vectors to zero. In particular, the map

$$\mathcal{M}_\pi \ni \phi \mapsto \pi_*(d_\pi\overline{\phi}) \in \mathrm{B}(\mathcal{H}_\pi) \tag{5.19}$$

is a semi-linear isomorphism.

By Proposition 5.37, the subspace $\mathcal{M}_\pi \subseteq L^1(G)$ is invariant under the left and right-regular representations, which implies that

$$G \ni g \longmapsto \left(\lambda_{h^{-1}}\rho_{h^{-1}}\pi_{m,n}(g) = \pi_{m,n}\big(hgh^{-1}\big)\right)$$

belongs to $\mathcal{M}_\pi$ for all $h \in G$. Taking the integral over $h \in G$ as in the definition of the averaging projection of Lemma 5.53 and applying the Schur orthogonality relations (Theorem 5.38) we see that

$$F = A(\pi_{m,n}) = \sum_{k,\ell} \beta_{k,\ell}\pi_{k,\ell} \in C(G^\sharp)$$

for some matrix $(\beta_{k,\ell}) \in \mathrm{Mat}_{d_\pi,d_\pi}(\mathbb{C})$. Since $\pi_*\big(d_\pi\overline{F}\big)$ is equivariant by Proposition 5.52, it follows by Schur's lemma (Theorem 1.27) that $\pi_*(d_\pi\overline{F}) = \alpha I$. Together with the properties of the map in (5.19), this gives $\overline{\beta_{k,\ell}} = \alpha\delta_{k,\ell}$ for all $k, \ell \in \{1,\dots,d_\pi\}$, and so $A(\pi_{m,n}) = \overline{\alpha}\chi_{[\pi]}$. To calculate α we take the trace in $\mathcal{H}_\pi$ and obtain

$$d_\pi\alpha = \mathrm{tr}(\alpha I) = \mathrm{tr}\big(\pi_*(d_\pi\overline{F})\big) = \mathrm{tr}\big(\pi_*(d_\pi\overline{\pi_{m,n}})\big) = \delta_{m,n}$$

by Lemma 5.53 and the above description of the linear map $\pi_*(d_\pi\overline{\pi_{m,n}})$ in (5.18). This gives the lemma. □

The characters also have very special properties for convolution.

Corollary 5.56 (Convolution of characters). *Let π, ρ be irreducible unitary representations of the compact group G. Then*

$$\chi_{[\pi]} * \chi_{[\pi]} = \tfrac{1}{d_\pi}\chi_{[\pi]}$$

and if $[\pi] \neq [\rho]$ are inequivalent, then

$$\chi_{[\pi]} * \chi_{[\rho]} = 0.$$

PROOF. Suppose first that $[\pi] \neq [\rho] \in \widehat{G}$ are inequivalent. By Proposition 5.37, both $\mathcal{M}_{[\pi]}$ and $\mathcal{M}_{[\rho]}$ are two-sided ideals, which implies that

$$\chi_{[\pi]} * \chi_{[\rho]} \in \mathcal{M}_{[\pi]} \cap \mathcal{M}_{[\rho]}$$

for $\chi_{[\pi]} \in \mathcal{M}_{[\pi]}$ and $\chi_{[\rho]} \in \mathcal{M}_{[\rho]}$. However, by the Schur orthogonality relation (Theorem 5.38) we have $\mathcal{M}_{[\pi]} \perp \mathcal{M}_{[\rho]}$ and hence $\chi_{[\pi]} * \chi_{[\rho]} = 0$.

For the former claim we let $[\pi] \in \widehat{G}$ and note that

$$f = \chi_{[\pi]} * \chi_{[\pi]} \in \mathcal{M}_{[\pi]}$$

satisfies

$$\pi_*(d_\pi\overline{f}) = \pi_*\big(d_\pi\overline{\chi_{[\pi]}} * \overline{\chi_{[\pi]}}\big) = \tfrac{1}{d_\pi}\pi_*\big(d_\pi\overline{\chi_{[\pi]}}\big)\pi_*\big(d_\pi\overline{\chi_{[\pi]}}\big).$$

However, in the proof of Proposition 5.55 we saw that

$$\mathcal{M}_\pi \ni \phi \longmapsto \pi_*\big(d_\pi\overline{\phi}\big)$$

is a semi-linear isomorphism sending $\chi_{[\pi]}$ to the identity. This gives

$$\pi_*(d_\pi\overline{f}) = \tfrac{1}{d_\pi}I$$

and so $\overline{f} = \frac{1}{d_\pi}\chi_{[\pi]}$, which concludes the proof. □

Exercise 5.57. Let $[\pi] \in \widehat{G}$ be an irreducible representation of the compact group G. Let $\pi_{m,n}$ for $m, n \in \{1, \dots, d_\pi\}$ be as in the Schur orthogonality relations (Theorem 5.38). Calculate $\pi_{m,n} * \pi_{k,\ell}$ for all $m, n, k, \ell \in \{1, \dots, d_\pi\}$, and deduce the formula in Corollary 5.56 for $\chi_{[\pi]} * \chi_{[\pi]}$ from this.

5.5.4 Dense Algebra

Corollary 5.58 (Dense algebra). *For the compact group G the linear span*

$$\mathcal{M}(G^\sharp) = \langle \chi_\pi \mid [\pi] \in \widehat{G} \rangle = \langle \chi_\rho \mid \rho \text{ a finite-dimensional representation of } G \rangle$$

is a dense sub-algebra of $C(G^\sharp)$ with respect to pointwise multiplication.

Proof. The equivalence of the two descriptions of $\mathcal{M}(G^\sharp)$ is a consequence of the fact that every irreducible representation is finite-dimensional by Theorem 5.26, and since a finite-dimensional representation ρ has $\mathcal{H}_\rho = \bigoplus_{j=1}^n \mathcal{H}_{\pi_j}$ for some irreducible representations $\pi_1, \dots, \pi_n$ so that the character decomposes as

$$\chi_\rho = \sum_{j=1}^n \chi_{\pi_j}. \tag{5.20}$$

Suppose now that $f \in C(G^\sharp)$ and $\varepsilon > 0$. Then by Theorem 5.40 there exist finitely many irreducible representations $\pi_1, \dots, \pi_n$ and vectors $v_j, w_j \in \mathcal{H}_{\pi_j}$ for $j = 1, \dots, n$ so that

$$\Big\| f - \sum_{j=1}^n \alpha_j \varphi^{\pi_j}_{v_j,w_j} \Big\|_\infty < \varepsilon \tag{5.21}$$

for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$.

Applying the averaging projection $A\colon C(G) \to C(G^\sharp)$ from Lemma 5.53 we obtain the corollary. Indeed,

$$A\big(\varphi^{\pi_j}_{v_j,w_j}\big) = \beta_j \chi_{[\pi_j]}$$

for some $\beta_j \in \mathbb{C}$ and $j = 1, \dots, n$. Hence

$$\left\| f - \sum_{j=1}^{n} \alpha_j \beta_j \chi_{[\pi_j]} \right\|_\infty < \varepsilon,$$

which gives the corollary, as $\varepsilon > 0$ was arbitrary.

To see that $\mathcal{M}(G^\sharp)$ is a sub-algebra we note that, for finite-dimensional representations ρ_1, ρ_2 of G, we have

$$\chi_{\rho_1 \otimes \rho_2}(g) = \operatorname{tr}\big(\rho_1(g) \otimes \rho_2(g)\big) = \operatorname{tr}\big(\rho_1(g)\big) \operatorname{tr}\big(\rho_2(g)\big) = \chi_{\rho_1}(g)\chi_{\rho_2}(g)$$

for all $g \in G$. □

5.5.5 Characters as an Orthonormal Basis

Corollary 5.59 (Characters form an orthonormal basis). *The characters $\chi_{[\pi]}$ for $[\pi] \in \widehat{G}$ form an orthonormal basis of $L^2(G^\sharp)$.*

PROOF. For inequivalent $[\pi] \neq [\pi'] \in \widehat{G}$ we have $\mathcal{M}_\pi \perp \mathcal{M}_{\pi'}$ by the Schur orthogonality relation (Theorem 5.38). With $\chi_\pi \in \mathcal{M}_\pi$ and $\chi_{\pi'} \in \mathcal{M}_{\pi'}$ this implies that $\langle \chi_\pi, \chi_{\pi'} \rangle = 0$ (where it does not matter whether we take the inner product in $L^2(G)$ or in $L^2(G^\sharp)$). For a given $[\pi] \in \widehat{G}$ we have

$$\chi_\pi = \pi_{1,1} + \pi_{2,2} + \cdots + \pi_{d_\pi, d_\pi},$$

where as before $\pi_{m,n} = \varphi_{w_m,w_n}$ for an orthonormal basis $w_1, \dots, w_{d_\pi}$ of $\mathcal{H}_\pi$. Since $\|\pi_{m,m}\| = \frac{1}{\sqrt{d_\pi}}$ it follows that $\|\chi_\pi\| = 1$. By Corollary 5.58 we know that the collection $\chi_{[\pi]}$ for $[\pi] \in \widehat{G}$ forms an orthonormal basis of $L^2(G^\sharp)$. □

Exercise 5.60. What additional consequences about $(d_\pi \mid [\pi] \in \widehat{G})$ can be derived from Corollary 5.59 if G is finite? (See also Exercise 5.45.)

5.6 Summary and Outlook

As we have seen in this chapter, unitary representations of compact groups have many special properties, since

- all unitary representations are (at most countable) direct sums of irreducible finite-dimensional representations;

- the unitary dual is discrete; and
- the regular representation contains all irreducible representations with prescribed multiplicity.

In the next chapter we will discuss the group $\mathrm{SU}_2(\mathbb{R})$ as a particularly important example for the material of this chapter, and in particular will describe its conjugacy classes and its characters.

This will lead us to the topic of unitary representations of simple Lie groups, which will be the topic of Chapters 6 to 9.