Chapter 5
Action of Horospherical Subgroups

The inheritance property of ergodicity in the Mautner phenomenon (Proposition 2.11, the general Theorem 2.15; also see Exercise 2.4.4) established in Chapter 2 already gives the equidistribution of many orbits.

Indeed, if a simple Lie group G acts ergodically on (X, μ) and

$$\{g_t \mid t \in \mathbb{R}\} \subseteq G$$

is an unbounded one-parameter subgroup, then

$$\frac{1}{T}\int_0^T f(g_t \cdot x)\,\mathrm{d}t \to \int_X f\,\mathrm{d}\mu$$

for μ-almost every $x \in X$, for any $f \in C_c(X)$. This is a straightforward application of the pointwise ergodic theorem (see [54, Cor. 8.15 and Sec. 4.4.2] and Lemma 6.12 below). A point $x \in X$ with this property is called *generic* for μ and the one-parameter subgroup $\{g_t \mid t \in \mathbb{R}\}$.

In this short chapter we start the discussion of unipotent dynamics by considering the case of horospherical actions. For those actions we will show unique ergodicity, and sometimes 'almost unique ergodicity', and we will understand the distribution of orbits of *any* given point.

5.1 Dynamics on Hyperbolic Surfaces

Let us start by discussing briefly the case of the geodesic flow and the horocycle flow on quotients of $\mathrm{SL}_2(\mathbb{R})$ as introduced in Section 1.2

We note first that for the geodesic flow it is not possible to make a more general statement about the equidistribution of orbits by relaxing the requirement that the point be μ-typical. Indeed, if $g_t = a_t$ is diagonalizable, then the flow is partially hyperbolic and as a result X contains many irregular orbits. As this result can be considered of negative type we will not prove it here, but see Example 5.1 resp. [54, Sec. 9.7.2] for a more detailed discussion of the case of the geodesic flow on the modular surface.

Example 5.1. [20]For a compact quotient $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$, the action of the one-parameter subgroup

$$\left\{a_t = \begin{pmatrix} \mathrm{e}^{-t/2} & \\ & \mathrm{e}^{t/2} \end{pmatrix} \mid t \in \mathbb{R}\right\}$$

has many orbits that stay on one side of the dotted line in Figure 5.1.

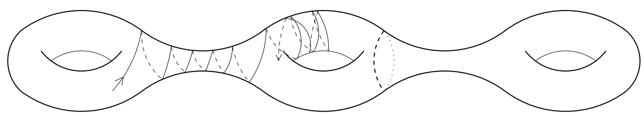

Fig. 5.1: There are many orbits under the action of A that stay on one side of the dotted line furthest to the right.

We also refer to Exercises 5.1.1–5.1.4 for the behaviour of the geodesic flow and higher dimensional analogues.

This is in stark contrast to the behaviour of orbits of the horocycle subgroup

$$\left\{u_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \mid s \in \mathbb{R}\right\}$$

in the compact surface X: the orbit of every point under this group action visits the right-hand side at some point (much more is true, as we will show below).

In fact Hedlund [77] showed in 1936 that the horocycle flow on any compact quotient $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ is *minimal* (that is, has no non-trivial closed invariant subsets) and that Haar measure is ergodic. This was strengthened by Furstenberg [68] in 1972 and by Dani [18] in 1978, who showed the following theorems.

Theorem 5.2 (Unique ergodicity of horocycle flow). *If $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ is compact, then the horocycle flow (i.e. the action of the subgroup)*

$$\left\{u_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \mid s \in \mathbb{R}\right\}$$

is uniquely ergodic.

Theorem 5.3 (Almost unique ergodicity of horocycle flow). *If*

$$X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$$

(or another non-compact quotient of finite volume) then a probability measure m on X that is invariant and ergodic for the action of

$$\left\{u_s = \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \mid s \in \mathbb{R}\right\}$$

is either

- *the Haar measure m_X on X inherited from the Haar measure on $\mathrm{SL}_2(\mathbb{R})$; or*
- *a one-dimensional Lebesgue measure supported on a periodic orbit of the action.*

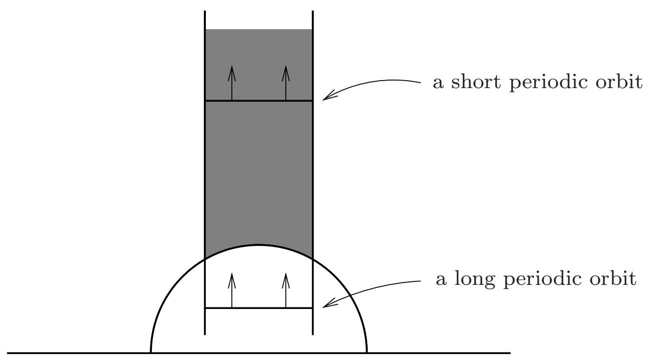

Fig. 5.2: In the upper half-plane model of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$, the observed speed of a periodic horocycle orbit increases with the height, so the two different periodic orbits shown are of different lengths. The longer periodic orbit could also be drawn in the fundamental domain, but it would look very complicated.

We will prove Theorem 5.2 in Section 5.2 and give an outline of the proof of Theorem 5.3 in Section 5.3.

Exercises for Section 5.1

Exercise 5.1.1. (Anosov shadowing for $\mathrm{SL}_2(\mathbb{R})$) Let $X = \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ be the quotient of $\mathrm{SL}_2(\mathbb{R})$ by a discrete subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$.
(a) Let $x \in X$, $T > 0$, $\varepsilon > 0$ and $y \in X$ be chosen with $\mathsf{d}(a_T \bullet x, y) < \varepsilon$. Then there exists a point $z \in X$ with $\mathsf{d}(x, z) \ll \mathrm{e}^{-T}\varepsilon$ (and so $\mathsf{d}(a_t \bullet x, a_t \bullet z) \ll \varepsilon$ for $t \in [0, T]$) and $\mathsf{d}(a_t \bullet y, a_{T+t} \bullet z) \ll \varepsilon$ for all $t \geqslant 0$. Also show that there exists some δ with $|\delta| \ll \varepsilon$ such that $\mathsf{d}(a_{t+\delta} \bullet y, a_{T+t} \bullet z) \ll \mathrm{e}^{-t}$ for all $t \geqslant 0$.

(b) Assume now that X is compact (for example, as in Figure 5.1) and use (a) to construct non-periodic orbits as in Example 5.1.

Exercise 5.1.2. (Anosov closing for $\mathrm{SL}_2(\mathbb{R})$) Let $X = \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ be as in Exercise 5.1.1. Let $x \in X$ and $T \geqslant 1$ be chosen so that $\mathsf{d}(a_T\bullet x, x) \leqslant \varepsilon < 1$. Show that there exists a point $z \in X$ which is periodic with period T_z satisfying

$$|T_z - T| \ll \varepsilon$$

and

$$\mathsf{d}(a_t\bullet x, a_t\bullet z) \ll \varepsilon$$

for all $t \in [0, T]$.

Exercise 5.1.3. (Anosov shadowing for G) Let G be a connected Lie group, let $\Gamma < G$ be a discrete subgroup, let $X = \Gamma\backslash G$, and let $a \in G$ be such that Ad_a is diagonalizable with positive eigenvalues.
(a) Let $x \in X$, $N > 1$, $\varepsilon > 0$ and $y \in X$ be such that $\mathsf{d}(a^N\bullet x, y) < \varepsilon$. Then there exists a point $z \in X$, some $\lambda < 1$ (independent of x, y and Γ) with

$$\mathsf{d}(a^n\bullet x, a^n\bullet z) \ll \lambda^{N-n}\varepsilon$$

for $n = 0, \dots, N$ and

$$\mathsf{d}(a^{N+n}\bullet z, a^n\bullet y) \ll \varepsilon$$

for all $n \geqslant 0$.
(b) Assume that X has finite volume and a acts mixingly on X with respect to m_X. Construct non-periodic irregular orbits by iterating (a).

Exercise 5.1.4 (Anosov closing for $X = \Gamma\backslash\mathrm{SL}_d(\mathbb{R})$). We let $X = \Gamma\backslash\mathrm{SL}_d(\mathbb{R})$ be any quotient by a discrete subgroup $\Gamma < G = \mathrm{SL}_d(\mathbb{R})$, and let A be the subgroup of G of positive diagonal matrices. Let $a \in A$ be a non-trivial element.
(a) Suppose that $x \in X$ and $N \geqslant 1$ are such that $\mathsf{d}(a^N, I) \geqslant 1$ but $\mathsf{d}(a^N\bullet x, x) \leqslant \varepsilon < 1$. Assume that ε is sufficiently small and that N is sufficiently large. Show that there exists some $z \in X$ and some $c \in \mathrm{SL}_d(\mathbb{R})$ with $ac = ca$, $\mathsf{d}(a^N, c) \ll \varepsilon$, $c\bullet z = z$ and

$$\mathsf{d}(a^n\bullet x, a^n\bullet z) \ll \varepsilon$$

for $n = 0, \dots, N$.
(b) Suppose that a is generic (that is, no two eigenvalues are the same) and X is compact. Show that z as in (a) is a periodic point for A.
(c) Suppose $d = 3$ and a is generic and a does not have 1 as an eigenvalue, and

$$X = \mathsf{X}_3 = \mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R}).$$

Show again that the point z as in (a) is periodic for A.
(d) Repeat (c) for

$$X = \mathsf{X}_d = \mathrm{SL}_d(\mathbb{Z})\backslash\mathrm{SL}_d(\mathbb{R}),$$

assuming that $a \in A$ has the property that no product over a proper non-empty subset of the eigenvalues of a equals 1.
(e) In the setting of (b), (c), and of (d), show that periodic A-orbits are dense in X.
(f) Generalize the statement in (b) to any semi-simple group†.

† In that sense Poincaré recurrence can be used to construct anisotropic tori (see Section 7.3).

5.2 Horospherical Actions on Compact Quotients

As we will show now the unique ergodicity of the horocycle flow on compact quotients of $\mathrm{SL}_2(\mathbb{R})$ as in Theorem 5.2 holds for other Lie groups and their horospherical subgroups as well.

Suppose that G is a connected Lie group and let $a \in G$ be an $\mathbb{R}$-diagonalizable element that acts as a mixing transformation† on all the quotients of G appearing below. Let

$$G_a^- = \{g \in G \mid a^n g a^{-n} \to I \text{ as } n \to \infty\}$$

be the stable horospherical subgroup of a. The 'general method' discussed below gives a way to classify the G_a^--invariant ergodic probability measures on X.

Theorem 5.4 (Unique ergodicity of horospherical actions[(21)]). *Let G be a linear Lie group, $\Gamma < G$ be a uniform lattice, and let $a \in G$ be $\mathbb{R}$-diagonalizable. Suppose a acts mixingly on $X = \Gamma\backslash G$. Then the action of G_a^- is uniquely ergodic.*

Proof of Theorem 5.4 (and hence of 5.2). Let us assume compatibility of the Haar measures in the sense that $m_X(\pi(B)) = m_G(B)$ for any injective Borel subset $B \subseteq G$, and that $m_X(X) = 1$.

Since a is diagonalizable and G is linear, the subgroups G_a^- and

$$P_a = \{g \in G \mid a^n g a^{-n} \text{ stays bounded as } n \to -\infty\}$$

can easily be defined in terms of the vanishing of certain matrix entries, and so are closed subgroups. Together they define a local coordinate system, in the sense that $P_a G_a^-$ contains an open neighborhood of the identity‡, and the implied representation of elements of G in that neighborhood is unique. In fact, if $u_1 p_1 = u_2 p_2$ with $u_1, u_2 \in G_a^-$ and $p_1, p_2 \in P_a$ then

$$g = u_2^{-1} u_1 = p_2 p_1^{-1}$$

has $a^n g a^{-n} \to I$ as $n \to \infty$ and stays bounded as $n \to -\infty$, which together show§ that $g = I$. Moreover, the Haar measure of G restricts to the product

† Unless a specific other probability measure is identified, a property of a transformation on a homogeneous space like ergodicity, mixing, and so on, is meant with respect to the measure induced by the Haar measure on G.

‡ This can be quickly checked using the Lie algebras of G_a^- (and of P_a), which are simply the sum of the eigenspaces of Ad_a for all eigenvalues of absolute value less than one (respectively greater or equal than one).

§ This is a consequence of considering the eigenvalue decomposition of the matrix $g - I \in \mathrm{Mat}_d(\mathbb{R})$ for the linear map

$$\mathrm{Mat}_d(\mathbb{R}) \ni v \longmapsto a v a^{-1}.$$

of a Haar measure on G_a^- and a left Haar measure on P_a (see Proposition 1.6 and Lemma 1.22).

We let $B_0 \subseteq G_a^-$ be a neighborhood of the identity with compact closure such that $m_{G_a^-}(\partial B_0) = 0$ and define $B_n = a^{-n} B_0 a^n$. We claim that

$$\frac{1}{m_{G_a^-}(B_n)} \int_{B_n} f(u \centerdot x) \, \mathrm{d}m_{G_a^-}(u) \longrightarrow \int_X f \, \mathrm{d}m_X \tag{5.1}$$

for any $f \in C(X)$ and any $x \in X$.

Assuming this for now, it follows that $\mu = m_X$ is the only G_a^--invariant probability measure. Indeed if μ is another such measure then

$$\int_X f \, \mathrm{d}\mu = \int_X \frac{1}{m_{G_a^-}(B_n)} \int_{B_n} f(u \centerdot x) \, \mathrm{d}m_{G_a^-}(u) \, \mathrm{d}\mu(x) \longrightarrow \int_X f \, \mathrm{d}m_X$$

by dominated convergence. As this would hold for any $f \in C(X)$ we deduce that $\mu = m_X$, as claimed.

Now fix a point $x \in X = \Gamma \backslash G$ and a function $f \in C(X)$. By compactness f is uniformly continuous, so given $\varepsilon > 0$ there is a $\delta > 0$ for which

$$\mathsf{d}(h, I) < \delta \implies |f(h \centerdot y) - f(y)| < \varepsilon \tag{5.2}$$

where d is a left-invariant metric on G (giving rise to the metric on X). Now we can choose a compact neighborhood $V \subseteq P_a$ of the identity whose boundary has measure zero with

$$\mathsf{d}(a^{-n} h a^n, I) < \delta$$

for $h \in V$ and $n \geqslant 0$. Then

$$\frac{1}{m_{G_a^-}(B_n)} \int_{B_n} f(u \centerdot x) \, \mathrm{d}m_{G_a^-}(u)$$

is within ε of

$$\frac{1}{m_{G_a^-}(B_n) m_{P_a}(a^{-n} V a^n)} \int_{B_n} \int_{a^{-n} V a^n} f(hu \centerdot x) \, \mathrm{d}m_{P_a}(h) \, \mathrm{d}m_{G_a^-}(u)$$

because of (5.2). Using $B_n = a^{-n} B_0 a^n$, the latter may in turn be written as

$$\frac{1}{m_G(V B_0)} \int_{V B_0} f(a^{-n} g a^n \centerdot x) \, \mathrm{d}m_G(g), \tag{5.3}$$

since m_G is locally the product of m_{P_a} and $m_{G_a^-}$. Now notice that

$$G_a^- \ni u \mapsto u \centerdot x$$

is injective for all $x \in X$, for otherwise the injectivity radius at $a^n \bullet x$ would shrink to zero, contradicting the compactness of X (see Proposition 1.11 and Lemma 10.8). By a simple compactness argument, we may assume that the above δ is small enough to ensure that the map

$$VB_0 \ni g \mapsto g \bullet x$$

is injective for all $x \in X$. Thus (5.3) can also be written as

$$\frac{1}{m_G(VB_0)} \int_X f(a^{-n}y) \mathbb{1}_{VB_0 a^n \bullet x}(y) \, \mathrm{d}m_X(y). \tag{5.4}$$

In the sequence (or in any of its subsequences) $(a^n \bullet x)_{n \geqslant 1}$ we can find (by compactness) a subsequence $(a^{n_k} \bullet x)_{k \geqslant 1}$ converging to some $z \in X$. Since[†]

$$\| \mathbb{1}_{VB_0 a^{n_k} \bullet x} - \mathbb{1}_{VB_0 \bullet z} \|_2 \longrightarrow 0$$

by dominated convergence as $k \to \infty$, we see that the expression (5.4) converges to

$$\frac{1}{m_G(VB_0)} \int_X f \, \mathrm{d}m_X \int \mathbb{1}_{VB_0 \bullet z} \, \mathrm{d}m_X$$

as $n \to \infty$ because a defines a mixing transformation. This proves (5.1) for the given function f up to an error of ε. Since f and $\varepsilon > 0$ were both arbitrary, the theorem follows. □

Notice that once unique ergodicity is proved (by using the Følner sequence $(a^{-n} B_0 a^n)$) then the pointwise everywhere convergence of the ergodic averages also follows for other Følner sets (see Exercises 5.2.1–5.2.2).

Exercises for Section 5.2

Exercise 5.2.1. We let $B_n = a^{-n} B_0 a^n$ be as in the proof of Theorem 5.4 with

$$m_{G_a^-}(\partial B_0) = 0.$$

Show that this sequence is a Følner sequence in G_a^-, that is a sequence satisfying

$$\frac{m_{G_a^-}(B_n \triangle (KB_n))}{m_{G_a^-}(B_n)} \longrightarrow 0$$

as $n \to \infty$ for every compact subset $K \subseteq G_a^-$.

Exercise 5.2.2. Let a and X be as in Theorem 5.4. Let $F_n \subseteq G_a^-$ be any Følner sequence and show that

[†] Here we use $m_G(\partial(VB_0)) = 0$ which follows since m_G is the product measure in the local coordinate system $G_a^- P_a$ of G that we use and since $m_{G_a^-}(\partial B_0) = 0$ and $m_{P_a}(\partial V) = 0$.

$$\frac{1}{m_{G_a^-}(F_n)}\int_{F_n} f(u\bullet x)\,\mathrm{d}m_{G_a^-}(u) \to \int_X f\,\mathrm{d}m_X$$

as $n\to\infty$, for any $f\in C(X)$ and any $x\in X$.

5.3 Almost Unique Ergodicity on Non-Compact Quotients with Finite Volume

We now explain, guided by examples, how the presence of a cusp (that is, the lack of compactness of the quotient) and the presence of horospherical invariant measures other than the Haar measure are related to each other.

5.3.1 Horocycle Action on Non-Compact Quotients

Indeed for the horocycle flow a quotient $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ is non-compact with finite volume if and only if Γ is a lattice and contains a unipotent element $\gamma\in\Gamma$. In that case the unipotent γ is conjugated to an element of the horocycle subgroup U, that is there exists some $g\in\mathrm{SL}_2(\mathbb{R})$ with $g\gamma g^{-1}\in U$. This shows that Γg^{-1} is periodic under U and hence there exists an invariant measure other than the Haar measure.

Outline Proof of Theorem 5.3. The argument used for the proof of Theorem 5.4 may also be used for non-compact quotients. Indeed, if μ is an invariant and ergodic probability measure on $X=\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ for the horocycle flow $U=G_a^-$ and $x\in X$ is generic for μ. Then either x is periodic under U and the geodesic orbit $a^n\bullet x$ necessarily diverges into one of the cusps of X as $n\to\infty$ (because a shrinks the length of the periodic orbit and so the injectivity radius at $a^n\bullet x$ goes to zero), or x is not periodic under U and the geodesic orbit $a^n\bullet x$ visits a fixed compact set of X infinitely often. Using this subsequence of times n_j and the corresponding pieces of the horocycle orbit $a^{-n_j}B_0a^{n_j}\bullet x$ for the argument we see by the argument in the proof of Theorem 5.4 that the ergodic average for the horocycle orbit and x converges to the integral of the test function with respect to the Haar measure. As x was chosen to be generic for μ we also have that the averages converge to the integral with respect to μ.

Therefore μ is either the Lebesgue measure on a periodic orbit or the Haar measure on X. (See [54, Ch. 11] for more details.) □

We would also like to point out that the same argument can be used to prove the following theorem† of Sarnak [161] (see Exercise 5.3.1).

† Sarnak also gives an error rate in this equidistribution result — obtaining this (or even any) error estimate requires more sophisticated methods than we will discuss here.

Theorem 5.5 (Equidistribution of long periodic orbits of the horocycle flow). *Let $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ be a non-compact quotient of finite volume. Let $a = a_1$ be the element corresponding to the time-one map for the geodesic flow corresponding the diagonal subgroup $A = \{a_t \mid t \in \mathbb{R}\}$. Let $x \in X$ be a periodic orbit for the horocycle flow $U = G_a^-$ and let μ be the normalized Lebesgue measure on the one-dimensional orbit xU. Then the periodic orbit measures $(a_{t_n})_*\mu$ diverge to infinity if $t_n \to \infty$ (in which case the periodic orbit $a_{t_n}\cdot(xU)$ becomes shorter and shorter) and equidistribute with respect to the Haar measure if $t_n \to -\infty$ (in which case the periodic orbit $a_{t_n}\cdot(xU)$ become longer and longer).*

5.3.2 The General Case

By analyzing the proof of Theorem 5.4 more carefully we identify the places where we used that X is compact:

- We used test functions $f \in C(X)$ and that these are uniformly continuous.
- We used that any subsequence of the sequence of points $a^n\cdot x$ converges along some subsequence to some $z \in X$ (equivalently that the injectivity radius at these points is bounded away from zero).

The first point is trivial to fix: we just work with functions $f \in C_c(X)$. These are still uniformly continuous and together are still dense in $L^2_\mu(X)$ for any probability measure on X.

The second point is of course the reason why the argument fails to prove unique ergodicity for non-compact spaces: It is entirely possible for the sequence $a^n\cdot x$ to go to infinity. In fact, for $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ this happens precisely for points with a periodic orbit for the horocycle flow. Even so, for the horocycle flow we found that for a non-periodic point at least one has some subsequence that stays within a compact subset and that this was sufficient for our purposes. However, for other spaces the divergence properties of a sequence of the form $a^n\cdot x$ are potentially much more complicated, and in particular a clear equivalence to rational properties of the starting point x may not hold. For that reason we are going to invoke the *Margulis–Dani–Kleinbock non-divergence* (Theorem 4.9) to obtain rational constraints on μ (rather than on[†] x).

Before we state the (somewhat inductive) description of invariant measures for horospherical subgroups we state the general version of the argument that was used in the proof of Theorem 5.4.

Proposition 5.6 (Mixing argument for G_a^-). *Let $X = G\cdot x_0 \subseteq \mathsf{X}_d$ be a finite volume orbit for a closed connected subgroup $G \leqslant \mathrm{SL}_d(\mathbb{R})$. Let $a \in G$*

[†] See the next section where we go further and describe properties of a given x.

be diagonalizable over $\mathbb{R}$*, and suppose that a acts as a mixing transformation on* X *with respect to* m_X*. Let* G_a^- *be the stable horospherical subgroup for a, and let* B_0 *be a neighborhood of* $I \in G_a^-$ *with compact closure and a boundary of zero Haar measure. For any* $f \in C_c(X)$*, any compact set* $K \subseteq X$*, and every* $\varepsilon > 0$ *there exists an integer* k_0 *such that*

$$\left| \frac{1}{m_{G_a^-}(a^{-k}B_0a^k)} \int_{a^{-k}B_0a^k} f(u\bullet x)\,\mathrm{d}m_{G_a^-}(u) - \int_X f\,\mathrm{d}m_X \right| < \varepsilon$$

for all $k \geqslant k_0$ *whenever* $a^k\bullet x \in K$.

Proof. We show how, after minor modifications, the argument for the proof of Theorem 5.4 also proves the proposition.

As $f \in C_c(X)$ is uniformly continuous, we can choose $V \subseteq P_a$ as in the proof of Theorem 5.4, again with a boundary of zero Haar measure. This shows that $VB_0 \subseteq P_aG_a^- \subseteq G$ has a boundary of zero measure. Therefore, there exists a compact set $C \subseteq (VB_0)^o$ and an open set $O \supseteq \overline{VB}$ such that $m_G(O\smallsetminus C) < \varepsilon^2$. If now $\delta > 0$ is sufficiently small, then $CB_\delta^G \subseteq VB_0$ and $\overline{VB_0}B_\delta^G \subseteq O$. By compactness of K it follows that

$$K \subseteq \bigcup_{i=1}^n B_\delta^G\bullet x_i$$

for some finite collection $x_1, \dots, x_n \in K$. This implies that for every $x \in K$ there is some x_i with $x \in B_\delta^G\bullet x_i$,

$$C\bullet x \subseteq CB_\delta^G\bullet x_i \subseteq VB_0\bullet x_i$$

and

$$O\bullet x \supseteq VB_0B_\delta^G\bullet x \supseteq VB_0\bullet x_i.$$

We also have trivially $C\bullet x \subseteq VB_0\bullet x \subseteq O\bullet x$, and so we get

$$\|\mathbb{1}_{VB_0\bullet x} - \mathbb{1}_{VB_0\bullet x_i}\|_2 < \varepsilon.$$

To summarise, we have shown that the set of characteristic functions $\mathbb{1}_{VB_0\bullet x}$ in $L^2(X, m_X)$ with $x \in K$ is totally bounded. By applying the mixing property to f and to $\mathbb{1}_{VB_0\bullet x_i}$ for $i = 1, \dots, n$, we may assume that mixing holds uniformly for f and for all $\mathbb{1}_{VB_0\bullet x}$ for $x \in K$. Now the argument used in the proof of Theorem 5.4 gives the proposition. □

Proposition 5.7. *Let* $X = G\bullet x_0 \subseteq \mathsf{X}_d$ *be a finite volume orbit of some closed connected subgroup* $G < \mathrm{SL}_d(\mathbb{R})$ *and some point* $x_0 \in \mathsf{X}_d$*. Let* $a \in G$ *and assume that the action of a on* X *is mixing with respect to the Haar measure* m_X*. Let* $U = G_a^- < G$ *be a horospherical subgroup defined by* $a \in G$*. Then any* U*-invariant and ergodic probability measure on* X *other than* m_X

is supported on a closed orbit $L{\scriptstyle\bullet}x$ for some closed connected proper subgroup $L < G$ and some $x \in X$.

For the proof the following will be helpful.(22)

Lemma 5.8. *Let U be a nilpotent connected Lie group acting ergodically on a locally compact metric space X with respect to some invariant probability measure μ. Then there exists a one-parameter subgroup of U that also acts ergodically.*

PROOF. We consider first the case where $U \cong \mathbb{R}^d$ is abelian, with $d \geqslant 1$, where we will apply the spectral theory of locally compact abelian groups (we refer to Folland [65] or [48] for the details). Let U act on X, and let μ be a U-invariant and ergodic probability measure. For each $u \in U$ let

$$\pi_u \colon \mathcal{H} = L^2(X, \mu) \longrightarrow \mathcal{H}$$

be the associated unitary representation defined by

$$\left(\pi_u(f)\right)(x) = f(u^{-1}{\scriptstyle\bullet}x).$$

By [48] or [65, Sec. 1.4] there exists a sequence† of spectral measures that completely describe the unitary representations. Specifically, there exists a sequence (ν_n) of finite measures on $\mathbb{R}^d$ such that π_u is unitarily isomorphic to the operator

$$\begin{aligned} M_u \colon \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^d, \nu_n) &\longrightarrow \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^n, \nu_n) \\ (f_n) &\longmapsto \left(M_{u,n}(f)\right)_n \end{aligned}$$

where

$$M_{u,n}(f_n)(t) = \mathrm{e}^{2\pi \mathrm{i}\langle u,t\rangle} f_n(t).$$

We may suppose that $\nu_0 = \delta_0$, with $M_{u,0}$ the trivial representation of U on $\mathbb{C}$ corresponding to the invariant subspace of constant functions in the Hilbert space $\mathcal{H} = L^2(X, \mu)$. By ergodicity of the U-action, we must have $\nu_n(\{0\}) = 0$ for $n \geqslant 1$, for otherwise we could find a non-trivial U-invariant L^2-function in the orthogonal complement of the constant functions. Therefore, we may push forward the measures ν_n for $n \geqslant 1$ from $\mathbb{R}^d \smallsetminus \{0\}$ to the projective space $\mathbb{P}^{d-1}(\mathbb{R}) \cong \mathbb{R}^d \smallsetminus \{0\}/{\sim}$, where $x \sim y$ if and only if there is some $\lambda \in \mathbb{R} \smallsetminus \{0\}$ with $x = \lambda y$. As these countably many finite measures can only have countably many atoms in total, and $\mathbb{P}^{d-1}(\mathbb{R})$ is uncountable as $d \geqslant 2$, there must be a point in $\mathbb{P}^{d-1}(\mathbb{R})$ of zero measure for all of the measures. Hence there is a line $\mathbb{R}v \leqslant \mathbb{R}^d$ with $\nu_n(\mathbb{R}v) = 0$ for $n \geqslant 1$. We claim that the

† Here we are using the fact that $\mathcal{H}$ is separable, which in turn follows from the fact that X is a locally compact σ-compact metric space.

restriction of the ergodic $U \cong \mathbb{R}^d$-action on (X,μ) to the hyperplane $v^\perp \leqslant \mathbb{R}^d$ is still ergodic. Indeed, if $f \in L^2(X,\mu)$ is orthogonal to the constant functions, then it corresponds to an element

$$(f_n) \in \bigoplus_{n \geqslant 1} L^2(\mathbb{R}^d, \nu_n),$$

and if f is $v^\perp$-invariant then the functions f_n for $n \geqslant 1$ would have to be supported on $\mathbb{R}v$ (since this is the subset of $\mathbb{R}^d$ where all the operators M_u with $u \in v^\perp$ act trivially). However, by the choice of v this forces $f_n = 0$ for $n \geqslant 1$ and hence forces f to be 0 (since it is assumed to be orthogonal to the constant functions).

The argument above shows that every ergodic action of $\mathbb{R}^d$ with $d \geqslant 2$ can be restricted to a hyperplane with the property that the restriction remains ergodic. By induction, this shows that the lemma holds for $U \cong \mathbb{R}^d$.

Now suppose that U is nilpotent but not abelian. We define

$$U' = [U,U]$$

so that $G = U/U' \cong \mathbb{R}^d$ for some $d \geqslant 2$. By assumption U acts ergodically by measure-preserving transformations of (X,μ). Recall from [54, Th. 8.20] (also see [54, Sec. 5.3]) that μ has an ergodic decomposition

$$\mu = \int_X \mu_x^{\mathcal{E}'} \,\mathrm{d}\mu(x) \tag{5.5}$$

into U'-ergodic components $\mu_x^{\mathcal{E}'}$ given by the conditional measures for the σ-algebras

$$\mathcal{E}' = \{B \subseteq X \mid \mu(u'\boldsymbol{\cdot} B \triangle B) = 0 \text{ for all } u' \in U'\}.$$

We now show that $\mathcal{E}'$ is U-invariant in the sense that $B \in \mathcal{E}'$ implies that $u\boldsymbol{\cdot} B \in \mathcal{E}'$ for all $u \in U$. Indeed, for $B \in \mathcal{E}'$, $u \in U$, and $u' \in U'$, we have

$$\mu\left(u'\boldsymbol{\cdot}(u\boldsymbol{\cdot} B)\triangle(u\boldsymbol{\cdot} B)\right) = \mu\left((u\boldsymbol{\cdot}((u''\boldsymbol{\cdot} B)\triangle B))\right) = 0,$$

where $u'' = u^{-1}u'u \in U''$. By [54, Cor. 5.24] this shows that

$$u_*\mu_x^{\mathcal{E}'} = \mu_{u\cdot x}^{\mathcal{E}'}$$

for almost every $x \in X$ and all $u \in U$. Hence the map

$$\begin{aligned} \phi\colon (X,\mu) &\longrightarrow \left(\mathcal{M}(X)^{U'}, \phi_*\mu\right) \\ x &\longmapsto \mu_x^{\mathcal{E}'} \end{aligned}$$

is a μ-almost everywhere well-defined factor map intertwining the U-action on (X,μ) and the induced U-action on $(\mathcal{M}(X)^{U'}, \phi_*\mu)$, where $\mathcal{M}(X)^{U'}$ denotes the space of U'-invariant measures with total mass no more than 1, and the induced action is defined by $u \cdot \nu = u_* \nu$ for $u \in U$ and $\nu \in \mathcal{M}(X)^{U'}$. Since by construction U' acts trivially on $\mathcal{M}(X)^{U'}$ we have obtained an ergodic[†] action of $G = U/U'$ on $(\mathcal{M}(X)^{U'}, \phi_* \mu)$. By the abelian case, there is a subgroup $H \leqslant G$ isomorphic to $\mathbb{R}$ which still acts ergodically on $(\mathcal{M}(X)^{U'}, \phi_* \mu)$. Let HU' denote the pre-image of H in U. We claim that the subgroup HU' still acts ergodically on (X, μ). Assuming this, the lemma follows by induction as HU' is a nilpotent group of smaller dimension than U.

Suppose now that $B \subseteq X$ is an HU'-invariant measurable set. In particular, it is U'-invariant and so we must have

$$\mu_x^{\mathcal{E}'}(B) \in \{0, 1\} \tag{5.6}$$

for μ-almost every $x \in X$. We define

$$B' = \{\nu \in \mathcal{M}(X)^{U'} \mid \nu(B) = 1\},$$

and notice that $\nu \in \mathcal{M}(X)^{U'}$ and $h \in HU'$ implies that

$$h_* \nu(B) = \nu(h^{-1} \cdot B) = \nu(B),$$

so that B' is an H-invariant set. By ergodicity of H on $(\mathcal{M}(X)^{U'}, \phi_* \mu)$, this shows that

$$\phi_* \mu(B') = \mu(\{x \mid \mu_x^{\mathcal{E}'}(B) = 1\}) \in \{0, 1\}.$$

By (5.5) and (5.6), this shows that $\mu(B) \in \{0, 1\}$ and so HU acts ergodically, which concludes the induction. □

Proof of Proposition 5.7. Let $X = G \cdot x_0$ and $a \in G$ be as in the theorem. Let μ be a G_a^--invariant ergodic probability measure on X. By Lemma 5.8 there is a one-parameter unipotent subgroup $U < G_a^-$ that acts ergodically with respect to μ.

Without loss of generality we may assume that $x_0 \in \operatorname{supp} \mu$ and that x_0 is U-generic (see Section 6.3.1 and [54, Sec. 4.4.2]). In particular, we have

$$\overline{x_0 U} = \operatorname{supp} \mu.$$

Let $g_0 \in \mathrm{SL}_d(\mathbb{R})$ be chosen with $x_0 = \Gamma g_0$, and let $\Lambda_0 = \mathbb{Z}^d g_0$ be the corresponding unimodular lattice in $\mathbb{R}^d$. For every Λ_0-rational subspace[‡] $V \subseteq \mathbb{R}^d$ we define

$$L_V = \{g \in G \mid V = Vg \text{ and } g|_V \text{ preserves the volume}\}.$$

[†] Ergodicity is automatic, as the system is exhibited as a factor of an ergodic system.

[‡] A subspace V is called Λ_0-rational if $V \cap \Lambda_0$ is also a lattice.

Applying Exercise 3.1.4 we see that $x_0 L_V$ is closed. If there exists one such proper subgroup $L_V \subsetneq G$ such that $\operatorname{supp}\mu \subseteq x_0 L_V$, then the theorem already holds for μ. Therefore, we assume that $\operatorname{supp}\mu \not\subseteq x_0 L_V$ for every $L_V \subsetneq G$.

We define

$$\eta = \min\left\{\operatorname{covol}(\Lambda_0 \cap V, V)^{1/\dim V} \mid V \text{ is } \Lambda_0\text{-rational}\right\}. \tag{5.7}$$

Applying the quantitative non-divergence theorem (Theorem 4.9) with D chosen for U and η as above, we find some $\varepsilon > 0$ with the following property: For any $x = \Gamma g \in \mathsf{X}_d$, any one-parameter subgroup U' with the same D as worked for U, we have that either U' fixes a $\mathbb{Z}^d g$-rational subspace V with

$$\operatorname{covol} V < \eta^{\dim V},$$

or there exists some T_x such that

$$\frac{1}{T}\left|\{t \in [0,T] \mid u'_t \cdot x \in \mathsf{X}_d(\varepsilon)\}\right| > \frac{9}{10}. \tag{5.8}$$

for $T > T_x$. We set $K = X \cap \mathsf{X}_d(\varepsilon)$. Applying (5.8) to $x = x_0$, $U' = U$, and $T \to \infty$, for example, gives $\mu(K) > \frac{9}{10}$.

We now let $n \geqslant 1$ and set $\mu_n = a^n_* \mu$. Notice that the subgroup $a^n U a^{-n}$ acts ergodically with respect to μ_n, and that $x_n = a^n \cdot x_0$ is generic for $a^n U a^{-n}$ and μ_n. Suppose that the lattice $\Lambda_n = \Lambda_0 a^{-n}$ corresponding to x_n has a Λ_n-rational subspace V with

$$\operatorname{covol}(V) < \eta^{\dim V}.$$

If the one-parameter subgroup $a^n U a^{-n}$ stabilizes V then $a^n U a^{-n} \leqslant L_V$, the orbit $x_n L_V$ is closed by Exercise 3.1.4, and so

$$\overline{a^n U a^{-n} \cdot x_n} = \operatorname{supp}\mu_n \subseteq x_n L_V$$

gives

$$\overline{U \cdot x_0} = \operatorname{supp}\mu \subseteq x_0 L_{V a^n}.$$

Since $a \in G \smallsetminus L_{V a^n}$, this would contradict our assumption on μ. As this holds for all subspaces as above, we see that $x = x_n$ and $U' = a^n U a^{-n}$ satisfy the assumptions that lead to (5.8). Letting $T \to \infty$ gives $\mu_n(K) \geqslant \frac{9}{10}$.

In other words, we have shown that $\frac{9}{10}$ of all $x \in X$ with respect to μ satisfy $a^n \cdot x \in K$. For any such x, any $f \in C_c(X)$, and any $\varepsilon > 0$, we may apply Proposition 5.6 to see that

$$\frac{1}{m_{G_a^-}(a^{-n} B_0 a^n)} \int_{a^{-n} B_0 a^n} f(h \cdot x)\, \mathrm{d}m_{G_a^-}(h) = \int_X f \,\mathrm{d}m_X + \mathrm{O}(\varepsilon)$$

if n is large enough. By Exercise 5.2.1, $(a^{-n} B_0 a^n)$ is a Følner sequence in G_a^-. Therefore, we may apply the mean ergodic theorem (see [54, Th. 8.13]) to see that for large enough n and $\frac{9}{10}$ of all $x \in X$ with respect to μ we have

$$\frac{1}{m_{G_a^-}(a^{-n}B_0a^n)} \int_{a^{-n}B_0a^n} f(h\centerdot x)\,\mathrm{d}m_{G_a^-}(h) = \int_X f\,\mathrm{d}\mu + \mathrm{O}(\varepsilon).$$

As $\varepsilon > 0$ and $f \in C_c(X)$ were arbitrary, we deduce that

$$\int_X f\,\mathrm{d}\mu = \int_X f\,\mathrm{d}m_X,$$

and conclude that $\mu = m_X$. □

Let us give a complete description in the case of X_3.

Theorem 5.9. *Let* $\mathsf{X}_3 = \mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})$. *Let* $U = G_a^-$ *be a horospherical subgroup of* $\mathrm{SL}_3(\mathbb{R})$ *defined by some* $\mathbb{R}$*-diagonalizable element* $a \in \mathrm{SL}_3(\mathbb{R})$. *Then any* U*-invariant and ergodic probability measure is* algebraic*, meaning that it is the Haar measure on a closed orbit of a closed connected subgroup* L *in* $\mathrm{SL}_3(\mathbb{R})$. *In fact, we could have either* $L = \mathrm{SL}_3(\mathbb{R})$, $L \simeq \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ *(which can be embedded into* $\mathrm{SL}_3(\mathbb{R})$ *in two non-conjugate ways), or* L *could be unipotent.*

Sketch of Proof. Let μ be a U-invariant and ergodic probability measure on X_3 as in the corollary. If $\mu \neq m_{\mathsf{X}_3}$ then there exists a subgroup $L < \mathrm{SL}_3(\mathbb{R})$ such that μ is supported by the closed orbit $L\centerdot x$ for some $x \in \mathsf{X}_3$. As the proof of Proposition 5.7 shows, we may assume that L is the stabilizer of a vector v or the stabilizer of a volume element $v_1 \wedge v_2$ in a plane in $\mathbb{R}^3$. Conjugating a and U we may suppose that $x = \mathrm{SL}_d(\mathbb{Z})$ is the identity coset, v resp. v_1, v_2 are integer vectors, and so $L\centerdot x$ is isomorphic to $\mathrm{ASL}_2(\mathbb{Z})\backslash\mathrm{ASL}(\mathbb{R})$. (This quotient can be embedded into X_3 in two ways depending on whether the radical is chosen to be represented by row vectors or by column vectors.)

If U is a two-dimensional horospherical subgroup, then either U can still be defined as a horospherical subgroup $U = L_{a'}^-$ for some $a' \in L$ or U equals the radical (which has a closed orbit for all points in xL). Hence we may repeat the argument or are done. If U is the Heisenberg group, then $U < L$ is not a horospherical subgroup within L, but U contains the full radical. Taking the quotient by the radical we obtain a measure on the modular surface that is invariant under the horocycle flow. In each of these cases we obtain that μ is algebraic and find that L is one of the mentioned subgroups. □

Exercises for Section 5.3

Exercise 5.3.1. Prove Theorem 5.5 using the method of proof from Theorem 5.4.

5.4 Equidistribution for Non-Compact Quotients

Dani and Smillie showed in [29] that even for non-compact quotients

$$X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$$

a rather strong equidistribution theorem holds: a horocycle orbit is either periodic or it equidistributes with respect to the uniform measure m_X.

For higher dimensional non-compact quotients $X = \Gamma\backslash G$ and their horospherical actions other possibilities can occur. For the following characterization of whether or not a horospherical orbit equidistributes we specialize to the case where the horospherical subgroup is abelian.

Before we can prove Theorem 5.11 we need to extend the non-divergence result to actions of more general unipotent groups. For simplicity we give only the version needed for the case at hand.

Corollary 5.10. *Let U be an abelian unipotent subgroup of $\mathrm{SL}_d(\mathbb{R})$, and fix some coordinate system identifying U with $\mathbb{R}^k$ and with respect to which we can describe 'blocks' whose edges are parallel to the coordinate axes. Then for every $\eta > 0$ and every $\delta > 0$ there exists a compact subset $K \subseteq \mathsf{X}_d$ with the property that for any $x \in \mathsf{X}_d$ either*

- *there is a Λ_x-rational U-invariant subspace V with $\operatorname{covol}(V) < \eta^{\dim V}$, or*
- *for any symmetric block $F \subseteq U$ with sufficiently large width we have*

$$\frac{1}{m_U(F)} m_U\left(\{u \in F \mid u\bullet x \in K\}\right) > 1 - \delta.$$

Proof. The corollary follows quite directly from Theorem 4.9. Let D be chosen so that we may apply Theorem 4.9 for any one-parameter subgroup of U, and let $x \in \mathsf{X}_d$ be arbitrary.

If U fixes a Λ_x-rational subspace $V \subseteq \mathbb{R}^d$ with $\operatorname{covol}(V) < \eta^{\dim V}$, then there is nothing to prove. So suppose that this is not the case. As there are only finitely many Λ_x-rational subspaces with co-volume less than $\eta^{\dim V}$, and for each such subspace the subgroup of U that fixes V is of codimension at least 1, there exists a one-parameter subgroup

$$U' = \{u'(t) \mid t \in \mathbb{R}\} \subseteq U$$

that does not fix any of these subspaces. Applying Theorem 4.9 to $p(t) = gu'(t)$ with $x = \mathrm{SL}_d(\mathbb{Z})g$, η as above, some $\varepsilon_0 > 0$ (depending on η and the implicit constant in (4.19) only), and some possibly very large T (depending on x), it follows that there exists at least one $t \in \mathbb{R}$ with

$$x' = u'(t)\bullet x \in \mathsf{X}_d(\varepsilon_0).$$

Now let $\{u_1(t_1) \mid t_1 \in \mathbb{R}\}, \{u_2(t_2) \mid t_2 \in \mathbb{R}\}, \ldots, \{u_k(t_k) \mid t_k \in \mathbb{R}\}$ be the one-parameter subgroups of U corresponding to the chosen coordinate system in $U \cong \mathbb{R}^k$.

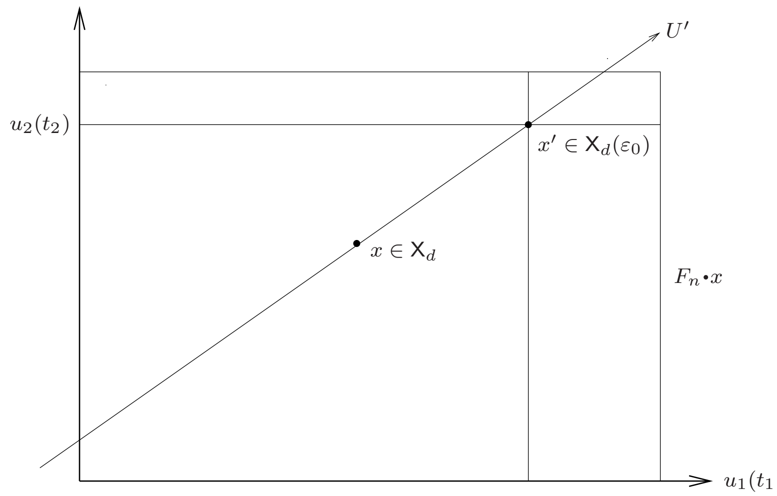

Fig. 5.3: The symmetric box $F_n \cdot x$ inside the U-orbit has to contain $x' \in \mathsf{X}_d(\varepsilon)$ if the width of F_n is sufficiently large.

We can now split $F_n{\centerdot}x$ as indicated in Figure 5.3 above into 2^k sets of the form $F{\centerdot}x'$ where each F is a block with the origin as one corner. For simplicity we consider only the block in the positive quadrant where

$$F = \{u_1(t_1)u_2(t_2)\cdots u_k(t_k) \mid 0 \leqslant t_i \leqslant T_i, i = 1, \ldots, k\} \subseteq U,$$

the blocks in the other quadrants can be dealt with in the same way. We now successively choose $\varepsilon_1, \ldots, \varepsilon_k$ (depending only on ε) such that

$$\frac{1}{T_1} \left|\{t_1 \in [0, T_1] \mid u_1(t_1){\centerdot}x' \notin \mathsf{X}_d(\varepsilon_1)\}\right| < \frac{\delta}{k},$$

and, if $u_1(t_1){\centerdot}x' \in \mathsf{X}_d(\varepsilon_1)$,

$$\frac{1}{T_2} \left|\{t_2 \in [0, T_2] \mid u_1(t_1)u_2(t_2){\centerdot}x' \notin \mathsf{X}_d(\varepsilon_2)\}\right| < \frac{\delta}{k},$$

and so on, ending with

$$\frac{1}{T_k} \left|\{t_k \in [0, T_k] \mid u_1(t_1)\cdots u_k(t_k){\centerdot}x' \notin \mathsf{X}_d(\varepsilon_k)\}\right| < \frac{\delta}{k}$$

if $u_1(t_1)\cdots u_{k-1}(t_{k-1}) \in \mathsf{X}_d(\varepsilon_{k-1})$. We set $K = \mathsf{X}_d(\varepsilon_k)$, and the corollary follows. □

Theorem 5.11. *Let $G\bullet x_0 \subseteq \mathsf{X}_d = \mathrm{SL}_d(\mathbb{Z})\backslash\mathrm{SL}_d(\mathbb{R})$ be a finite volume orbit for some closed connected subgroup $G \leqslant \mathrm{SL}_d(\mathbb{R})$ and some point $x_0 \in \mathsf{X}_d$. Let $a \in G$ be diagonalizable over $\mathbb{R}$ so that the action of a is mixing with respect to $m_{G\bullet x_0}$. Let $U = G_a^-$ be the stable horospherical subgroup of a and suppose that it is abelian. Let (F_n) be a Følner sequence in U consisting of blocks whose sides are parallel to some fixed coordinate system spanned by some eigenvectors for the conjugation map by a. Then for every $x \in G\bullet x_0$ the following are equivalent*

(1) *The U-orbit through x is equidistributed, meaning that*

$$\frac{1}{m_U(F_n)}\int_{F_n} f(u_t)\,\mathrm{d}m_U(t) \to \int_{\mathsf{X}_d} f\,\mathrm{d}m_{\mathsf{X}_d}$$

for any $f \in C_c(\mathsf{X}_d)$.

(2) *The orbit $U\bullet x$ is not contained in a closed orbit $L\bullet x$ for some proper connected subgroup $L < \mathrm{SL}_d(\mathbb{R})$.*

If, in addition, $G = \mathrm{SL}_d(\mathbb{R})$ then we also have the equivalence to the next property.

(3) *Let $x = \mathrm{SL}_d(\mathbb{Z})g$ for some $g \in \mathrm{SL}_d(\mathbb{R})$. Then there is no rational subspace $V \subseteq \mathbb{R}^d$ for which Vg is fixed by U and contracted by a.*

Proof. We let $x \in G\bullet x_0$ be as in the theorem. If the G_a^--orbit of x is contained in a closed orbit of a proper connected subgroup $L < G$ as in (2), then clearly we cannot have equidistribution of the G_a^--orbit as in (1). This shows that (1) implies (2), so we now assume that (2) holds.

Fix some $f \in C_c(\mathsf{X}_d)$ and $\varepsilon > 0$. We let $x_0 = \Gamma g_0$, $\Lambda_0 = \mathbb{Z}^d g_0$ be the lattice corresponding to x_0, and define η as in (5.7). By quantitative non-divergence for the action of $U = G_a^-$ there exists some compact set $K \subseteq \mathsf{X}_d$ with the property as in Corollary 5.10 with $\delta = \varepsilon$. We let B_0 be the symmetric unit cube (that is, centered at the origin) in $G_a^- \cong \mathbb{R}^k$. Applying Proposition 5.6 to f, KB_0, and ε we find some $k \geqslant 1$ such that

$$\left|\frac{1}{m_{G_a^-}(a^{-k}B_0a^k)}\int_{a^{-k}B_0a^k} f(u\bullet y)\,\mathrm{d}m_{G_a^-}(u) - \int_X f\,\mathrm{d}m_X\right| < \varepsilon \tag{5.9}$$

whenever $a^k\bullet y \in (KB_0)\cap X$ (or equivalently whenever $B_0a^k\bullet y$ intersects K).

Now let $x' = a^k\bullet x$ and notice that it may not belong to K. Since (F_n) is chosen to be a Følner sequence consisting of symmetric blocks, the same is true for $a^kF_na^{-k}$. If $U = G_a^-$ fixes a $\Lambda_{x'}$-rational subspace V of co-volume $< \eta^{\dim V}$, then we can define the subgroup

$$L' = \mathrm{Stab}^1_G(V) = \{g \in G \mid Vg = V \text{ and } g|_V \text{ has determinant } 1\} \lneq G.$$

Exercise 3.1.4 shows that $x'L'$ is closed, which shows that

$$x'L'a^k = x'a^k a^{-k} L' a^k = xL,$$

where $L = a^{-k}L'a^k$, is a closed orbit which contradicts the assumption that (2) holds if L is a proper subgroup of G.

It follows that U does not fix any Λ_x-rational subspaces that are not already fixed by G. Applying Corollary 5.10 we see now that for large enough n we have

$$\frac{1}{m_{G_a^-}(a^k F_n a^{-k})} m_{G_a^-}\left(\{u \in a^k F_n a^{-k} \mid u\boldsymbol{\cdot} x' \notin K\}\right) < \varepsilon. \tag{5.10}$$

We now split $a^k F_n a^{-k}$ into translates of the form $B_0 u_\ell$ for $\ell = 1, \dots, L$ of the unit cube B_0. Ignoring the effects of the boundary which contribute no more than $\mathrm{o}_f(1)$ to the ergodic average as $n \to \infty$, we now have

$$\frac{1}{m_{G_a^-}(F_n)} \int_{F_n} f(u\boldsymbol{\cdot} x)\,\mathrm{d}m_{G_a^-}$$

$$= \frac{1}{L}\sum_{\ell=1}^{L} \frac{1}{m_{G_a^-}(a^{-k}B_0 a^k)} \int_{a^{-k}B_0 a^k} f\left(u a^{-k} u_\ell a^k \boldsymbol{\cdot} x\right)\,\mathrm{d}m_{G_a^-} + \mathrm{o}_f(1).$$

For all those ℓ for which $B_0 u_\ell a^k \boldsymbol{\cdot} x$ intersects K the corresponding average is ε-close to $\int_X f\,\mathrm{d}m_X$ by (5.9). However, the number of boxes $B_0 u_\ell \boldsymbol{\cdot} x'$ that do not intersect K is controlled by (5.10), and gives

$$\frac{1}{m_{G_a^-}(F_n)} \int_{F_n} f(u\boldsymbol{\cdot} x)\,\mathrm{d}m_{G_a^-}(u) = \int_X f\,\mathrm{d}m_X + \mathrm{o}_f(1) + \mathrm{O}_f(\varepsilon).$$

As $\varepsilon > 0$ and $f \in C_c(X)$ were arbitrary, this shows (1).

Now suppose that $G = \mathrm{SL}_d(\mathbb{R})$ and

$$a = \begin{pmatrix} \lambda^n I_m & \\ & \lambda^{-m} I_n \end{pmatrix} \in \mathrm{SL}_d(\mathbb{R})$$

for some $\lambda > 1$ so that

$$G_a^- = \left\{ \begin{pmatrix} I_m & \\ * & I_n \end{pmatrix} \right\}$$

is indeed abelian. (Up to signs in the entries of a and the choice of m and n this is the only choice of a for which G_a^- is abelian.) If

$$L = \mathrm{Stab}^1_{\mathrm{SL}_d(\mathbb{R})}(V)$$

for some proper subspace V is G_a^--invariant, then either $V \subseteq \mathbb{R}^m \times \{0\}^n$ or V contains some $v = (v_m, v_n)$ with $v_m \in \mathbb{R}^m$ and $v_n \in \mathbb{R}^n \smallsetminus \{0\}$, which implies that $\mathbb{R}^m \times \{0\}^n \subseteq V$. In both cases $Va^{-1} = V$ and the restriction of a^{-1} to V has determinant smaller than 1. □

In the exercises we outline how one can remove the assumptions on commutativity of U.

Exercises for Section 5.4

Exercise 5.4.1. Let $G{\scriptstyle\bullet}x_0 \subseteq \mathsf{X}_d = \mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$ be a finite volume orbit for some closed connected subgroup $G \leqslant \mathrm{SL}_d(\mathbb{R})$ and some point $x_0 \in \mathsf{X}_d$.

Let U be a unipotent subgroup and let $U_1, \ldots, U_\ell < U$ be one-parameter subgroups such that $U = U_1 \cdots U_\ell$. Suppose $F_n = F_{1,n} \cdots F_{\ell,n}$ is a Følner sequence with respect to left and right translation where $F_{i,n} \subseteq U_i$ corresponds to an interval in U_i.

Prove that

$$\frac{1}{m_U(F_n)} m_U(\{u \in F_n \mid u{\scriptstyle\bullet}x \notin \mathsf{X}_d(\delta)\}) \ll \delta^\kappa + o(1)$$

for $n \to \infty$ and some $\kappa > 0$ (depending on $U_1, \ldots, U_\ell$) in the following two cases:

(a) x belongs to a fixed compact subset and the implicit constant is allowed to depend on the compact subset, and
(b) x is arbitrary but U does not fix any Λ_x-rational subspace V of co-volume $\eta^{\dim V}$ and the implicit constant is allowed to depend on η.

Exercise 5.4.2. Let $G{\scriptstyle\bullet}x_0 \subseteq \mathsf{X}_d = \mathrm{SL}_d(\mathbb{Z})\backslash \mathrm{SL}_d(\mathbb{R})$ be a finite volume orbit for some closed connected subgroup $G \leqslant \mathrm{SL}_d(\mathbb{R})$ and some point $x_0 \in \mathsf{X}_d$. Let $a \in G$ be diagonalizable over $\mathbb{R}$ so that the action of a is mixing with respect to $m_{G{\scriptstyle\bullet}x_0}$. Let $U = G_a^-$ be the stable horospherical subgroup of a and let F_n be as in Exercise 5.4.1. Let $x \in G{\scriptstyle\bullet}x_0$. Suppose that U does not fix any Λ_x-rational subspace which is not also fixed by G. Show that $F_n{\scriptstyle\bullet}x$ equidistributes in $X = G{\scriptstyle\bullet}x_0$, i.e.

$$\frac{1}{m_U(F_n)} \int_{F_n} f(u{\scriptstyle\bullet}x) dm_U(u) \longrightarrow \int_X f dm_X$$

as $n \to \infty$ for any $f \in C_c(X)$.

Notes to Chapter 5

(20) (Page 160) We refer to [54, Ex. 3.3.1, 9.6.3] for one example of such a construction. McMullen [131] gives explicit constructions of bounded geodesics of arbitrary length associated to elements of any given quadratic field, and relates the construction to continued fractions.
(21) (Page 163) This is an example of a circle of results developed among others by Dani [20], [26] and Veech [182].
(22) (Page 169) A much stronger form of this result is obtained by Pugh and Shub [146], where it is shown that if T is an ergodic measure-preserving action of $\mathbb{R}^d$ on a Borel probability space, then there is a countable collection $\{H_n \mid n \in \mathbb{N}\}$ of hyperplanes with

the property that for any $g \in \mathbb{R}^d \smallsetminus \bigcup_{n\in\mathbb{N}} H_n$ the measure-preserving transformation T_g is ergodic. In our setting, Lemma 5.8 can be avoided by using finitely many one-parameter subgroups as in the proof of Theorem 4.11 for unipotent subgroups on page 155.