Introduction

These notes will eventually become a volume on the interaction between number theory and dynamics via homogeneous spaces.

A big portion of this text (specifically, most of Part I and Part III) should be digestible for a reader who is familiar with measure theory and the basics of functional analysis. We will make efforts to avoid going beyond these prerequisites (except for in Part II — see below). There are additional prerequisites at various places of the text, and some of these can be avoided. For example, because the development of the theory is not strictly linear, the reader could skip some sections of the text that need more powerful background without losing track of the main ideas. We have marked those sections by footnotes.

The basics of Lie theory including, for instance, basic facts about the Lie algebra, the exponential map, and the adjoint representation, will be assumed after a brief review in Chapter 2. Unless the reader strives for maximal generality in her understanding, the more concrete case of linear groups (that is, subgroups of $\mathrm{SL}_d(\mathbb{R})$) is quite sufficient both in breadth of applications and in terms of issues arising. On the other hand, at certain places in Chapter 2 we will use the full force of Lie theory (including the Cartan decomposition, the Levi decomposition, and the Jacobson–Morozov theorem). However, that portion of Chapter 2 could also be skipped, and is not used later.

This text is part of a larger project that started with the book *'Ergodic theory with a view towards Number theory'* [53] and is (at modest but positive speed) being developed in parallel with *'Entropy in ergodic theory and homogeneous dynamics'* [47]. We will not repeat material from the other two volumes, and will refer to them as needed. Initially we need very few facts from [53], namely the Poincaré recurrence theorem and the pointwise ergodic theorem.

The text *'Entropy in ergodic theory and homogeneous dynamics'* only becomes relevant in Part II of the current notes (to be honest, some of the theorems in Chapter 6 will use the results of Part II to give the reader motivation for the latter).

In order to focus on the theory we plan to be quite brief about historical remarks throughout the text, and apologize in advance for any missing references that we should have included. We also try to develop the theory partly from a logical and partly from an instructional point of view. Historically, many theorems in these notes were much harder to prove initially, and as we have no desire to suffer ourselves, nor to cause suffering, where it is not strictly necessary we take the logically simpler route (even at the price of ignoring some interesting connections to other topics). On the other hand, from a purely logical point of view we should start immediately with homogeneous spaces defined using algebraic groups over local fields of zero or positive characteristic and also develop the entropy theory much more generally, so that the case of smooth maps on manifolds is included. However, as such a text would be quite hard to read for anyone who does not already know the field that we hope to introduce, we instead start with homogeneous spaces defined using linear groups and introduce the language of algebraic groups relatively slowly (mostly over local fields of zero characteristic), starting in Chapter 3. Moreover, we only develop the entropy theory for homogeneous dynamics (which in many ways is easier than the entropy theory for smooth maps).

We hope you will enjoy these notes and the theory that they introduce.

Notation

The symbols $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{Z}$ denote the natural numbers, non-negative integers and integers; $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the rational numbers, real numbers and complex numbers; the multiplicative and additive circle are denoted $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ respectively. The real and imaginary parts of a complex number are denoted $\Re(x+\mathrm{i}y)$ and $\Im(x+\mathrm{i}y)$. The order of growth of real- or complex-valued functions f, g defined on $\mathbb{N}$ or $\mathbb{R}$ with $g(x) \neq 0$ for large x is compared using Landau's notation:

$$f \sim g \text{ if } \left|\frac{f(x)}{g(x)}\right| \longrightarrow 1 \text{ as } x \to \infty;$$
$$f = \mathrm{o}(g) \text{ if } \left|\frac{f(x)}{g(x)}\right| \longrightarrow 0 \text{ as } x \to \infty.$$

For functions f, g defined on $\mathbb{N}$ or $\mathbb{R}$, and taking values in a normed space, we write $f = \mathrm{O}(g)$ if there is a constant $A > 0$ with $\|f(x)\| \leqslant A\|g(x)\|$ for all x. In particular, $f = \mathrm{O}(1)$ means that f is bounded. Where the dependence of the implied constant A on some set of parameters $\mathcal{A}$ is important, we write $f = \mathrm{O}_{\mathcal{A}}(g)$. The relation $f = \mathrm{O}(g)$ will also be written $f \ll g$, particularly when it is being used to express the fact that two functions are commensurate, $f \ll g \ll f$. A sequence $a_1, a_2, \dots$ will be denoted (a_n). Unadorned norms $\|x\|$ will only be used when x lives in a Hilbert space (usually L^2) and always refer to the Hilbert space norm. For a topological space X, $C(X)$, $C_{\mathbb{C}}(X)$, $C_c(X)$ denote the space of real-valued, complex-valued, compactly supported continuous functions on X respectively, with the supremum norm. For sets A, B, denote the set difference by

$$A \smallsetminus B = \{x \mid x \in A, x \notin B\}.$$

Additional specific notation introduced in the text is collected in an index of notation on page 470.

Conventions

Throughout we assume that groups (denoted G, H, and so on) are metric, σ-compact, locally compact and equipped with a left-invariant metric (denoted d_G, d_H, and so on) giving rise to the topology of G. The identity of G will be denoted e, unless G is a matrix group in which case the identity will be denoted I. Compact and locally compact topological spaces are always assumed to be Hausdorff. In most settings the measure spaces we deal with will be Borel subsets of separable metric spaces with the Borel σ-algebra. These conventions will be recalled at times for clarity.

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August 13, 2018 ME, Zürich
TW, Leeds

[†] Your name may still appear here if you send us useful remarks on the text.