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The space of group automorphisms

Tom Ward, 12 October 2017, Leeds

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The point is to vary G as well as T.

	0-dim.	connected
entropy	rigid	conjecturally rigid: 'Lehmer's problem'
growth rate of periodic points (cheating)	flexible	?
growth rate of periodic points (honest)	?	?
Mertens' constant	?	flexible
analytic properties of zeta function	rigid-ish	conjecturally rigid: 'Polya–Carlson dichotomy'

Entropy

Entropy

The entropy of a group automorphism T is the rate of decay of volume of a Bowen-Dinaburg ball:

$$h(T) = \lim_{\epsilon \searrow 0} \lim_{n \to \infty} -\frac{1}{n} \log m \left(\bigcap_{i=0}^{n-1} T^{-i} B_\epsilon(0) \right).$$

Imagine a toral automorphism has eigenvalues λ_i with

$$|\lambda_1| \leqslant \cdots \leqslant |\lambda_s| \leqslant 1 < |\lambda_{s+1}| \leqslant \cdots \leqslant |\lambda_d|.$$

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So we expect

$$h(T) = \sum_{i=s+1}^{d} \log |\lambda_i| = \sum_{i=1}^{d} \log^+ |\lambda_i| = \int_0^1 \log |f(\mathrm{e}^{2\pi \mathrm{i} t})| \mathrm{d}t,$$

the *Mahler measure* $m(f)$ of the characteristic polynomial.

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The 0-dimensional case is easier, and the possible values are of the form $\log k$ with $k \in \mathbb{N}$.

Periodic points

Periodic points

Example: For any $C \in [0, \infty]$ there is a compact group automorphism $T : X \to X$ with

$$\frac{1}{n} \log |\{x \in X \mid T^n x = x\}| \to C$$

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Mertens' constant

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Let

$$M_T(N) = \sum_{|\tau| \leqslant N} \frac{1}{e^{h|\tau|}},$$

where $|\tau|$ denotes the length of a closed orbit τ, and h is the topological entropy (that is, a normalization by the expected 'usual' growth rate).

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We claim that κ is a flexible invariant (joint work with Baier, Jaidee, Stevens).

Theorem: For any $\kappa \in (0, 1)$ there is an ergodic compact connected group automorphism $T : X \to X$ with $M_T(N) \sim \kappa \log N$.

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Theorem: For any $\delta \in (0,1)$ and $k > 0$ there is an ergodic compact connected group automorphism $T : X \to X$ with $M_T(N) \sim k(\log N)^\delta$.

Constructions in 1-solenoids

The simplest connected groups are the one-dimensional solenoids, which are in 1-to-1 correspondence with subgroups of $\mathbb{Q}$. These are easy to describe (*unlike the subgroups of* $\mathbb{Q}^2$).

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So the construction boils down to statements about sets of primes.

These ‘exotic’ solenoids are not really all that exotic: they appear as minimal sets of generic Hamiltonian dynamical systems on symplectic manifolds (Markus & Meyer 1980).

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- Given $\delta \in (0,1)$, find a set L of primes so that

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- in such a way that for any $c > 1$ there is a set $L' \subset L$ with

$$\prod_{p \in L'} \left(1 + \frac{1}{p}\right) = c$$

and

$$\sum_{p \in L'} \frac{\log p}{p} < \infty.$$

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It is easy to find other phenomena: there is a C^∞ diffeomorphism of $\mathbb{T}^2$ with $Fix(n) = \binom{2n}{n}$ for all $n \geqslant 1$ and hence with

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General problem: Is the set of zeta functions of bijections of $\mathbb{N}$ the same as the set of zeta functions of C^∞ diffeomorphisms of $\mathbb{T}^2$ (yes) that are topologically transitive (?), area-preserving (?),... (Hunt & Kaloshin 2001 show that a stretched exponential rate is prevalent)

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Cautionary example: The function $f(z) = \frac{1}{(1-z)(1-z^5)}$ is the dynamical zeta function of the permutation $\tau = (1)(23456)$ on the set $\{1, 2, 3, 4, 5, 6\}$. The sequence ($Fix_\tau(n)$) is a linearly recurrent divisibility sequence, but f is not the zeta function of any group automorphism.

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Problem: Is there a combinatorial characterization for being the zeta function of a group automorphism?

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$$Fix(n) = (2^n - 1)|2^n - 1|_3,$$

and we outline a proof that $|z| = \frac{1}{2}$ is a natural boundary for its zeta function.

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Let $R(z) = \sum_{n \geqslant 1} |2^n - 1|_3 z^n$, so that

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We claim that $|z| = 1$ is a natural boundary for R, and hence $|z| = \frac{1}{2}$ is one for F (and hence for ζ).

Write

$$R(z) = \frac{1}{3} \sum_{2|n} |n|_3 z^n + \sum_{2\nmid n} z^n,$$

so $R(z) = \frac{1}{3} G(z^2) + H_2(z)$, where $G(z) = \sum_{n\geqslant 1} |n|_3 z^n$.

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Writing $n = 3^e k$, where $e \geqslant 0$ and $3 \nmid k$, gives

$$G(z) = \sum_{e \geqslant 0} \frac{1}{3^e} \sum_{3 \nmid k} z^{3^e k} = \sum_{e \geqslant 0} \frac{1}{3^e} H_3(z^{3^e})$$

$$= H_3(z) + \frac{1}{3} \sum_{e \geqslant 0} \frac{1}{3^e} H_3(z^{3^{e+1}}).$$

It follows that

$$G(z) = H_3(z) + \frac{1}{3}G(z^3).$$

Using this functional equation inductively, we deduce that there are dense singularities of G on the unit circle, occurring at 3^e-th roots of unity, $e \in \mathbb{N}$.

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Remark: This is not a reasonable proof – its only method is luck.

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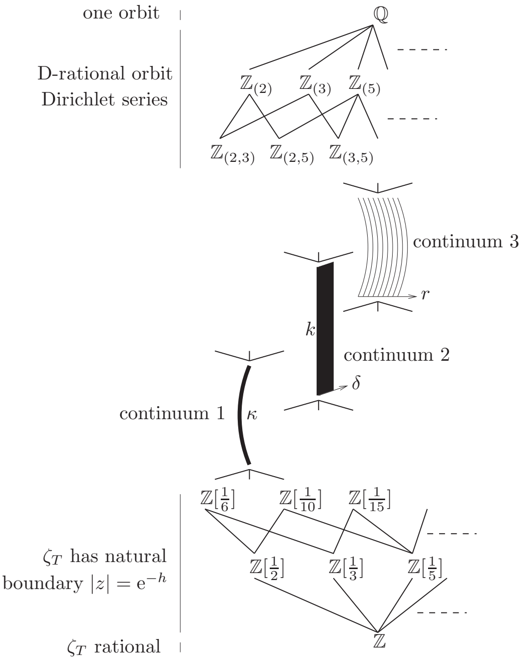
Question: Do zeta functions for compact group automorphisms enjoy the same dichotomy?

Baby case: draw a portrait of $x \to 2x$ on (some) one-dimensional solenoids up to topological conjugacy and identify what we know...



one orbit
$\mathbb{Q}$
D-rational orbit
Dirichlet series
$\mathbb{Z}_{(2)}$
$\mathbb{Z}_{(3)}$
$\mathbb{Z}_{(5)}$
$\mathbb{Z}_{(2,3)}$
$\mathbb{Z}_{(2,5)}$
$\mathbb{Z}_{(3,5)}$
continuum 3
r
k
continuum 2
δ
continuum 1
κ
$\mathbb{Z}[\frac{1}{6}]$
$\mathbb{Z}[\frac{1}{10}]$
$\mathbb{Z}[\frac{1}{15}]$
ζ_T has natural
boundary $|z| = \mathrm{e}^{-h}$
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Hadamard: Let $\mathbb{K}$ be a field of characteristic zero, and suppose that $\sum_{n \geqslant 0} b_n z^n$ and $\sum_{n \geqslant 1} c_n z^n$ in $\mathbb{K}[[z]]$ are expansions of rational functions. If there is a finitely-generated ring R over $\mathbb{Z}$ with $a_n = \frac{b_n}{c_n} \in R$ for all $n \geqslant 1$, then $\sum_{n \geqslant 0} a_n z^n$ is also the expansion of a rational function.

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Fabry: If $0 < p_1 < p_2 < \cdots$ are integers with $\frac{p_n}{n} \to \infty$ as $n \to \infty$ and (a_n) is a sequence of complex numbers for which $\sum_{n\geqslant 1} a_n z^{p_n}$ has radius of convergence 1, then the series admits $|z| = 1$ as a natural boundary.

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Warning: The radius of convergence of the zeta function of a group automorphism is rarely 1, and is usually unknown.

The simplest case is to assume that X is a one-dimensional solenoid, so (roughly) the automorphism is dual to the map $x \mapsto rx$ on the ring $R = \mathbb{Z}[\frac{1}{p} : p \in S]$ for some subset S of the primes.

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Write $f_S(n) = |r^n - 1| \cdot |r^n - 1|_S$ and $F_S(z) = \sum_{n \geqslant 1} f_S(n) z^n$, where $|x|_S = \prod_{p \in S} |x|_p$.

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To see how Hadamard arises, we claim that F_S is rational if and only if $|r|_p \neq 1$ for all $p \in S$ ('hyperbolicity').

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Arithmetic arguments can then be used to show that f takes on infinitely many values infinitely often, which is impossible.

For S co-finite it is easy to show that the Pólya–Carlson dichotomy holds because the theorem itself applies.

With more effort we can show that the Pólya–Carlson dichotomy holds in several cases:

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- If $\dim(X) \leqslant 3$ and S is finite then the Pólya–Carlson dichotomy holds.