Chapter 8
Temperedness and Tempered Decay for $\mathbf{SL_2}(\mathbb{R})$

In Chapter 2 we have seen that for abelian non-compact groups no irreducible representation can be contained in the regular representation.† For compact groups, we have seen in Chapter 5 that every irreducible representation is contained in the regular representation.

However, as we will see in Section 8.4, for non-abelian non-compact groups it is possible for certain irreducible unitary representations to be contained in the regular representation. Before we get to these examples, we classify such representations from an abstract point of view, and relate weak containment in the regular representation to almost square integrability of diagonal matrix coefficients.

However, our main purpose in this chapter will be to start our discussion of the group $\mathrm{SL}_2(\mathbb{R})$, and discuss its discrete series and tempered representations.

8.1 Discrete Series Representations

Definition 8.1. An irreducible unitary representation π of the group G is called a *discrete series representation* if π is contained in the regular representation of G.

Theorem 8.2 (Characterization of discrete series). *Suppose the group G is unimodular, and let π be an irreducible unitary representation of G. Then the following are equivalent.*[(12)]

(1) *π is a discrete series representation (that is, $\pi < \lambda$).*
(2) *There exist vectors $u, v \in \mathcal{H}_\pi \smallsetminus \{0\}$ such that the matrix coefficient $\varphi^\pi_{u,v}$ is square integrable (that is, $\varphi^\pi_{u,v} \in L^2(G)$).*

† For this, see Exercise 1.14 and its hint on p. 473, Exercise 2.13, Lemma 2.17, and the characterization of containment in Proposition 2.54.

(3) *The matrix coefficient* $\varphi^\pi_{u,v}$ *is square integrable for any pair of vectors* $u,v\in\mathcal{H}_\pi$.

Proof. We first show that (1) implies (2). Suppose that $\pi<\lambda$ and so $\mathcal{H}_\pi=\mathcal{V}$ for an invariant supspace $\mathcal{V}\subseteq L^2(G)$ as in (1). Let $v\in\mathcal{V}\smallsetminus\{0\}$ and let u be the projection $P(f)$ where $P\colon L^2(G)\to\mathcal{V}\subseteq L^2(G)$ is the orthogonal projection operator and f is a function in $C_c(G)$. For $f\in C_c(G)$ and $g\in G$, we also define $\widetilde{f}(g)=f(g^{-1})$. Since $\mathcal{V}$ is invariant, P is equivariant and so

$$\begin{aligned}\varphi^\pi_{u,v}(g)=\langle\lambda_gPf,v\rangle=\langle\lambda_gf,v\rangle&=\int f(g^{-1}h)\overline{v}(h)\,\mathrm{d}m(h)\\&=\int\overline{v}(h)\widetilde{f}(h^{-1}g)\,\mathrm{d}m(h)=\overline{v}*\widetilde{f}(g)\end{aligned}$$

for all $g\in G$. Now recall that $\overline{v}*\widetilde{f}\in L^2(G)$ (see [21, Lem. 3.75] or generalize Lemma 2.16), and note that density of $C_c(G)\subseteq L^2(G)$ implies that there exists some $f\in C_c(G)$ such that $u=Pf\in\mathcal{V}\smallsetminus\{0\}$. This shows (2).

Notice that (3) clearly implies (2), since an irreducible representation π by definition satisfies $\mathcal{H}_\pi\neq\{0\}$. Hence the following step will finish the proof of the theorem.

We will now show that (2) implies (1) and (3). So suppose that u_0,v_0 in $\mathcal{H}_\pi\smallsetminus\{0\}$ have $\varphi^\pi_{u_0,v_0}\in L^2(G)$. We fix u_0 and consider the subspace

$$D_{u_0}=\{v\in\mathcal{H}_\pi\mid\varphi^\pi_{u_0,v}\in L^2(G)\}$$

as the domain of the (possibly unbounded) operator T from $\mathcal{H}_\pi$ into $L^2(G)$ defined by

$$T(v)=\overline{\varphi^\pi_{u_0,v}}\in L^2(G)$$

for $v\in D_{u_0}$. As the main step of the remaining proof, we claim that T is a densely defined closed equivariant operator.

Equivariance follows from the properties of the matrix coefficients. Indeed, for $v\in D_{u_0}$ and $g_0\in G$ we have

$$\begin{aligned}\lambda_{g_0}\overline{\varphi^\pi_{u_0,v}}(g)=\overline{\varphi^\pi_{u_0,v}(g_0^{-1}g)}&=\overline{\left\langle\pi(g_0^{-1}g)u_0,v\right\rangle}\\&=\overline{\langle\pi(g)u_0,\pi(g_0)v\rangle}=\overline{\varphi^\pi_{u_0,\pi(g_0)v}}(g)\end{aligned}$$

for every $g\in G$, equivalently $\lambda_{g_0}T(v)=T(\pi_{g_0}(v))\in L^2(G)$ and $\pi_{g_0}v\in D_{u_0}$.

Since $v_0\in D_{u_0}\smallsetminus\{0\}$ and D_{u_0} is invariant under π by the above, we see that $\overline{D_{u_0}}=\mathcal{H}_\pi$ by irreducibility of π. In other words, T is densely defined. To complete the proof of the claim, it remains to show that T is a closed operator. Suppose therefore that (v_n) is a sequence in D_{u_0} with

$$(v_n,Tv_n)\longrightarrow(v,f)\in\mathcal{H}_\pi\times L^2(G)$$

as $n \to \infty$. Choosing a subsequence if necessary we may upgrade the L^2 convergence and suppose without loss of generality that $Tv_n \to f$ almost everywhere (see e.g. [20, Cor. A.12] for this argument). Furthermore, notice that $v_n \to v \in \mathcal{H}_\pi$ implies

$$(Tv_n)(g) = \overline{\varphi^\pi_{u_0,v_n}}(g) = \overline{\langle \pi(g)u_0, v_n\rangle} \longrightarrow \overline{\langle \pi(g)u_0, v\rangle}$$

as $n \to \infty$ for all $g \in G$. Together we see that

$$f(g) = \lim_{n\to\infty} (Tv_n)(g) = \overline{\langle \pi(g)u_0, v\rangle}$$

for almost every $g \in G$. However, since $f \in L^2(G)$ we see that $v \in D_{u_0}$ belongs to the domain of T and $f = Tv$. In particular, $(v, f) \in \mathrm{Graph}(T)$ as required.

The claim above, the assumed irreducibility of π, and Schur's lemma in the form of Corollary 1.36 together imply that $D_{u_0} = \mathcal{H}_\pi$ and either $T = 0$ or T is a scalar multiple of a unitary isomorphism. Using $u_0 \in D_{u_0} = \mathcal{H}_\pi$ we see that $Tu_0 = \overline{\varphi^\pi_{u_0,u_0}} \neq 0$ and obtain $\pi < \lambda$ as in (1).

To prove (3), let $u_1, v_1 \in \mathcal{H}_\pi$ be any vectors. By the argument above we already have $\varphi_{u_0,v_1} \in L^2(G)$ and repeat the argument as follows. Define

$$\widetilde{D}_{v_1} = \{u \in \mathcal{H}_\pi \mid \varphi^\pi_{u,v_1} \in L^2(G)\}$$

and $\widetilde{T}(u) = \varphi^\pi_{u,v_1}$ for $u \in \widetilde{D}_{v_1}$. Now recall that

$$\rho_{g_0}(\widetilde{T}(u))(g) = \langle \pi(gg_0)u, v_1\rangle = \langle \pi(g)\pi(g_0)u, v_1\rangle = \widetilde{T}(\pi(g_0)u)(g)$$

for $g, g_0 \in G$. Since G is assumed to be unimodular, we may apply the argument above to deduce that $\widetilde{T}$ is a densely defined closed equivariant operator from $\mathcal{H}_\pi$ to $L^2(G)$ with the latter equipped with the right-regular representation. As above, this implies by Schur's lemma (Corollary 1.36) that $D_{v_1} = \mathcal{H}_\pi$, and in particular $\varphi^\pi_{u_1,v_1} \in L^2(G)$. As $u_1, v_1 \in \mathcal{H}_\pi$ were arbitrary, (3) follows. □

To see an example of a discrete series representation, the reader may continue with Section 8.4 (which only builds on this section and some preparations concerning hyperbolic geometry in Section 8.3). As one of our motivations for this volume is to better understand decay of matrix coefficients, Theorem 8.2 is of interest to us. Indeed, the requirement to belong to $L^2(G)$ is a requirement on the average decay of the matrix coefficients. This observation will become very important for us after replacing the strong condition of containment by weak containment as in the next section.

8.2 Almost Square Integrable Matrix Coefficients

We now present a theorem of Cowling, Haagerup and Howe [13] that will allow us to characterize in Section 8.6 the so called *tempered representations* for the group $\mathrm{SL}_2(\mathbb{R})$.

Definition 8.3 (Tempered representations). Let π be a unitary representation of G. We say that π is *tempered* if it is weakly contained in the regular representation λ of G.

Our interest in tempered representations comes from our wish to understand decay of matrix coefficients. We will see the connection explicitly in Section 8.6, where the following theorem will be a crucial ingredient.

The reader may also wonder whether non-tempered representations actually exist, since so far we have only encountered tempered representations (without pointing this out). However, we will see that (for example) the trivial representation is not always tempered (which relates to the definition of amenability in Section 4.2.).

Definition 8.4 (Almost square integrability). Let π be a unitary representation of G. We say that π is *almost square integrable* if there exists a dense subset $\mathcal{V} \subseteq \mathcal{H}_\pi$ such that $\varphi_v^\pi \in L^{2+\varepsilon}(G)$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$.

Theorem 8.5 (Almost square integrable matrix coefficients [13]). *Let G be a locally compact group and π a unitary representation. Suppose that $v \in \mathcal{H}_\pi$ has the property that the diagonal matrix coefficient*

$$\varphi_v^\pi(g) = \langle \pi_g v, v \rangle$$

for $g \in G$ belongs to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. Then the cyclic representation generated by v is weakly contained in the regular representation of G. Moreover, if π is almost square integrable, then π is tempered.

Along the way, and as a warm-up for the proof of the theorem, we will obtain the following result which is of independent interest.

Proposition 8.6 (Sub-exponential growth). *Suppose that G has sub-exponential growth, meaning that for every compact subset $K \subseteq G$ we have*

$$\lim_{n\to\infty} m\left(K^n\right)^{\frac{1}{n}} = 1. \tag{8.1}$$

Then every unitary representation is tempered.

For the proof of the theorem and the proposition we are going to use Lemma 4.33 and its twin for the regular representation below and the equivalent definition $(\pi \prec_{\mathrm{op}} \rho)$ of weak containment from Theorem 4.28.

Lemma 8.7 (2-norm formula for operator norm). *Let $f \in C_c(G)$. Then for the left-regular representation λ of G we have*

$$\|\lambda_*(f)\|_{\mathrm{op}} = \lim_{n\to\infty} \|(f^* * f)^{*n}\|_2^{\frac{1}{2n}}.$$

PROOF. By Lemma 4.33 we have

$$\|\lambda_*(f)\| = \sup_{f_0\in L^2(G)} \lim_{n\to\infty} \left(\int_G (f^* * f)^{*n} \varphi_{f_0}\,\mathrm{d}m\right)^{\frac{1}{2n}}$$

where $\varphi_{f_0}(g) = \langle \lambda_g f_0, f_0\rangle$ for $f_0 \in L^2(G)$. Also, by Lemma 4.33, we may also restrict to functions $f_0 \in C_c(G)$ in which case $\varphi_{f_0} \in C_c(G)$. Noticing that for $f_0 \in C_c(G)$

$$\int_G (f^* * f)^{*n}\varphi_{f_0}\,\mathrm{d}m \leqslant \|(f^* * f)^{*n}\|_2 \|\varphi_{f_0}\|_2,$$

we get

$$\|\lambda_*(f)\|_{\mathrm{op}} \leqslant \liminf_{n\to\infty} \|(f^* * f)^{*n}\|_2^{\frac{1}{2n}}. \tag{8.2}$$

On the other hand, setting $f_0 = f^* * f \in C_c(G)$ gives

$$\begin{aligned}(f^* * f)^{*n}(g) &= (f^* * f)^{*(n-1)} * f_0(g)\\ &= \int_G (f^* * f)^{*(n-1)}(h) f_0(h^{-1}g)\,\mathrm{d}m(h)\\ &= \left(\lambda_*\left((f^* * f)^{*(n-1)}\right) f_0\right)(g)\end{aligned}$$

for all $g \in G$, which leads to

$$\begin{aligned}\|(f^* * f)^{*n}\|_2 &\leqslant \|\lambda_*(f^* * f)^{*(n-1)}\|_{\mathrm{op}} \|f_0\|_2\\ &\leqslant \|\lambda_*(f)\|_{\mathrm{op}}^{2(n-1)} \|f_0\|_2\end{aligned}$$

and in turn to

$$\limsup_{n\to\infty} \|(f^* * f)^{*n}\|_2^{\frac{1}{2n}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}.$$

Together with (8.2), this gives the lemma. □

PROOF OF PROPOSITION 8.6. Let G have sub-exponential growth, let π be a unitary representation of G, and f a function in $C_c(G)$. We wish to show that $\|\pi_*(f)\|_{\mathrm{op}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}$, and will use Lemmas 4.33 and 8.7 to prove this. So set $f_0 = f^* * f$ and $K = \operatorname{supp} f_0$. Then $\operatorname{supp} f_0^{*n} \subseteq K^n$, and we may use Cauchy–Schwarz for any $v \in \mathcal{H}_\pi$ to obtain

$$\left|\int_G f_0^{*n}\varphi_v^\pi \,\mathrm{d}m\right|^2 \leqslant \int_{K^n}|f_0^{*n}|^2\,\mathrm{d}m\int_{K^n}|\varphi_v^\pi|^2\,\mathrm{d}m.$$

Taking the $4n$th root and using $\|\varphi_v^\pi\|_\infty \leqslant \|v\|^2$, this gives

$$\lim_{n\to\infty}\left|\int_G f_0^{*n}\varphi_v^\pi\,\mathrm{d}m\right|^{\frac{1}{2n}} \leqslant \lim_{n\to\infty}\|f_0^{*n}\|_2^{\frac{1}{2n}}\lim_{n\to\infty}\left(\|v\|^4 m(K^n)\right)^{\frac{1}{4n}} = \|\lambda_*(f)\|_{\mathrm{op}}$$

by (8.1) and Lemma 8.7. Taking the supremum over $v\in\mathcal{H}_\pi$, Lemma 4.33 shows that $\|\pi_*(f)\|_{\mathrm{op}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}$. As $f\in C_c(G)$ was arbitrary, Theorem 4.28 shows that $\pi\prec\lambda$. □

In general, groups may not satisfy sub-exponential growth (see Exercise 8.9 below), but the following is a general estimate.

Lemma 8.8 (At most exponential growth). *Let $K\subseteq G$ be a compact subset with non-empty interior. Then there exist constants $c>0$, $M>1$ (depending on K) with $m_G(K^n)\leqslant cM^n$ for all $n\geqslant 1$.*

Proof. By compactness, there exists a finite collection $x_1,\dots,x_M\in G$ with the property that

$$K^2\subseteq\bigcup_{j=1}^M x_jK.$$

By induction, this implies that

$$K^n\subseteq\bigcup_{j_1,\dots,j_{n-1}=1}^M x_{j_1}\cdots x_{j_{n-1}}K,$$

which gives the lemma with $c=\frac{m_G(K)}{M}$. □

We are now ready to prove the main result of this section.

Proof of Theorem 8.5. Suppose that π is a unitary representation of G and $v\in\mathcal{H}_\pi$ has the property that the matrix coefficient $\varphi_v^\pi(g)=\langle\pi(g)v,v\rangle$ belongs to $L^{2+\varepsilon}(G)$ for all $\varepsilon>0$. Let us now bound the expressions appearing in Lemma 4.33. To do this, fix $\varepsilon>0$, let $f\in C_c(G)$, write $f_0=f^* * f$ and set $K=\operatorname{supp} f_0$. This implies that $\operatorname{supp} f_0^{*n}\subseteq K^n$, and so

$$\begin{aligned}\left|\int_{K^n} f_0^{*n}(g)\varphi_v^\pi\,\mathrm{d}m\right|^2 &\leqslant \int_{K^n}|f_0^{*n}|^2\,\mathrm{d}m\int_{K^n}|\varphi_v^\pi|^2\,\mathrm{d}m\\ &\leqslant \|f_0^{*n}\|_2^2\left(\int_{K^n}|\varphi_v^\pi|^{2+\varepsilon}\,\mathrm{d}m_G\right)^{\frac{2}{2+\varepsilon}}\left(\int_{K^n}\mathrm{d}m_G\right)^{\frac{\varepsilon}{2+\varepsilon}}\\ &\leqslant \|f_0^{*n}\|_2^2\|\varphi_v^\pi\|_{2+\varepsilon}^2(cM^n)^{\frac{\varepsilon}{2+\varepsilon}},\end{aligned}$$

where we have used the Cauchy–Schwarz and Hölder inequalities with the conjugate pair of exponents $\left(\frac{2+\varepsilon}{2}, \frac{2+\varepsilon}{\varepsilon}\right)$. Taking the $4n$th root and letting n go to infinity, it follows that

$$\begin{aligned}\lim_{n\to\infty}\left(\int_G (f^* * f)^{*n}\varphi_v^\pi \,\mathrm{d}m\right)^{\frac{1}{2n}} &\leqslant \lim_{n\to\infty}\|(f^* * f)^{*n}\|_2^{\frac{1}{2n}} M^{\frac{\varepsilon}{4(2+\varepsilon)}}\\ &= \|\lambda_*(f)\|_{\mathrm{op}} M^{\frac{\varepsilon}{4(2+\varepsilon)}}\end{aligned}$$

by Lemma 8.7. Since this holds for all $\varepsilon > 0$, we also obtain

$$\lim_{n\to\infty}\left(\int_G (f^* * f)^{*n}\varphi_v^\pi \,\mathrm{d}m\right)^{\frac{1}{2n}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}.$$

If there is a dense set of vectors $v \in \mathcal{H}_\pi$ for which the matrix coefficients are almost square integrable, then Lemma 4.33 gives $\|\pi(f)\|_{\mathrm{op}} \leqslant \|\lambda_*(f)\|_{\mathrm{op}}$ for all $f \in C_c(G)$ and Theorem 4.28 gives $\pi \prec \lambda$, proving the last part of the theorem.

Now suppose that $v \in \mathcal{H}_\pi$ has almost square integrable matrix coefficients. To prove the first part of the theorem assume without loss of generality that $\mathcal{H}_\pi$ is a cyclic representation with generator v. Hence vectors of the form $w = \sum_{j=1}^n \alpha_j \pi_{g_j} v$ for $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $g_1, \dots, g_n \in G$ are dense in $\mathcal{H}_\pi$. We claim that the matrix coefficients of such vectors also belong to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. To see this let w be as above and calculate

$$\begin{aligned}\varphi_w^\pi(g) = \langle \pi_g w, w\rangle &= \sum_{j,k=1}^n \alpha_j \overline{\alpha_k}\left\langle \pi_g \pi_{g_j} v, \pi_{g_k} v\right\rangle\\ &= \sum_{j,k=1}^n \alpha_j \overline{\alpha_k}\left\langle \pi_{g_k^{-1} g g_j} v, v\right\rangle\\ &= \sum_{j,k=1}^n \alpha_j \overline{\alpha_k}\varphi_v^\pi(g_k^{-1} g g_j).\end{aligned}$$

Clearly the left-regular representation preserves the p-norm for $p \in [1,\infty]$ and so $\|\varphi_v^\pi(g_k^{-1}\cdot g_j)\|_p = \|\varphi_v^\pi(\cdot g_j)\|_p$. Furthermore,

$$\|\varphi_v^\pi(\cdot g_j)\|_p^p = \int_G |\varphi_v^\pi(g g_j)|^p \,\mathrm{d}m(g) = \int |\varphi_v^\pi(g)|^p \,\mathrm{d}m(g)\Delta(g_j)^{-1} < \infty$$

by Lemma 1.12. It follows that $\varphi_w^\pi \in L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$, and the theorem follows from the case considered above. □

It is clear that, unlike the statement for discrete series representations in Theorem 8.2, the condition of almost square integrability is not a characterization for temperedness (for example, for $G = \mathbb{R}$). However, as we will see in the next chapter it is a characterization for some groups (for example, for $G = \mathrm{SL}_2(\mathbb{R})$)

Exercise 8.9. Show that the '$ax + b$'-group from Section 3.3.2 does not have sub-exponential growth, but that all its irreducible unitary representations (as in Proposition 3.18) are tempered.

8.3 The Group $\mathrm{SL}_2(\mathbb{R})$

To give interesting examples of discrete series representations and of tempered representations, we need to study non-compact semi-simple Lie groups. The principal example — and hence the first group we should try to understand — is $\mathrm{SL}_2(\mathbb{R})$.

8.3.1 The Action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$

As we will see in the discussions below, and in the following chapters, the group $\mathrm{SL}_2(\mathbb{R})$ and its unitary representations are intimately linked to hyperbolic geometry. We briefly review the link between $\mathrm{SL}_2(\mathbb{R})$ and hyperbolic geometry, and refer to [20, Ch. 9] or the survey by Cannon *et al.* [7] for example, for more details.

The upper half-plane model $\mathbb{H}$ of the hyperbolic plane is defined by

$$\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$$

together with the hyperbolic metric and volume. The hyperbolic Riemannian metric is defined by

$$\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

and the hyperbolic area measure is defined by

$$\mathrm{dvol} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

at any point $z = x + \mathrm{i}y \in \mathbb{H}$ with $x = \Re(z) \in \mathbb{R}$ and $y = \Im(z) > 0$. The group $\mathrm{SL}_2(\mathbb{R}) = \{g \in \mathrm{Mat}_{2,2} \mid \det g = 1\}$ acts on $\mathbb{H}$ via Möbius transformations as

$$g\cdot z = \frac{az + b}{cz + d} \tag{8.3}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

A calculation reveals that

$$\Im(g\bullet z) = \frac{\Im(z)}{|cz+d|^2}, \tag{8.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(g\bullet z) = \frac{a(cz+d)-(az+b)c}{(cz+d)^2} = \frac{1}{(cz+d)^2}, \tag{8.5}$$

and that g preserves both the hyperbolic Riemannian metric and the hyperbolic area measure. We also note that the action of $\mathrm{SL}_2(\mathbb{R})$ is transitive and that

$$\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\mathrm{i}) = \mathrm{SO}_2(\mathbb{R}),$$

so that $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

Essential Exercise 8.10. (a) Show (8.4) and conclude that $g\bullet z \in \mathbb{H}$ for any $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$.
(b) Show that the Möbius transformations in (8.3) define a continuous action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$.
(c) Show that the Möbius transformation (8.3) preserves the hyperbolic Riemannian metric and the hyperbolic area measure for any $g \in \mathrm{SL}_2(\mathbb{R})$.
(d) Show that $\mathrm{SO}_2(\mathbb{R}) = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(\mathrm{i})$ is the stabilizer of the point $\mathrm{i} \in \mathbb{H}$.
(e) Show that the action of the subgroup

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}$$

on $\mathbb{H}$ is simply transitive and deduce that the action of $\mathrm{SL}_2(\mathbb{R})$ is transitive.

By (8.5) and the chain rule, the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ extends via taking the derivative to an action of $\mathrm{SL}_2(\mathbb{R})$ on the tangent bundle $\mathrm{T}\mathbb{H}$. Since the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{T}\mathbb{H} = \mathbb{H}\times\mathbb{C}$ preserves the Riemannian metric, this action restricts to an action of $\mathrm{SL}_2(\mathbb{R})$ on the unit tangent bundle

$$\mathrm{T}^1\mathbb{H} = \left\{(z,v) \mid z \in \mathbb{H}, \|v\|_z = \tfrac{\|v\|_2}{\Im(z)} = 1\right\}.$$

Moreover, this action is transitive, $-I \in \mathrm{SL}_2(\mathbb{R})$ acts trivially,† and the action descends to a simply transitive action of $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ on $\mathrm{T}^1\mathbb{H}$.

As we have already seen in Exercise 8.10(e), the action is transitive on $\mathbb{H}$ and so it suffices to consider all unit tangent vectors at a given point in $\mathbb{H}$. Hence we fix attention on the point $\mathrm{i} \in \mathbb{H}$. Here the action of $K = \mathrm{SO}_2(\mathbb{R})$ and Exercise 8.10(d) help. In fact, for

$$k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K$$

† Note that $(-I)\bullet z = \frac{-z}{-1} = z$ for all $z \in \mathbb{H}$, so $-I$ acts trivially on $\mathbb{H}$ and on $\mathrm{T}^1\mathbb{H}$.

and $z \in \mathbb{H}$ we have

$$k_\theta \bullet z = \frac{\cos\theta z - \sin\theta}{\sin\theta z + \cos\theta}.$$

Using (8.5) at $z = \mathrm{i}$, this gives

$$\Big(\frac{\mathrm{d}}{\mathrm{d}z}(k_\theta \bullet z)\Big)\Big|_{z=\mathrm{i}} = \frac{1}{(\cos\theta + \mathrm{i}\sin\theta)^2} = \mathrm{e}^{-2\mathrm{i}\theta},$$

which in words means that the action of $\mathrm{SO}_2(\mathbb{R})$ fixes i but rotates the unit tangent vector at i at double speed clockwise. This implies the simple transitivity of the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathrm{T}^1\mathbb{H}$.

8.3.2 The Action of $\mathrm{SU}_{1,1}(\mathbb{R})$ on $\mathbb{D}$

The Poincaré disk model $\mathbb{D}$ of the hyperbolic plane is defined by

$$\mathbb{D} = \{w \in \mathbb{C} \mid |w| < 1\}$$

together with the hyperbolic metric and volume. In the disk model, the hyperbolic Riemannian metric is defined by

$$4\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1-|w|^2)^2} \tag{8.6}$$

for $w = x + \mathrm{i}y$, and the associated hyperbolic area measure is defined by

$$\mathrm{dvol} = 4\frac{\mathrm{d}x\,\mathrm{d}y}{(1-|w|^2)^2}. \tag{8.7}$$

For this model it is more convenient to use the special unitary group

$$\mathrm{SU}_{1,1}(\mathbb{R}) = \left\{ \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} \text{ with } |\alpha|^2 - |\beta|^2 = 1 \right\},$$

which is defined as the stabilizer inside $\mathrm{SL}_2(\mathbb{C})$ of the Hermitian form

$$\mathbb{C}^2 \ni (z_1, z_2) \longmapsto |z_1|^2 - |z_2|^2$$

of signature $(1,1)$. In fact

$$h = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{R})$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 = 1$ acts on $w \in \mathbb{D}$ by the Möbius transformation

$$h \bullet w = \frac{\alpha w + \overline{\beta}}{\beta w + \overline{\alpha}}. \tag{8.8}$$

A calculation reveals that

$$1 - |h \bullet w|^2 = \frac{1 - |w|^2}{|\beta w + \overline{\alpha}|^2}, \tag{8.9}$$

and that h preserves both the hyperbolic Riemannian metric and the hyperbolic area measure.

Essential Exercise 8.11. (a) Show (8.9) and deduce that $h \bullet w \in \mathbb{D}$ for any $h \in \mathrm{SU}_{1,1}(\mathbb{R})$ and $w \in \mathbb{D}$.
(b) Show that the Möbius transformations in (8.3) define a continuous action of $\mathrm{SU}_{1,1}(\mathbb{R})$ on $\mathbb{D}$.
(c) Show that the Möbius transformation (8.8) preserves the hyperbolic Riemannian metric and the hyperbolic area measure on $\mathbb{D}$ for any $h \in \mathrm{SU}_{1,1}(\mathbb{R})$.
(d) Show that

$$K' = \left\{ k'_\theta = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} = \mathrm{Stab}_{\mathrm{SU}_{1,1}(\mathbb{R})}(0)$$

is the stabilizer of the point $0 \in \mathbb{D}$. Show also that the action of $k'_\theta \in K'$ rotates unit tangent vectors at 0 at double speed clockwise.

8.3.3 Equivalence of the Models

These two models of the hyperbolic plane and their associated isometry groups are in fact equivalent. We let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and note that any $g \in \mathrm{GL}_2(\mathbb{C})$ acts as a Möbius transformation on $\overline{\mathbb{C}}$. The map

$$\Phi\colon \overline{\mathbb{C}} \ni w \longmapsto \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \bullet w = \frac{w + \mathrm{i}}{\mathrm{i}w + 1} \in \overline{\mathbb{C}}$$

satisfies, for $w = x + \mathrm{i}y \in \mathbb{C}$, the equation

$$\Im \Phi(x + \mathrm{i}y) = \Im \frac{(x + \mathrm{i}y + \mathrm{i})(-\mathrm{i}x - y + 1)}{(\mathrm{i}x - y + 1)(-\mathrm{i}x - y + 1)} = \frac{1 - x^2 - y^2}{x^2 + (y-1)^2},$$

and so $\Phi(\mathbb{D}) \subseteq \mathbb{H}$. Conversely,

$$\Psi\colon \overline{\mathbb{C}} \ni z \longmapsto \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \bullet z = \frac{z - \mathrm{i}}{-\mathrm{i}z + 1} \in \overline{\mathbb{C}}$$

satisfies for $z = x + \mathrm{i}y \in \mathbb{H}$ that

$$|\Psi(x+\mathrm{i}y)|^2 = \left|\frac{x+\mathrm{i}y-\mathrm{i}}{-\mathrm{i}x+y+1}\right|^2 = \frac{x^2+(y-1)^2}{x^2+(y+1)^2}$$

and so $\Psi(\mathbb{H}) \subseteq \mathbb{D}$.

We also note that for

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

and $z \in \overline{\mathbb{C}}$ we have

$$\begin{aligned} g_1 \centerdot (g_2 \centerdot z) = g_1 \centerdot \frac{a_2 z + b_2}{c_2 z + d_2} &= \frac{a_1 \frac{a_2 z+b_2}{c_2 z+d_2} + b_1}{c_1 \frac{a_2 z+b_2}{c_2 z+d_2} + d_1} \\ &= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)} = (g_1 g_2) \centerdot z, \end{aligned}$$

and for

$$g = \begin{pmatrix} a & \\ & a \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

we have $g \centerdot z = z$. Together with

$$\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

this implies that $\Phi \circ \Psi = \Psi \circ \Phi = I$, and hence that $\mathbb{D}$ and $\mathbb{H}$ are biholomorphic under these maps.

Let us also verify that the Riemannian metric on $\mathbb{D}$ and on $\mathbb{H}$ are mapped to each other with respect to Φ and Ψ. For this we let $w = x + \mathrm{i}y \in \mathbb{D}$ and $v \in \mathrm{T}_w\mathbb{D}$ be a tangent vector and calculate

$$\Phi'(w) = \frac{2}{(\mathrm{i}w+1)^2}.$$

In other words, the complex derivative $\Phi'(w)$ corresponds as a derivative from $\mathbb{R}^2$ to $\mathbb{R}^2$ to a rotation and a scalar multiplication by

$$\frac{2}{|\mathrm{i}w+1|^2} = \frac{2}{x^2+(y-1)^2}.$$

Therefore, the squared length of the image vector $\Phi'(w)v$ as an element of $\mathrm{T}_{\Phi(w)}\mathbb{H}$ is given by

$$\frac{|\Phi'(w)v|^2}{(\Im\Phi(w))^2} = \frac{\frac{4|v|^2}{\left(x^2+(y-1)^2\right)^2}}{\left(\frac{1-x^2-y^2}{x^2+(y-1)^2}\right)^2} = \frac{4|v|^2}{1-|w|^2}.$$

Therefore the two notions of hyperbolic Riemannian metric, and so also the two notions of hyperbolic volume, are equivalent.

Finally, we calculate for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

the product

$$\frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+d+\mathrm{i}(b-c) & b+c-\mathrm{i}(d-a) \\ b+c+\mathrm{i}(d-a) & a+d-\mathrm{i}(b-c) \end{pmatrix} = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$$

where $\alpha = \frac{a+d+\mathrm{i}(b-c)}{2}$ and $\beta = \frac{b+c+\mathrm{i}(d-a)}{2}$ satisfy

$$|\alpha|^2 - |\beta|^2 = \tfrac{1}{4}\left[(a+d)^2 + (b-c)^2 - (b+c)^2 - (d-a)^2\right] = ad - bc = 1.$$

Conversely, for

$$h = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{R})$$

we can use the formulas

$$\begin{aligned} a &= \Re\alpha - \Im\beta, \\ b &= \Im\alpha + \Re\beta, \\ c &= -\Im\alpha + \Re\beta, \\ d &= \Re\alpha + \Im\beta \end{aligned}$$

to define

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$

giving back h under the conjugation above. It follows that the groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}_{1,1}(\mathbb{R})$ are conjugated, and that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ corresponds to the action of $\mathrm{SU}_{1,1}(\mathbb{R})$ on $\mathbb{D}$. Moreover $\Psi(\mathrm{i}) = 0$, $\Phi(0) = \mathrm{i}$, and the compact subgroup $\mathrm{SO}_2(\mathbb{R}) \subseteq \mathrm{SL}_2(\mathbb{R})$ is conjugated to the diagonal compact subgroup

$$K' = \left\{ k'_\theta = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} < \mathrm{SU}_{1,1}(\mathbb{R}).$$

In fact for $g = k_\theta$ we have

$$\tfrac{1}{2}\begin{pmatrix}1 & -\mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix} = \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta}\end{pmatrix}$$

and obtain that $k_\theta \in K$ is mapped to $k'_\theta \in K'$ with matching parametrization.

Exercise 8.12. The subgroup $K = \mathrm{SO}_2(\mathbb{R}) < \mathrm{SL}_2(\mathbb{R})$ (and, similarly, the diagonal subgroup K' of $\mathrm{SU}_{1,1}(\mathbb{R})$) is often called a *maximal* compact subgroup. Justify this by showing that if $H < \mathrm{SL}_2(\mathbb{R})$ is a subgroup properly containing $\mathrm{SO}_2(\mathbb{R})$, then $H = \mathrm{SL}_2(\mathbb{R})$.

8.3.4 Two Geometric Decompositions of $\mathrm{SL}_2(\mathbb{R})$

The action of $\mathrm{SL}_2(\mathbb{R})$ on the hyperbolic plane is fundamental to hyperbolic geometry, and is also a valuable tool for understanding the group $\mathrm{SL}_2(\mathbb{R})$ itself, and (as we will see) for understanding its unitary representations. This geometric viewpoint allows us to decompose $\mathrm{SL}_2(\mathbb{R})$ in two different ways.

The first decomposition that will be important to us is the *Iwasawa decomposition* of $\mathrm{SL}_2(\mathbb{R})$, using, in addition to the maximal compact subgroup $K = \mathrm{SO}_2(\mathbb{R})$, the diagonal subgroup

$$A = \left\{ a_t = \begin{pmatrix}\mathrm{e}^t & \\ & \mathrm{e}^{-t}\end{pmatrix} \mid t \in \mathbb{R}\right\}$$

and the upper unipotent subgroup

$$U = \left\{ u_x = \begin{pmatrix}1 & x\\ & 1\end{pmatrix} \mid x \in \mathbb{R}\right\}.$$

With this notation, the Iwasawa decomposition states that every $g \in \mathrm{SL}_2(\mathbb{R})$ can be written in a unique way as a product

$$g = u_x a_t k_\theta,$$

where $u_x \in U$, $a_t \in A$, and $k_\theta \in K$ for $x, t \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. Exercise 8.10(d)–(e) reveals the geometric meaning of this decomposition, and also provides a proof. Indeed, $g\raisebox{0.2ex}{$\scriptstyle\bullet$}\mathrm{i} \in \mathbb{H}$ and by transitivity of the action of the subgroup $B = UA$ on $\mathbb{H}$ there exists $x \in \mathbb{R}$ and $t > 0$ with

$$g\bullet\mathrm{i} = \underbrace{\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\mathrm{e}^t & \\ & \mathrm{e}^{-t}\end{pmatrix}}_{=b}\bullet\mathrm{i} = x + \mathrm{e}^{2t}\mathrm{i}.$$

Applying b^{-1} and using the fact that the stabilizer of i is $\mathrm{SO}_2(\mathbb{R})$, we obtain $b^{-1}g\bullet\mathrm{i} = \mathrm{i}$ and hence $b^{-1}g = k_\varphi$ for some φ, giving $g = u_x a_t k_\varphi$ as required. Moreover, it follows that $x = \Re(g\bullet\mathrm{i})$, $t = \frac{1}{2}\log\Im(g\bullet\mathrm{i})$, and that the angle of the vector $g\bullet(\mathrm{i},\mathrm{i})$ is determined by 2θ measuring clockwise from the vector pointing upwards. Taking inverses, we see that the Iwasawa decomposition also holds in the order KAU. We also refer to Exercise 6.4 for a different proof of the Iwasawa decomposition.

The second decomposition of $\mathrm{SL}_2(\mathbb{R})$ that we will use is the *Cartan decomposition*, which states that every $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as a product $g = k_\theta a_t k_\psi$ with $k_\theta, k_\psi \in K$ and $a_t \in A$. Moreover, we may assume that $t \geqslant 0$, and with this assumption the element $a_t \in A$ is uniquely determined by g. If $t > 0$ then the only ambiguity in the choice of k_φ and k_ψ is that we can multiply both simultaneously by the element $-I$ of the centre.

The existence of the Cartan decomposition follows from linear algebra (see the footnote on p. 59). The claimed uniqueness of a_t follows, since left and right multiplication by $k_\theta, k_\psi \in K = \mathrm{SO}_2(\mathbb{R})$ preserves the Hilbert–Schmidt norm $|\cdot| = \|\cdot\|_{\mathrm{HS}}$, which gives

$$|g| = \left|k_\theta a_t k_\psi\right| = |a_t| = \sqrt{\mathrm{e}^{2t} + \mathrm{e}^{-2t}} = \sqrt{2\cosh(2t)},$$

and so uniquely determines $t \geqslant 0$.

To practise using the disk model a little more, we explain the geometric meaning of the Cartan decomposition of $\mathrm{SU}_{1,1}(\mathbb{R}) = K'AK'$ using $\mathbb{D}$. Here we use

$$K' = \left\{ k'_\theta = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

as in Exercise 8.11(d) and

$$A' = \left\{ a'_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

We note that this subgroup $A' < \mathrm{SU}_{1,1}(\mathbb{R}) \cap \mathrm{SL}_2(\mathbb{R})$, which is conjugated to our previous choice $A < \mathrm{SL}_2(\mathbb{R})$ by an element of K, could also have been used in the Cartan decomposition for $\mathrm{SL}_2(\mathbb{R})$, and that our isomorphism between $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}_{1,1}(\mathbb{R})$ fixes A'. So suppose that $g \in \mathrm{SU}_{1,1}(\mathbb{R})$ and that $w = g \cdot 0 = \rho \mathrm{e}^{-2\mathrm{i}\theta} \in \mathbb{D}$ for some $\rho \in [0, 1)$ and $\theta \in [0, \pi)$. Applying the inverse $k'_{-\theta}$ of k'_θ to w gives

$$k'_{-\theta} g \cdot 0 = k'_{-\theta} \cdot w = \rho \geqslant 0.$$

We define $t = \operatorname{arctanh} \rho \in \mathbb{R}_{\geqslant 0}$ and obtain

$$k'_{-\theta} g \cdot 0 = \tanh t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \cdot 0.$$

Using Exercise 8.11(d), we find some $k'_\psi \in K'$ with $\psi \in [0, 2\pi)$ and

$$a'_{-t} k'_{-\theta} g = k'_\psi$$

so that

$$g = k'_\theta \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} k'_\psi$$

as claimed. In this way, $k'_\theta \in K'$ and $t \geqslant 0$ correspond to the position of the point $g\cdot 0 \in \mathbb{D}$. In fact $t \geqslant 0$ is uniquely determined, and if $t > 0$ then $k'_\theta\{\pm I\} \in K'/\{\pm I\}$ is uniquely determined by g. Moreover, $k'_{\theta+\psi}\{\pm I\}$ corresponds to the direction of the unit tangent vector when g is applied to the tangent vector 1 at 0.

8.3.5 Volume Calculations

As we will see, it will be important to be able to explicitly calculate integrals over $\mathrm{SL}_2(\mathbb{R})$. For instance, to prove or disprove that certain irreducible unitary representations are a discrete series representation in the sense of Definition 8.1. Moreover, some crucial results in the next chapter will depend on playing the description of the Haar measure on G expressed in terms of the Iwasawa and Cartan decompositions against each other. Hence we wish to describe the Haar measure m of $\mathrm{SL}_2(\mathbb{R})$ using both decompositions.

For the Iwasawa decomposition, this is achieved using the following general fact applied to the subgroups $L = B$ and $R = K$ of $G = \mathrm{SL}_2(\mathbb{R})$.

Lemma 8.13 (Product decomposition of Haar measure). *Let G be a locally compact, σ-compact, metric, unimodular group and suppose that L, R are closed subgroups of G such that LR is a neighbourhood of the identity and the map $\Phi\colon L \times R \ni (\ell, r) \mapsto \ell r \in LR \subseteq G$ is a homeomorphism. Then the Haar measure m_G restricted to LR is proportional to the push-forward of the product measure $m_L \times m_R^{(\mathrm{r})}$ of the left Haar measure m_L on L and the right Haar measure $m_R^{(\mathrm{r})}$ on R.*

We refer to [20, Lem. 11.31] for the details (or [21, Lem. 10.57] for the case of interest here), but point out the main idea of the proof: Indeed, a consequence of the fact that m_G is left and right invariant is that $\Psi_*^{-1}\bigl(m_G|_{LR}\bigr)$ is a left Haar measure on $L \times R$, where $\Psi\colon L \times R \ni (\ell, r) \mapsto \ell r^{-1} \in G$.

Exercise 8.14. Show that $\mathrm{SL}_d(\mathbb{R})$ is unimodular for all $d \geqslant 2$.

To describe the Haar measure m of $\mathrm{SL}_2(\mathbb{R})$ using Lemma 8.13, we need to calculate the left Haar measure of B and the right Haar measure of K. For the former we use Exercise 8.10(e) to identify B with $\mathbb{H}$ and move the hyperbolic area measure from $\mathbb{H}$ to B. We use the coordinates $z = x + \mathrm{i}y \in \mathbb{H}$ and

$$b = u_x a_t = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix} \in B$$

which gives, with $z = b\cdot \mathrm{i}$ that $x = \Re z$ and $y = \Im z = \mathrm{e}^{2t}$. With $\mathrm{dvol} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$, and $\mathrm{d}y = 2\mathrm{e}^{2t}\,\mathrm{d}t$ this leads to the description

$$\mathrm{d}m_B = 2\,\mathrm{d}x\mathrm{e}^{-2t}\,\mathrm{d}t. \tag{8.10}$$

Using now the coordinates $g = u_x a_t k_\psi \in \mathrm{SL}_2(\mathbb{R})$ and Lemma 8.13, we obtain for the Haar measure m on $\mathrm{SL}_2(\mathbb{R})$ that

$$\mathrm{d}m = 2\,\mathrm{d}x\mathrm{e}^{-2t}\,\mathrm{d}t\,\mathrm{d}m_K(k_\psi)$$

by using $\mathrm{d}m_K(k_\psi) = \frac{1}{2\pi}\,\mathrm{d}\psi$ as the Haar measure on K.

We now wish to calculate the Haar measure in terms of the Cartan decomposition, using once more the conjugated group $\mathrm{SU}_{1,1}(\mathbb{R})$. For this, we make the choices

$$g = k'_\theta a'_t k'_\psi \tag{8.11}$$

with $\theta \in [0,\pi)$, $t > 0$, $\psi \in [0, 2\pi)$ to uniquely parametrize $\mathrm{SU}_{1,1}(\mathbb{R}) \smallsetminus K'$. Using again the hyperbolic plane, but this time in the disk model, we have that

$$w = x + \mathrm{i}y = g\cdot 0 = k'_\theta a'_t \cdot 0 = k'_\theta \cdot \tanh t = \mathrm{e}^{-2\theta\mathrm{i}} \tanh t.$$

Using Euclidean polar coordinates with radius $r = |w| = \tanh t$, and angle $\gamma = -2\theta$, the hyperbolic area on $\mathbb{D}$ is therefore given by

$$\begin{aligned}\mathrm{dvol} = 4\frac{\mathrm{d}x\,\mathrm{d}y}{(1-r^2)^2} = 4r\frac{\mathrm{d}r\,\mathrm{d}\gamma}{(1-r^2)^2} &= 4\tanh t\frac{(\tanh t)'\,\mathrm{d}t\,2\,\mathrm{d}\theta}{(1-\tanh^2 t)^2}\\ &= 8\sinh t\cosh t\,\mathrm{d}t\,\mathrm{d}\theta\\ &= 4\sinh 2t\,\mathrm{d}t\,\mathrm{d}\theta,\end{aligned}$$

since

$$\begin{aligned}(\tanh t)' &= \frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t} = \frac{1}{\cosh^2 t},\\ (1-\tanh^2 t)^2 &= \left(\frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t}\right)^2 = \frac{1}{\cosh^4 t},\end{aligned}$$

and

$$\sinh 2t = 2\sinh t\cosh t.$$

We also note that the Haar measure m on $\mathrm{SU}_{1,1}(\mathbb{R})$ is invariant under multiplication on the right by $k'_\psi \in K'$. Hence the Haar measure on $\mathrm{SU}_{1,1}(\mathbb{R})$, and equivalently on $\mathrm{SL}_2(\mathbb{R})$, is given (in the Cartan decomposition (8.11)) by

$$\mathrm{d}m = \frac{2}{\pi}\sinh 2t\,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}\psi, \tag{8.12}$$

where as before $g = k_\theta a_t k_\psi$ with $\theta \in [0,\pi)$, $t > 0$, and $\psi \in [0, 2\pi)$.

To practice using the above formulas, we calculate some volumes in the next lemma.

Lemma 8.15 (Volume of hyperbolic and norm balls). *We have*

$$B_R^{\mathbb{D}}(0) = \bigl\{w \in \mathbb{D} \mid \|w\| < \tanh\bigl(\tfrac{R}{2}\bigr)\bigr\} \tag{8.13}$$

and

$$\mathrm{vol}\bigl(B_R^{\mathbb{H}}(z)\bigr) = \mathrm{vol}\bigl(B_R^{\mathbb{D}}(w)\bigr) = 2\pi\,(\cosh(R) - 1)$$

for all $R > 0$, $z \in \mathbb{H}$, and $w \in \mathbb{D}$. Moreover,

$$m_{\mathrm{SL}_2(\mathbb{R})}\bigl(B_r^{|\cdot|}\bigr) = \pi\bigl(r^2 - 2\bigr)$$

for all $r \geqslant \sqrt{2}$, where

$$B_r^{|\cdot|} = \{g \in \mathrm{SL}_2(\mathbb{R}) \mid |g| \leqslant r\}.$$

Proof. By the discussion in Section 8.3.3, the volume calculation may be carried out using the disk model $\mathbb{D} = \{w \in \mathbb{C} \mid |w| < 1\}$ of the hyperbolic plane, which carries the Riemannian metric

$$4\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{(1 - r^2)^2}$$

at the point $w = x + \mathrm{i}y$, where $r^2 = x^2 + y^2$. Moreover, the transitive action of $\mathrm{SU}_{1,1}(\mathbb{R})$ on $\mathbb{D}$ preserves the hyperbolic Riemannian metric, and hence also the hyperbolic area. Therefore it suffices to consider the point $w = 0$ and calculate the volume of the ball of (hyperbolic) radius R around 0, which by invariance of $B_R^{\mathbb{D}}(0)$ under the diagonal subgroup K' is a disk of some radius ρ around 0 in the Euclidean metric as well. The Euclidean radius ρ may be calculated using the relation

$$R = 2\int_0^\rho \frac{\mathrm{d}r}{1 - r^2} = \log\left(\frac{1 + r}{1 - r}\right)\Big|_0^\rho = \log\left(\frac{1 + \rho}{1 - \rho}\right)$$

or, equivalently,

$$\rho = \frac{\mathrm{e}^R - 1}{\mathrm{e}^R + 1} = \frac{\mathrm{e}^{\frac{R}{2}} - \mathrm{e}^{-\frac{R}{2}}}{\mathrm{e}^{\frac{R}{2}} + \mathrm{e}^{-\frac{R}{2}}} = \tanh\bigl(\tfrac{R}{2}\bigr).$$

This gives (8.13). Therefore, the volume of the hyperbolic ball of radius R around $w = 0$ is given by

$$\begin{aligned}\operatorname{vol}\Big(B_R^{\mathbb{D}}(0)\Big) &= 4\int_0^\rho \int_0^{2\pi} \frac{r\,\mathrm{d}r\,\mathrm{d}\gamma}{(1-r^2)^2} = 4\pi \int_0^{\rho^2} \frac{\mathrm{d}u}{(1-u)^2} \\ &= 4\pi \frac{1}{1-u}\Big|_0^{\rho^2} = 4\pi \left(\frac{1}{1-\rho^2} - 1\right) \\ &= 2\pi \left(\frac{2(\mathrm{e}^R+1)^2}{(\mathrm{e}^R+1)^2 - (\mathrm{e}^R-1)^2} - 2\right) = 2\pi\left(\cosh(R) - 1\right).\end{aligned}$$

We note that $g = k_\theta a_t k_\psi \in B_r^{\|\cdot\|}$ if and only if $\mathrm{e}^{2t} + \mathrm{e}^{-2t} \leqslant r^2$ (that is, $\cosh 2t \leqslant \frac{r^2}{2}$). Hence we can use (8.12) and calculate

$$\begin{aligned}m\big(B_r^{\|\cdot\|}\big) &= \int_0^{2\pi} \int_0^{\frac{1}{2}\operatorname{arcosh}\frac{r^2}{2}} \int_0^\pi \frac{2}{\pi} \sinh 2t\,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}\psi \\ &= 2\pi \int_0^{\operatorname{arcosh}\frac{r^2}{2}} \sinh u\,\mathrm{d}u \\ &= 2\pi\big(\tfrac{r^2}{2} - 1\big) = \pi(r^2-2).\end{aligned}$$

Alternatively, we could have put $R = \operatorname{arcosh}\frac{r^2}{2}$ into the calculation of the hyperbolic R-ball to obtain $m\big(B_r^{\|\cdot\|}\big) = \operatorname{vol}\big(B_R^{\mathbb{H}}(\mathrm{i})\big) = \pi\big(r^2-2\big)$. □

8.3.6 The Hyperbolic Metric on $\mathbb{H}$

For some of our later discussions (specifically, those in Section 9.4.5), we will need to describe the hyperbolic metric more concretely than we did in Section 8.3.1.

Lemma 8.16 (Hyperbolic metric). *The hyperbolic distance between the points $z_1 = x_1 + \mathrm{i}y_2$ and $z_2 = x_2 + \mathrm{i}y_2$ in $\mathbb{H}$ is given by*

$$\mathsf{d}_{\mathrm{hyp}}(z_1, z_2) = \operatorname{arcosh}\left(1 + \frac{\|z_1 - z_2\|_2^2}{2y_1y_2}\right).$$

Proof. For the proof, we will again use the isometric action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$. In fact, we claim that the expression

$$\frac{\|z_1 - z_2\|^2}{y_1 y_2} \tag{8.14}$$

is also invariant under the simultaneous action of $g \in \mathrm{SL}_2(\mathbb{R})$ on z_1 and z_2. Assuming the claim (which will follow from a calculation) for the moment, we now prove the lemma. By transitivity of the isometric action of $\mathrm{SL}_2(\mathbb{R})$ we may assume that $z_1 = \mathrm{i}$. Without moving z_1 we may also apply an element of $\mathrm{SO}_2(\mathbb{R})$, allowing us to assume that $z_2 = \mathrm{i}y_2$. Indeed, the action of $\mathrm{SO}_2(\mathbb{R})$ fixes i and acts transitively on unit tangent vectors in $\mathrm{T}_\mathrm{i}\mathbb{H}$, while the geodesic through z_1 and z_2 is uniquely determined by z_1 and the unit tangent vector in $\mathrm{T}_{z_1}\mathbb{H}$ in the direction of the geodesic. Since the action is isometric, this does not change $\mathsf{d}_{\mathrm{hyp}}(z_1, z_2)$, nor the expression in (8.14) by the claim. We now calculate the latter for $z_1 = \mathrm{i}$ and $z_2 = \mathrm{i}y_2$, and obtain

$$\frac{\|z_1 - z_2\|^2}{y_1 y_2} = \frac{(y_2 - 1)^2}{y_2} = \frac{y_2^2 - 2y_2 + 1}{y_2}.$$

For the term $\operatorname{arcosh}\bigl(1 + \frac{\|z_1 - z_2\|^2}{2y_1 y_2}\bigr)$, this gives

$$\operatorname{arcosh}\left(\frac{y_2 + y_2^{-1}}{2}\right) = |\log y_2|.$$

On the other hand, the geodesic path from $z_1 = \mathrm{i}$ to $z_2 = \mathrm{i}y_2$ (for $y_2 \neq 1$) is parametrized by $[0, 1] \ni t \mapsto \mathrm{i}y_2^t$, which gives

$$\mathsf{d}_{\mathrm{hyp}}(\mathrm{i}, \mathrm{i}y_2) = \int_0^1 \frac{|\log y_2| y_2^t}{y_2^t}\,\mathrm{d}t = |\log y_2|.$$

Thus the claim that the expression in (8.14) is invariant under the action of $\mathrm{SL}_2(\mathbb{R})$ implies the lemma.

For the claim, we first notice that the action of

$$u_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in U$$

on $z \in \mathbb{H}$ satisfies $u_x \cdot z = z + x$, does not change $y = \Im(z)$, nor the difference $z_1 - z_2$ if applied to both z_1 and z_2. This already gives part of the claim. Since $\mathrm{SL}_2(\mathbb{R})$ is generated by the subgroups U and

$$\left\{ \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$

it is enough to show that the simultaneous action of

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$$

on $z_1, z_2 \in \mathbb{H}$ does not change the term (8.14). Since

$$\begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for all

$$u_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \in U,$$

it suffices to consider the action of

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Recall that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bullet z = -\frac{1}{z} = \frac{-\overline{z}}{\|z\|^2} = \frac{-x + \mathrm{i}y}{\|z\|^2}$$

for all $z = x + \mathrm{i}y \in \mathbb{H}$. We now apply this to $z_1, z_2 \in \mathbb{H}$ and calculate

$$\begin{aligned} \frac{\left\| \frac{-\overline{z_1}}{\|z_1\|^2} - \frac{-\overline{z_2}}{\|z_2\|^2} \right\|^2}{\frac{y_1}{\|z_1\|^2} \frac{y_2}{\|z_2\|^2}} &= \frac{\|z_1\|^2 \|z_2\|^2 \left\| \frac{z_1 \|z_2\|^2 - z_2 \|z_1\|^2}{\|z_1\|^2 \|z_2\|^2} \right\|^2}{y_1 y_2} \\ &= \frac{\left\| z_1 \|z_2\|^2 - z_2 \|z_1\|^2 \right\|^2}{y_1 y_2 \|z_1\|^2 \|z_2\|^2}. \end{aligned}$$

The numerator is equal to

$$\begin{aligned} &\left(x_1 \|z_2\|^2 - x_2 \|z_1\|^2\right)^2 + \left(y_1 \|z_2\|^2 - y_2 \|z_1\|^2\right)^2 \\ &\quad = x_1^2 \|z_2\|^4 + x_2^2 \|z_1\|^4 + y_1^2 \|z_2\|^4 + y_2^2 \|z_1\|^4 \\ &\qquad\qquad - 2x_1 x_2 \|z_1\|^2 \|z_2\|^2 - 2y_1 y_2 \|z_1\|^2 \|z_2\|^2 \\ &\quad = \|z_1\|^2 \|z_2\|^4 + \|z_1\|^4 \|z_2\|^2 - 2x_1 x_2 \|z_1\|^2 \|z_2\|^2 - 2y_1 y_2 \|z_1\|^2 \|z_2\|^2 \end{aligned}$$

or, equivalently,

$$\|z_1\|^2 \|z_2\|^2 \underbrace{(x_2^2 + y_2^2 + x_1^2 + y_1^2 - 2x_1 x_2 - 2y_1 y_2)}_{=\left((x_1 - x_2)^2 + (y_1 - y_2)^2\right) = \|z_1 - z_2\|^2}.$$

This gives

$$\frac{\|z_1\|^2 \|z_2\|^2 \|z_1 - z_2\|^2}{y_1 y_2 \|z_1\|^2 \|z_2\|^2} = \frac{\|z_1 - z_2\|^2}{y_1 y_2}$$

for the quotient. It follows that neither the elements of U nor the element

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

changes the term (8.14) when applied to both z_1 and z_2 in $\mathbb{H}$. As explained above, this establishes the claim and hence the lemma. □

We will also use the terminology introduced in Section 7.2.2 for the compact subgroup $K' < \mathrm{SU}_{1,1}(\mathbb{R})$ and unitary representations of $\mathrm{SU}_{1,1}(\mathbb{R})$. We note that in the following arguments we will always study either $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SU}_{1,1}(\mathbb{R})$, but not both at the same time. Hence we will simplify the notation for

$$K = \left\{ k_\theta = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} < \mathrm{SU}_{1,1}(\mathbb{R}).$$

8.4 (Mock) Discrete Series Representations for $\mathbf{SL_2}(\mathbb{R})$

We will describe in this section the first and second types of irreducible unitary representations for the group $G = \mathrm{SL}_2(\mathbb{R})$. As we will see, these are intimately connected to complex analysis and will also give examples of discrete series representations (in the sense of Definition 8.1) and tempered almost square integrable representations (in the sense of Definition 8.3 and Definition 8.4).

8.4.1 The Discrete Series Representation

The discussion in Section 8.3.3 shows that we can work either with the upper half-plane model or the disk model of the hyperbolic plane. As the description of a particular important orthogonal basis will be more convenient in the disk model, we will work here with $\mathbb{D}$ and the group $\mathrm{SU}_{1,1}(\mathbb{R})$ instead of $\mathbb{H}$ and the isomorphic group $\mathrm{SL}_2(\mathbb{R})$.

We fix an integer $n \geqslant 2$, and define the Hilbert space

$$V_n = L^2\left(\mathbb{D}, (1-|z|^2)^n \,\mathrm{dvol}\right) \tag{8.15}$$

where $\mathrm{dvol} = \frac{4\,\mathrm{d}x\,\mathrm{d}y}{(1-|z|^2)^2}$ denotes the hyperbolic volume on $\mathbb{D}$. We note that, unlike the infinite measure vol, the finite measure

$$(1-|z|^2)^n \,\mathrm{dvol} = 4(1-|z|^2)^{n-2}\,\mathrm{d}x\,\mathrm{d}y$$

is not invariant under $\mathrm{SU}_{1,1}(\mathbb{R})$, but is instead only quasi-invariant. We define the representation π^n of $\mathrm{SU}_{1,1}(\mathbb{R})$ on V_n by the formula

$$\left(\pi^n_g f\right)(z) = (-\beta z + \alpha)^{-n} f(g^{-1}\bullet z)$$

where

$$g = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{R}),$$

and $f \in V_n$, $z \in \mathbb{D}$, and the factor $(-\beta z + \alpha)$ is the denominator of

$$g^{-1} \bullet z = \frac{\overline{\alpha} z - \overline{\beta}}{-\beta z + \alpha}.$$

As indicated in Section 8.3.2, this factor $-\beta z + \alpha$ also appears in the formula for the complex derivative of $z \mapsto g^{-1} \bullet z$ given by

$$\left(g^{-1} \bullet z\right)' = \frac{\overline{\alpha}(-\beta z + \alpha) - (\overline{\alpha} z - \overline{\beta})(-\beta)}{(-\beta z + \alpha)^2} = \frac{1}{(-\beta z + \alpha)^2}$$

and hence also in the Jacobian determinant $|-\beta z + \alpha|^{-4}$ of the map $z \mapsto g^{-1} \bullet z$ considered as a real differentiable map from an open subset of $\mathbb{R}^2$ to itself.

Using the measure-preserving substitution $w = g^{-1} \bullet z$ for the measure vol, we wish to calculate

$$\|\pi_g^n f\|_{V_n}^2 = \int_{\mathbb{D}} |-\beta z + \alpha|^{-2n} |f(g^{-1} \bullet z)|^2 \left(1 - |z|^2\right)^n \,\mathrm{dvol}.$$

In fact

$$\begin{aligned} 1 - |w|^2 &= 1 - \frac{(\overline{\alpha} z - \overline{\beta})(\alpha \overline{z} - \beta)}{(-\beta z + \alpha)(-\overline{\beta}\overline{z} + \overline{\alpha})} \\ &= \frac{(-\beta z + \alpha)(-\overline{\beta}\overline{z} + \overline{\alpha}) - (\overline{\alpha} z - \overline{\beta})(\alpha \overline{z} - \beta)}{|-\beta z + \alpha|^2} \\ &= \frac{1 - |z|^2}{|-\beta z + \alpha|^2}, \end{aligned}$$

and so we obtain

$$\begin{aligned} \|\pi_g^n f\|_{V_n}^2 &= \int_{w \in \mathbb{D}} |f(w)|^2 \left(1 - |w|^2\right)^n \underbrace{\left(\frac{1 - |z|^2}{1 - |w|^2} |-\beta z + \alpha|^{-2}\right)^n}_{=1} \mathrm{dvol} \\ &= \int_{w \in \mathbb{D}} |f(w)|^2 \left(1 - |w|^2\right)^n \,\mathrm{dvol} = \|f\|_{V_n}^2. \end{aligned}$$

We also need to verify that $\pi_g^n \pi_h^n = \pi_{gh}^n$ for all $g, h \in \mathrm{SU}_{1,1}(\mathbb{R})$. By definition, we have

$$\pi_h^n f(w) = (-\gamma w + \eta)^{-n} f(h^{-1} \bullet w)$$

for

$$h = \begin{pmatrix} \eta & \overline{\gamma} \\ \gamma & \overline{\eta} \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{R})$$

and $f \in V_n$ and $w \in \mathbb{D}$. For

$$g = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$$

we now have for $z \in \mathbb{D}$ and $w = g^{-1} \bullet z \in \mathbb{D}$ that

$$\begin{aligned} \left(\pi_g^n \pi_h^n(f)\right)(z) &= (-\beta z + \alpha)^{-n} \pi_h^n f(g^{-1} \bullet z) \\ &= (-\beta z + \alpha)^{-n} (-\gamma w + \eta)^{-n} f(h^{-1} \bullet (g^{-1} \bullet z)) \\ &= (-\beta z + \alpha)^{-n} \left(-\gamma \frac{\overline{\alpha} z - \overline{\beta}}{-\beta z + \alpha} + \eta\right)^{-n} f\left((gh)^{-1} \bullet z\right) \\ &= \left(-\gamma(\overline{\alpha} z - \overline{\beta}) + \eta(-\beta z + \alpha)\right)^{-n} f\left((gh)^{-1} \bullet z\right) \\ &= \left(-(\gamma \overline{\alpha} + \eta \beta) z + (\gamma \overline{\beta} + \eta \alpha)\right)^{-n} f\left((gh)^{-1} \bullet z\right). \end{aligned}$$

Since

$$\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \eta & \overline{\gamma} \\ \gamma & \overline{\eta} \end{pmatrix} = \begin{pmatrix} \alpha\eta + \overline{\beta}\gamma & \alpha\overline{\gamma} + \overline{\beta}\overline{\eta} \\ \beta\eta + \overline{\alpha}\gamma & \beta\overline{\gamma} + \overline{\alpha\eta} \end{pmatrix},$$

it follows that $\pi_g^n \pi_h^n = \pi_{gh}^n$ for all $g, h \in \mathrm{SU}_{1,1}(\mathbb{R})$. Therefore π^n is a unitary representation of $\mathrm{SU}_{1,1}(\mathbb{R})$ (see also Exercise 8.17).

Exercise 8.17. Either mimic the argument from Proposition 1.5 to prove the continuity requirement for π^n, or show that the above is a special case of a twisted normalized representation in Proposition 1.5.

Definition 8.18 (Bergman space and discrete series representations). Let n be a natural number with $n \geqslant 2$, and let

$$\mathrm{Hol}(\mathbb{D}) = \{f \colon \mathbb{D} \longrightarrow \mathbb{C} \mid f \text{ is holomorphic}\}.$$

The subspace $A_n(\mathbb{D}) = V_n \cap \mathrm{Hol}(\mathbb{D}) < V_n$ is called a *weighted Bergman space*. The restriction of π^n to $A_n(\mathbb{D})$ is called the *holomorphic discrete series representation* $\delta^{n,+}$.

Lemma 8.19 (Holomorphic discrete series representations). *Let $n \geqslant 2$ be a natural number. Then the weighted Bergman space $A_n(\mathbb{D})$ is closed in V_n and is invariant under π^n.*

Proof. We first show that $A_n(\mathbb{D}) \subseteq V_n$ is closed. For this, we claim that if a sequence in $A_n(\mathbb{D})$ converges with respect to $\|\cdot\|_{V_n}$ to some element of V_n, then the convergence also holds uniformly on compact subsets of D. The claim implies that the limit is holomorphic.

Define the compact subset $K = \overline{B_r(0)}$ for $r \in (0,1)$. For $f \in A_n(\mathbb{D})$, $z_0 \in K$, and $R \in (r, 1)$ we have

$$f(z_0) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=R} \frac{f(z)}{z - z_0} \, \mathrm{d}z$$

by the Cauchy residue formula, where $\oint$ denotes the positively oriented complex path integral along the circle defined by $|z| = R$. Equivalently, we have

$$f(z_0) = \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{f(R\mathrm{e}^{\mathrm{i}\theta})}{R\mathrm{e}^{\mathrm{i}\theta} - z_0} R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta.$$

We now fix two additional real parameters s, t with $0 < r < s < t < 1$, and vary R within $[s, t]$ to obtain

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi(t-s)} \int_s^t \int_0^{2\pi} \frac{f(R\mathrm{e}^{\mathrm{i}\theta})}{R\mathrm{e}^{\mathrm{i}\theta} - z_0} \mathrm{e}^{\mathrm{i}\theta} R\,\mathrm{d}\theta\,\mathrm{d}R \\ &= \frac{1}{2\pi(t-s)} \int_{s\leqslant |z|\leqslant t} \frac{f(z)}{z - z_0} \frac{z}{|z|(1-|z|^2)^{n-2}} \underbrace{\left(1-|z|^2\right)^{n-2}\mathrm{d}x\,\mathrm{d}y}_{=\frac{1}{4}(1-|z|^2)^n\,\mathrm{dvol}} \end{aligned}$$

by using polar coordinates for $z = x + \mathrm{i}y = R\mathrm{e}^{\mathrm{i}\theta}$. This shows that the map

$$A_n(\mathbb{D}) \ni f \mapsto f(z_0) \in \mathbb{C}$$

is continuous with respect to $\|\cdot\|_{V_n}$ for all $z_0 \in \overline{B_r(0)}$, and in fact also gives

$$\left\| f|_{\overline{B_r(0)}} \right\|_\infty \ll_{s,t} \|f\|_{V_n}.$$

However, this proves the claim and hence shows that $A_n(\mathbb{D})$ is a closed subspace of V_n.

For the invariance, let $f \in A_n(\mathbb{D})$, $g \in \mathrm{SU}_{1,1}(\mathbb{R})$ and recall from above that we already know that $\pi_g^n f \in V_n$. However, the definition of $\pi_g^n f$ also shows that $\pi_g^n f$ is holomorphic on $\mathbb{D}$, which gives $\pi_g^n f \in A_n(\mathbb{D})$ as required. □

Definition 8.20 (Anti-holomorphic discrete series representations). Let $n \geqslant 2$ be a natural number. The *anti-holomorphic discrete series representation* $\delta^{n,-}$ is defined as the restriction of the unitary representation $\overline{\pi}^n$ defined by

$$\overline{\pi}_g^n(f)(z) = (-\overline{\beta}\overline{z} + \overline{\alpha})^{-n} f(g^{-1}\bullet z)$$

for

$$g = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{R})$$

and $f \in V_n$, $z \in \mathbb{D}$ to the conjugated subspace

$$\overline{A_n(\mathbb{D})} = V_n \cap \overline{\mathrm{Hol}(\mathbb{D})}$$

of anti-holomorphic functions in V_n.

Exercise 8.21. Show, for any integer $n \geqslant 2$, that $\delta^{n,+}$ is the contragredient representation to $\delta^{n,-}$.

What follows holds similarly for the anti-holomorphic discrete series representations, but we will only consider the holomorphic discrete series representations.

Lemma 8.22 (K-weights in $\delta^{n,+}$). *Let $n \geqslant 2$ be a natural number. The discrete series representation $\delta^{n,+}$ contains K-eigenfunctions for the compact diagonal subgroup*

$$K = \left\{ k_\theta = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} < \mathrm{SU}_{1,1}(\mathbb{R})$$

for the weights $n, n+2, n+4, \dots$ each with multiplicity one, and only these. In fact the functions $e_\ell \colon \mathbb{D} \ni z \mapsto z^\ell$ for $\ell \in \mathbb{N}_0$ are mutually orthogonal, span $A_n(\mathbb{D})$, and

$$\delta^{n,+}_{k_\theta} e_\ell = \mathrm{e}^{\mathrm{i}(n+2\ell)\theta} e_\ell \tag{8.16}$$

for all $k_\theta \in K$ and $\ell \in \mathbb{N}_0$.

Proof. For any $\ell \geqslant 0$ and $k_\theta \in K$, we note that

$$\begin{aligned} \delta^{n,+}_{k_\theta} e_\ell(z) &= (-0z + \mathrm{e}^{-\mathrm{i}\theta})^{-n} e_\ell \left(\frac{\mathrm{e}^{\mathrm{i}\theta} z}{\mathrm{e}^{-\mathrm{i}\theta}} \right) \\ &= \mathrm{e}^{\mathrm{i}n\theta} (\mathrm{e}^{2\mathrm{i}\theta} z)^\ell \\ &= \mathrm{e}^{\mathrm{i}(n+2\ell)\theta} z^\ell = \mathrm{e}^{\mathrm{i}(n+2\ell)\theta} e_\ell(z) \end{aligned}$$

for all $z \in \mathbb{D}$, which gives (8.16). Since the weights are mutually distinct, we see that the non-zero vectors $e_0, e_1, e_2, \ldots \in A_n(\mathbb{D})$ are mutually orthogonal.

It remains to show that the closed linear hull of $\{e_\ell \mid \ell \in \mathbb{N}_0\}$ is $A_n(\mathbb{D})$, as the other claims in the lemma then follow from this. So suppose that f lies in $A_n(\mathbb{D})$. Then f is holomorphic on $\mathbb{D}$, and complex analysis therefore implies that

$$f(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$$

for all $z \in \mathbb{D}$, and that this convergence is uniform on compact subsets of $\mathbb{D}$. This gives, with $\mathrm{dvol} = 4\frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{(1-r^2)^2}$, that

$$\begin{aligned}\|f\|^2_{A_n(\mathbb{D})} &= 4\int_0^1 (1-r^2)^{n-2}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^2 \,\mathrm{d}\theta\, r\,\mathrm{d}r\\ &= 4\int_0^1 (1-r^2)^{n-2} r \underbrace{\int_0^{2\pi}\left|\sum_{\ell=0}^{\infty} a_\ell r^\ell \mathrm{e}^{\mathrm{i}\ell\theta}\right|^2 \mathrm{d}\theta}_{=\sum_{\ell=0}^{\infty}|a_\ell r^\ell|^2\cdot 2\pi}\,\mathrm{d}r\\ &= 4\sum_{\ell=0}^{\infty}\int_0^1 (1-r^2)^{n-2} r\int_0^{2\pi} |a_\ell r^\ell \mathrm{e}^{\mathrm{i}\ell\theta}|^2\,\mathrm{d}\theta\,\mathrm{d}r = \sum_{\ell=0}^{\infty}\|a_\ell e_\ell\|^2_{A_n(\mathbb{D})}\end{aligned}$$

by using the fact that the functions $[0,2\pi]\ni\theta\mapsto \mathrm{e}^{\mathrm{i}\ell\theta}$ for $\ell\in\mathbb{N}_0$ are mutually orthogonal. Therefore, the series $\sum_{\ell=0}^{\infty} a_\ell e_\ell$ also converges in $A_n(\mathbb{D})$, and the integral formulas in the proof of Lemma 8.19 show that the limit of the series in $A_n(\mathbb{D})$ equals f. □

Due to the conclusion of the previous lemma, we will also for $n\geqslant 2$ say that n is the *lowest K-weight* for $\delta^{n,+}$. Similarly, $-n$ is the *highest K-weight* for $\delta^{n,-}$. If we want to speak about the representations $\delta^{n,\pm}$ simultaneously, we also say that $\pm n$ is the *terminal K-weight*.

Theorem 8.23 (The discrete series representation $\delta^{n,+}$). *Let $n\geqslant 2$. Then the holomorphic discrete series representations $\delta^{n,+}$ is an irreducible unitary representation of $\mathrm{SU}_{1,1}(\mathbb{R})$ that is also a discrete series representation (in the sense of Definition 8.1). Moreover, $\delta^{n,+}$ has K-eigenfunctions with weights in $n+2\mathbb{N}_0$, each with multiplicity one, and only these. This holds similarly for $\delta^{n,-}$ and weights in $-(n+2\mathbb{N}_0)$.*

It remains to prove irreducibility and square integrability of at least one non-trivial principal matrix coefficient. For the former, we will use the Lie algebra element

$$\mathbf{d} = \begin{pmatrix} & 1\\ 1 & \end{pmatrix} \in \mathfrak{su}_{1,1}(\mathbb{R})$$

satisfying

$$\exp(t\mathbf{d}) = \begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}.$$

Proof of irreducibility of $\delta^{n,+}$. Suppose that $\mathcal{V}\subseteq A_n(\mathbb{D})$ is a non-trivial $\delta^{n,+}$-invariant closed subspace. As $K\cong\mathbb{S}^1$ is compact and abelian there exists some non-trivial eigenvector $v\in\mathcal{V}$ for $\delta^{n,+}|_K$ and some weight. By Lemma 8.22, this shows that $e_\ell\in\mathcal{V}$ for some $\ell\in\mathbb{N}_0$. We will now show that $\delta^{n,+}_\partial(\mathbf{d})e_\ell$ exists and conclude that $\delta^{n,+}_\partial(\mathbf{d})e_\ell\in\mathcal{V}$. By definition,

$$\delta^{n,+}_\partial(\mathbf{d})e_\ell = \lim_{t\to 0}\frac{1}{t}\left(\delta^{n,+}(\exp(t\mathbf{d}))e_\ell - e_\ell\right). \tag{8.17}$$

Now

$$\delta^{n,+}(\exp(t\mathbf{d}))e_\ell(z) = (-\sinh tz + \cosh t)^{-n}\left(\frac{\cosh tz - \sinh t}{-\sinh tz + \cosh t}\right)^{\ell}$$
$$= (\cosh tz - \sinh t)^{\ell}\,(-\sinh tz + \cosh t)^{-n-\ell}$$

and

$$\lim_{t\to 0}\frac{1}{t}\Big((\cosh tz - \sinh t)^{\ell}(-\sinh tz + \cosh t)^{-n-\ell} - z^{\ell}\Big)$$
$$= \frac{\partial}{\partial t}\Big|_{t=0}(\cosh tz - \sinh t)^{\ell}\,(-\sinh tz + \cosh t)^{-n-\ell}$$
$$= \ell z^{\ell-1}\cdot(-1)\cdot(1) + z^{\ell}\,(-n-\ell)\cdot 1\cdot(-z)$$
$$= -\ell z^{\ell-1} + (n+\ell)z^{\ell+1}.$$

The latter convergence holds for any $z\in\mathbb{C}$, uniformly on $\overline{\mathbb{D}}$. Moreover, the limit in (8.17) also exists with respect to $\|\cdot\|_{V_n}$, since

$$\int_{\mathbb{D}}(1-|z|^2)^n\,\mathrm{dvol} = 4\int_{\mathbb{D}}\left(1-|z|^2\right)^{n-2}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

In fact, we obtain

$$\delta_\partial^{n,+}(\mathbf{d})e_\ell = -\ell e_{\ell-1} + (n+\ell)e_{\ell+1}.$$

As $e_\ell\in\mathcal{V}$ and $\mathcal{V}$ is assumed to be invariant, we obtain $\delta_\partial^{n,+}(\mathbf{d})e_\ell\in\mathcal{V}$. As the vectors $e_0, e_1, e_2, \dots$ are K-eigenfunctions with different eigenvalues by Lemma 8.22, we also obtain $e_{\ell-1}\in\mathcal{V}$ if $\ell>0$ and $e_{\ell+1}\in\mathcal{V}$ (since $n+\ell$ is non-zero). Iterating this argument, we obtain $e_\ell\in\mathcal{V}$ for all $\ell\in\mathbb{N}_0$, and so $\mathcal{V}=A_n(\mathbb{D})$ by Lemma 8.22. It follows that $\delta^{n,+}$ is irreducible. □

To prove that $\delta^{n,+}$ is a discrete series representation in the sense of Definition 8.1, it suffices by Theorem 8.2 to find one non-zero vector in $A_n(\mathbb{D})$ for which the associated matrix coefficient belongs to $L^2(\mathrm{SU}_{1,1}(\mathbb{R}))$. For this reason we calculate the matrix coefficient of $e_0\in A_n(\mathbb{D})$. For this, we first consider

$$a_t = \exp(t\mathbf{d}) = \begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix}$$

for $t\in\mathbb{R}$, and calculate

$$\varphi_{e_0}(a_t) = \left\langle \delta_{a_t}^{n,+}e_0, e_0\right\rangle_{A_n(\mathbb{D})}$$
$$= \int_{\mathbb{D}}(-\sinh t\,z + \cosh t)^{-n}\left(1-|z|^2\right)^{n}\,\mathrm{dvol}$$
$$= 4(\cosh t)^{-n}\int_0^1(1-r^2)^{n-2}r\int_0^{2\pi}\left(1 - r\tanh t\,\mathrm{e}^{\mathrm{i}\theta}\right)^{-n}\,\mathrm{d}\theta\,\mathrm{d}r.$$

Since $|r \tanh t| < 1$ for all $r \in [0,1)$ and $t \in \mathbb{R}$, we can simply expand the innermost term in a power series. After integration with respect to $\theta \in [0, 2\pi]$, we therefore obtain the constant 2π for the inner integral, and hence

$$\varphi_{e_0}(a_t) = 4\pi(\cosh t)^{-n} \int_0^1 \left(1 - r^2\right)^{n-2} 2r \, \mathrm{d}r = \frac{4\pi}{n-1}(\cosh t)^{-n}.$$

Proof of square integrability. We recall from (8.11) that the Cartan decomposition of $\mathrm{SU}_{1,1}(\mathbb{R})$ is given by KAK, meaning that any $g \in \mathrm{SU}_{1,1}(\mathbb{R})$ can be written as $g = k_\theta a_t k_\psi$, where a_t is as above and

$$k_\theta = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & \\ & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix} \in K$$

for $\theta \in [0, \pi)$ and $\psi \in [0, 2\pi)$. With this, the matrix coefficient φ_{e_0} satisfies

$$|\varphi_{e_0}(g)| = \left| \left\langle \delta^{n,+}_{a_t k_\psi} e_0, \delta^{n,+}_{k_\theta^{-1}} e_0 \right\rangle \right| = \left| \mathrm{e}^{\mathrm{i}n\psi} \mathrm{e}^{\mathrm{i}n\theta} \underbrace{\left\langle \delta^{n,+}_{a_t} e_0, e_0 \right\rangle}_{=\varphi_{e_0}(a_t)} \right| = \frac{4\pi}{n-1}(\cosh t)^{-n}$$

by the calculation above.

To see that φ_{e_0} belongs to $L^2(\mathrm{SU}_{1,1}(\mathbb{R}))$, we need the description of the Haar measure m in the coordinates of the Cartan decomposition. This has the form

$$\mathrm{d}m = \frac{2}{\pi} \sinh(2t) \, \mathrm{d}\theta \, \mathrm{d}t \, \mathrm{d}\psi, \tag{8.18}$$

and so we obtain

$$\begin{aligned} \int_{\mathrm{SU}_{1,1}(\mathbb{R})} |\varphi_{e_0}(g)|^2 \, \mathrm{d}m &= \frac{2}{\pi} \int_0^{2\pi} \int_0^\infty \int_0^\pi |\varphi_{e_0}(a_t)|^2 \sinh(2t) \, \mathrm{d}\theta \, \mathrm{d}t \, \mathrm{d}\psi \\ &= 4\pi \left(\frac{4\pi}{n-1} \right)^2 \int_0^\infty (\cosh t)^{-2n} \sinh(2t) \, \mathrm{d}t. \end{aligned}$$

Since $(\cosh t)^{-1} \ll \mathrm{e}^{-t}$ and $\sinh 2t \ll \mathrm{e}^{2t}$, and since $n \geqslant 2$ we deduce that $\varphi_{e_0} \in L^2(\mathrm{SU}_{1,1}(\mathbb{R}))$. By Theorem 8.2, this implies that $\delta^{n,+}$ is a discrete series representation of $\mathrm{SU}_{1,1}(\mathbb{R})$. □

8.4.2 Mock Discrete Series Representations

The mock discrete series representation corresponds in a way to setting $n = 1$ in the discussions above. However, if we simply set $n = 1$ in the definition of V in (8.15), then the underlying measure would be an infinite invariant measure on $\mathbb{D}$. This would mean that the most basic holomorphic functions e_ℓ defined by $e_\ell(z) = z^\ell$ would not any longer be square-integrable for $\ell \in \mathbb{N}_0$. To guess how we could give a meaningful definition of a Hilbert space for $n = 1$, we allow for a moment $n \in (1, 2)$ and consider $L^2(\mathbb{D}, (1 - |z|^2)^{n-2}\,\mathrm{d}x\,\mathrm{d}y)$ for $n \searrow 1$. A calculation reveals that $(1 - |z|^2)^{n-2}\,\mathrm{d}x\,\mathrm{d}y$ defines in this case a finite measure with more and more total mass as n decreases to 1. If we normalize this measure to be a probability measure for each n, then those probability measures have a weak* limit in the space of probability measures on $\overline{\mathbb{D}} = \mathbb{D} \cup \mathbb{S}^1$ as $n \searrow 1$. In fact, this limit is given by the normalized arc length measure on $\partial\mathbb{D} = \mathbb{S}^1$. This motivates the following definition and the alternative name 'limit discrete series representation' for the 'mock discrete series representations' discussed here. The definition could be chosen closer to the discussion above, but the following will be easier to work with.

Definition 8.24 (Hardy space). The *Hardy space* $H(\mathbb{D})$ for $\mathbb{D}$ is defined to consist of all holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ for which the Hardy norm $\|f\|_{H(\mathbb{D})}$ is finite, where

$$\|f\|_{H(\mathbb{D})}^2 = \sup_{0 \leqslant r < 1} \int_0^1 |f(r\mathrm{e}^{2\pi\mathrm{i}\theta})|^2\,\mathrm{d}\theta. \tag{8.19}$$

Lemma 8.25 (Norm on Hardy space). *For elements $f \in H(\mathbb{D})$ the functions $\mathbb{S}^1 \ni z \mapsto f(rz)$ have a limit as $r \nearrow 1$ in $L^2(\mathbb{S}^1)$. Extending f to almost every $z \in \mathbb{S}^1$ by this limit, we also have*

$$\|f\|_{H(\mathbb{D})} = \|f|_{\partial\mathbb{D}}\|_{L^2(\mathbb{S}^1)},$$

where we use the normalized arc length measure to define $L^2(\mathbb{S}^1)$. In this way $H(\mathbb{D})$ has the functions $e_\ell\colon \mathbb{D} \ni z \mapsto z^\ell$ for $\ell \in \mathbb{N}_0$ as an orthonormal basis and can be identified with a subspace of $L^2(\mathbb{S}^1)$. In fact for a holomorphic function defined by

$$f(z) = \sum_{k=0}^\infty a_k z^k \tag{8.20}$$

we have

$$\|f\|_{H(\mathbb{D})}^2 = \sum_{k=0}^\infty |a_k|^2. \tag{8.21}$$

Proof. We first prove the last claim concerning $f \in \mathrm{Hol}(\mathbb{D})$ in (8.20)–(8.21). By definition of the Hardy norm, we have

$$\int_0^1 |f(r\mathrm{e}^{2\pi\mathrm{i}\theta})|^2\,\mathrm{d}\theta = \int_0^1 \left|\sum_{\ell=0}^\infty a_\ell r^\ell \mathrm{e}^{2\pi\mathrm{i}\ell\theta}\right|^2\,\mathrm{d}\theta = \sum_{k=0}^\infty |a_k|^2 r^{2k}$$

by orthogonality of $\mathrm{e}^{2\pi\mathrm{i}\ell\theta}$ for $\ell \in \mathbb{N}_0$. It follows that the integral appearing in (8.19) increases monotonically as $r \nearrow 1$ to $\sum_{k=0}^{\infty} |a_k|^2$, proving (8.21).

For $f \in H(\mathbb{D})$ the above shows for the coefficients of f that

$$\sum_{\ell=0}^{\infty} |a_\ell|^2 < \infty,$$

which implies that

$$\mathbb{S}^1 \ni z \longmapsto \sum_{\ell=0}^{\infty} a_\ell r^\ell z^\ell$$

converges for $r \nearrow 1$ in $L^2(\mathbb{S}^1)$ to the function

$$\mathbb{S}^1 \ni z \longmapsto \sum_{\ell=0}^{\infty} a_\ell z^\ell.$$

The remaining claims now follow from this. □

Exercise 8.26. Show that for any $f \in H(\mathbb{D})$ the values $f(z)$ for $z \in \mathbb{D}$ are determined by the extension of f to $\mathbb{S}^1 \subseteq \overline{\mathbb{D}}$ by the Cauchy integral formula

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\mathbb{S}^1} \frac{f(w)}{w - z}\,\mathrm{d}w.$$

Definition 8.27 (Mock discrete series representations). We define the mock discrete series representation $\delta^{1,+}$ on $H(\mathbb{D})$ for

$$g = \begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_{1,1}(\mathbb{R})$$

and $f \in H(\mathbb{D})$ by

$$\delta^{1,+}(g)(f)(z) = (-\beta z + \alpha)^{-1} f\left(\frac{\overline{\alpha} z - \beta}{-\beta z + \alpha}\right). \tag{8.22}$$

Lemma 8.28 (Holomorphic mock discrete series representation). *The mock discrete series representation $\delta^{1,+}$ is a unitary representation on the Hardy space $H(\mathbb{D})$.*

Proof. From (8.9) we already know that the Möbius transformation

$$\mathbb{C} \ni z \longmapsto g{\boldsymbol{\cdot}}z$$

for $g \in \mathrm{SU}_{1,1}(\mathbb{R})$ maps $\mathbb{S}^1$ into $\mathbb{S}^1$. The complex derivative of

$$z \longmapsto g^{-1}{\boldsymbol{\cdot}}z \longmapsto \frac{\overline{\alpha} z - \overline{\beta}}{-\beta z + \alpha}$$

is given by
$$\frac{\overline{\alpha}(-\beta z+\alpha)+(\overline{\alpha}z-\overline{\beta})\beta}{(-\beta z+\alpha)^2}=\frac{1}{(-\beta z+\alpha)^2}.$$
From this it is straightforward to verify that
$$\frac{\mathrm{d}g_* m}{\mathrm{d}m}(z)=|-\beta z+\alpha|^{-2}$$
for all $g\in\mathrm{SU}_{1,1}(\mathbb{R})$ and $z\in\mathbb{S}^1$. We set $c(g,z)=(-\beta z+\alpha)^{-1}$ and deduce that the twisted normalized representations in Proposition 1.5 agrees with the formula in (8.22) and so defines a unitary representation on $L^2(\mathbb{S}^1)$.

For one of the basis vectors $e_\ell\in H(\mathbb{D})$ with $\ell\in\mathbb{N}_0$ and $g\in\mathrm{SU}_{1,1}(\mathbb{R})$, we also have that $\delta_g^{1,+}e_\ell$ is given by
$$\begin{aligned}\delta_g^{1,+}e_\ell(z)&=(-\beta z+\alpha)^{-1}\left(\frac{\overline{\alpha}z-\overline{\beta}}{-\beta z+\alpha}\right)^\ell\\&=\alpha^{-\ell-1}\left(1-\frac{\beta}{\alpha}z\right)^{-(\ell+1)}\left(\overline{\alpha}z-\overline{\beta}\right)^\ell\\&=\alpha^{-\ell-1}\left(\sum_{j=0}^\infty\left(\frac{\beta}{\alpha}z\right)^j\right)^{\ell+1}\left(\overline{\alpha}z-\overline{\beta}\right)^\ell.\end{aligned}$$
Recalling that $|\alpha|^2-|\beta|^2=1$ we see that $|\beta|<|\alpha|$, and that this defines a holomorphic function on an open set O_g containing $\overline{\mathbb{D}}$. In particular, we see that $\delta_g^{1,+}e_\ell\in H(\mathbb{D})$. As this holds for all $g\in\mathrm{SU}_{1,1}(\mathbb{R})$ and $\ell\in\mathbb{N}_0$, it follows that $H(\mathbb{D})$ is invariant under the normalized unitary representation. This gives the lemma. □

Definition 8.29 (Anti-holomorphic mock discrete series representation). The *anti-holomorphic mock discrete series representation* $\delta^{1,-}$ is defined to be the contragredient representation to the holomorphic mock discrete series representation.

Theorem 8.30 (Mock discrete series representation). *The holomorphic and anti-holomorphic mock discrete series representations are irreducible tempered representations but not discrete series representations. The holomorphic mock discrete series representation $\delta^{1,+}$ has K-eigenfunctions with weights in $1+2\mathbb{N}_0$, and only those, each with multiplicity one. Similarly, the anti-holomorphic mock discrete series representation $\delta^{1,-}$ has K-eigenfunctions with weights $-1-2\mathbb{N}_0$, and only these, each with multiplicity one.*

We will again say that 1 is the *lowest K-weight* of $\delta^{1,+}$, -1 is the *highest K-weight* of $\delta^{1,-}$, and that ±1 are the *terminal K-weights* of $\delta^{1,\pm}$.

The last part of Theorem 8.30 follows immediately from studying the orthonormal basis of $H(\mathbb{D})$ comprising e_ℓ for $\ell \in \mathbb{N}_0$. Indeed,

$$\delta^{1,+}_{k_\theta} e_\ell(z) = (-0z + \mathrm{e}^{-\mathrm{i}\theta})^{-1} \left(\frac{\mathrm{e}^{\mathrm{i}\theta} z + 0}{0 + \mathrm{e}^{-\mathrm{i}\theta}} \right)^\ell = \mathrm{e}^{\mathrm{i}(1+2\ell)\theta} e_\ell(z)$$

for $\theta \in [0, 2\pi)$ shows that e_ℓ is a K-eigenfunction with weight $1 + 2\ell$. For the contragredient representation, this also implies that $e'_\ell \in H(\mathbb{D})'$ is a K-eigenfunction with weight $-1 - 2\ell$.

Proof of irreducibility. We again use the differential operator associated to

$$\mathbf{d} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the basis vectors e_ℓ for $\ell \in \mathbb{N}_0$. Indeed, if we suppose $\mathcal{V} \subseteq H(\mathbb{D})$ is a non-trivial closed invariant subspace. As $K < \mathrm{SU}_{1,1}(\mathbb{R})$ is a compact abelian subgroup, $\mathcal{V}$ must contain a K-eigenfunction for some weight $k \in \mathbb{Z}$. As the basis vectors are K-eigenfunctions of mutually different weights, it follows that $e_\ell \in \mathcal{V}$ for some $\ell \in \mathbb{N}_0$. We recall that

$$\delta^{1,+}_\partial(\mathbf{d})(e_\ell) = \lim_{t \to 0} \frac{1}{t} \left(\delta^{1,+}\left(\exp(t\mathbf{d})\right)(e_\ell) - e_\ell \right)$$

and note that

$$\begin{aligned} \delta^{1,+}\left(\exp(t\mathbf{d})\right) e_\ell(z) &= (-\sinh t z + \cosh t)^{-1} \left(\frac{\cosh t z - \sinh t}{-\sinh t z + \cosh t} \right)^\ell \\ &= (\cosh t z - \sinh t)^\ell \, (-\sinh t z + \cosh t)^{-(\ell+1)} \end{aligned}$$

has the partial derivative with respect to t at $t = 0$ and $z \in \mathbb{C}$ given by

$$-\ell z^{\ell-1} \cdot 1 + z^\ell \cdot (-(\ell+1)) \cdot (-z) = \left(-\ell e_{\ell-1} + (\ell+1) e_{\ell+1}\right)(z).$$

In fact the underlying convergence is uniform on (for example) the ball $B_2^{\mathbb{C}}$, and it follows that the convergence also takes place in $H(\mathbb{D})$. Since $\mathcal{V}$ is closed, it follows that

$$\delta^{1,+}_\partial(\mathbf{d})(e_\ell) = -\ell e_{\ell-1} + (\ell+1) e_{\ell+1} \in \mathcal{V}.$$

Since $e_{\ell-1}$ and $e_{\ell+1}$ have different weights, and $\mathcal{V}$ is T-invariant, we see that $e_{\ell+1} \in \mathcal{V}$ and if $\ell > 1$ we also have $e_{\ell-1} \in \mathcal{V}$. Therefore $\mathcal{V}$ contains all the basis vectors, so $\mathcal{V} = H(\mathbb{D})$, and the mock discrete series representation is irreducible. This implies the same statement for its contragredient representation. □

Proof of temperedness. As the matrix coefficients of the contragredient representation are the conjugates of the matrix coefficients of the original representation, it suffices to study the mock discrete series representation $\delta^{1,+}$. We again calculate the matrix coefficient φ_{e_0} first along the one-parameter subgroup corresponding to $\mathbf{d}$. This gives

$$\varphi_{e_0}(a_t) = \left\langle \delta^{1,+}_{a_t} e_0, e_0 \right\rangle = \int_0^1 \left(-\sinh t \mathrm{e}^{2\pi\mathrm{i}\theta} + \cosh t \right)^{-1} \cdot 1 \cdot 1 \,\mathrm{d}\theta = (\cosh t)^{-1}$$

for all $t \in \mathbb{R}$, where we once more expanded the expression

$$\left(-\sinh t \mathrm{e}^{2\pi\mathrm{i}\theta} + \cosh t \right)^{-1}$$

into a geometric series to calculate the integrals. Using the formula (8.18) for the Haar measure in the KAK coordinates, and the fact that e_0 is a K-eigenfunction for K with weight 1, we obtain

$$\begin{aligned}\int_G |\varphi_{e_0}(g)|^{2+\varepsilon} \,\mathrm{d}m(g) &= \frac{2}{\pi} \int_0^{2\pi} \int_0^\infty \int_0^\pi \underbrace{\left| \varphi_{e_0}\left(k_\theta a_t k_\psi \right) \right|}_{=(\cosh t)^{-1}}{}^{2+\varepsilon} \sinh 2t \,\mathrm{d}\theta \,\mathrm{d}t \,\mathrm{d}\psi \\ &= 4\pi \int_0^\infty (\cosh t)^{-(2+\varepsilon)} \sinh 2t \,\mathrm{d}t \begin{cases} = \infty & \text{if } \varepsilon = 0; \\ < \infty & \text{if } \varepsilon > 0. \end{cases}\end{aligned}$$

Using the case $\varepsilon = 0$, it follows from Theorem 8.2 that the mock discrete series representation is not a discrete series representation. Using $\varepsilon > 0$, we can use Theorem 8.5 to see that the mock discrete series representation is tempered (that is, weakly contained in the regular representation). □

8.5 Effective Decay for the Regular Representation of $\mathbf{SL_2}(\mathbb{R})$

The goal in this section is to prove the following effective decay property for the regular representation of $\mathrm{SL}_2(\mathbb{R})$. This material comes from work of Harish-Chandra [32]. We again use the terminology introduced in Section 7.2.2.

Theorem 8.31 (Effective decay for the regular representation). *Let $f_1, f_2 \in L^2(\mathrm{SL}_2(\mathbb{R}))$ be K-eigenfunctions for the regular representation. Then*

$$\left| \left\langle \lambda_g f_1, f_2 \right\rangle \right| \leqslant \Xi(g) \|f_1\| \|f_2\|,$$

where $\Xi\colon \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ is the Harish-Chandra spherical function, which satisfies

$$\Xi(g) \ll |g|^{-1} \log |g| \ll_\varepsilon |g|^{-1+\varepsilon}$$

for all $g \in \mathrm{SL}_2(\mathbb{R})$ *and all* $\varepsilon > 0$.

8.5.1 Subgroups and the Modular Character

Let us recall once more the notation for elements and subgroups of $\mathrm{SL}_2(\mathbb{R})$ from Section 8.3. We write

$$K = \mathrm{SO}_2(\mathbb{R}) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

for the usual maximal compact subgroup, and

$$A = \left\{ a_t = \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\}$$

for the connected diagonal subgroup. Note that the centre $M = \{\pm I\}$ of $\mathrm{SL}_2(\mathbb{R})$ is also the maximal compact subgroup of the centralizer AM of A, and that AM consists of all diagonal matrices (with positive and negative eigenvalues). We also define

$$U = \left\{ u_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

to be the upper unipotent matrix. Finally, we define the connected upper Borel subgroup

$$B = \left\{ \begin{pmatrix} \mathrm{e}^t & x \\ & \mathrm{e}^{-t} \end{pmatrix} \mid t \in \mathbb{R}, x \in \mathbb{R} \right\} = AU < \mathrm{SL}_2(\mathbb{R}).$$

We will often denote elements of the full upper triangular group BM by $u_x a_t m$ or $a_t m u_x$, with the implicit assumptions $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $m \in M$.

We now give a concrete description of the Haar measure on B and the modular character Δ_B. Notice that every element $b \in B = AU$ can be written uniquely as a product $b = u_x a_t$ of $u_x \in U$ and $a_t \in A$. We recall from (8.10) that in this coordinate system (that is, order of writing A and U) the Haar measure is given by

$$\mathrm{d}m_B = 2\mathrm{e}^{-2t}\,\mathrm{d}x\,\mathrm{d}t,$$

which we obtained by identifying $b \in B$ with $b \cdot \mathrm{i} \in \mathbb{H}$. More precisely, we have

$$\int_B f\,\mathrm{d}m_B = 2\int_{\mathbb{R}^2} f\big(u_x a_t\big)\mathrm{e}^{-2t}\,\mathrm{d}x\,\mathrm{d}t$$

for any integrable function f on B.

To calculate Δ_B, we first let $b_0 = u_{x_0} \in B$ and let $f \geqslant 0$ be any integrable function on B. Then we get for the right translated function that

$$\begin{aligned}\int_B f(bb_0^{-1})\,\mathrm{d}m_B(b) &= 2\int_{\mathbb{R}^2} f\bigl(u_x a_t u_{-x_0}\bigr)\mathrm{e}^{-2t}\,\mathrm{d}x\,\mathrm{d}t\\ &= 2\int_{\mathbb{R}^2} f\bigl(u_{x'}a_t\bigr)\mathrm{e}^{-2t}\,\mathrm{d}x'\,\mathrm{d}t\end{aligned}$$

by using the calculation

$$a_t u_{-x_0} = \begin{pmatrix}\mathrm{e}^t & \\ & \mathrm{e}^{-t}\end{pmatrix}\begin{pmatrix}1 & -x_0\\ & 1\end{pmatrix} = \begin{pmatrix}\mathrm{e}^t & -\mathrm{e}^t x_0\\ & \mathrm{e}^{-t}\end{pmatrix} = u_{-\mathrm{e}^{2t}x_0}a_t$$

and the substitution $x' = x - \mathrm{e}^{2t}x_0$ for the Lebesgue measure on $\mathbb{R}$. Therefore $\Delta_B(u_x) = 1$.

Suppose now that $b_0 = a_{t_0} \in B$ and that $f \geqslant 0$ is again an integrable function on B. Then we may use the substitution $t' = t - t_0$ to get

$$\begin{aligned}\int_B f(bb_0^{-1})\,\mathrm{d}m_B(b) &= 2\int_{\mathbb{R}^2} f(u_x a_t a_{-t_0})\mathrm{e}^{-2t}\,\mathrm{d}x\,\mathrm{d}t\\ &= 2\int_{\mathbb{R}^2} f(u_x a_{t'})\mathrm{e}^{-2t'-2t_0}\,\mathrm{d}x\,\mathrm{d}t'\\ &= \mathrm{e}^{-2t_0}\int_B f(b)\,\mathrm{d}m_B(b).\end{aligned}$$

Therefore the modular character Δ_B of B (which is characterized by this formula; see Lemma 1.12) is given by

$$\Delta_B\bigl(a_{t_0}u_{x_0}\bigr) = \Delta_B(a_{t_0}) = \mathrm{e}^{-2t_0} \tag{8.23}$$

for all $b_0 = a_{t_0}u_{x_0} \in B$.

For the right Haar measure $m_B^{(\mathrm{r})}$ we also have

$$\int_B f(b_0 b)\,\mathrm{d}m_B^{(\mathrm{r})}(b) = \Delta_B(b_0)\int_B f\,\mathrm{d}m_B^{(\mathrm{r})} = \mathrm{e}^{-2t_0}\int_B f\,\mathrm{d}m_B^{(\mathrm{r})}. \tag{8.24}$$

Indeed, applying Lemma 1.13 and the above, we obtain

$$\begin{aligned}\int_B f(b_0 b)\,\mathrm{d}m_B^{(\mathrm{r})}(b) &= \int_B f(b_0 b^{-1})\,\mathrm{d}m_B(b) = \int_B f\bigl((bb_0^{-1})^{-1}\bigr)\,\mathrm{d}m_B(b)\\ &= \Delta_B(b_0)\int_B f(b^{-1})\,\mathrm{d}m_B(b)\\ &= \mathrm{e}^{-2t_0}\int_B f\,\mathrm{d}m_B^{(\mathrm{r})}.\end{aligned}$$

We recall that we normalize the Haar measure m_K on K so that $m_K(K) = 1$.

8.5.2 A Particular Induced Representation

In order to define the Harish-Chandra spherical function we will need the notion of induction for representations. Given any abstract representation π_B on a (complex) vector space V_B of a subgroup B of a group G, we can define the *induced representation* to be the left-regular representation of G on the vector space

$$\left\{f\colon G \longrightarrow V_B \mid f(gb) = \pi_B(b)^{-1} f(g) \text{ for all } g \in G, b \in B\right\}.$$

The following exercise should clarify why the induced representation has the given form.

Exercise 8.32. Let G be a discrete group and $B < G$ be a subgroup. Given a representation π_B on a vector space V_B, show that the induced representation contains a B-invariant subspace $\mathcal{W}$ so that the restriction of the induced representation to B and this B-invariant subspace is isomorphic to the original representation π_B on V_B. Show also that if $B \lhd G$ is normal, then the above induced representation is a direct product of subspaces that are invariant under the restriction of the induced representation to B, one of them coincides with $\mathcal{W}$, and the subspaces are permuted under the induced representation.

We will be particularly interested in the case where π_B is the trivial representation, in which case the representation space above is simply the space

$$\left\{f\colon G/B \longrightarrow \mathbb{C}\right\}.$$

We hope that the short excursion above to the abstract setting of group representation theory will help the reader to digest the following definition concerning unitary representations (which is slightly more involved). In the context of unitary represenation it is of course natural to require in addition some square-integrability conditions. Thus, for example, if B is the upper Borel subgroup in $\mathrm{SL}_2(\mathbb{R})$ we recall that $\mathrm{SL}_2(\mathbb{R})/B \cong K$, and may consider the space $L^2(K)$. However, this definition together with the left-regular representation has a fundamental flaw: There is no $\mathrm{SL}_2(\mathbb{R})$-invariant measure on $\mathrm{SL}_2(\mathbb{R})/B \cong K$. To rectify this and obtain a unitary representation, we make the following definition.†

Definition 8.33 (Unitary induction for $B < \mathrm{SL}_2(\mathbb{R})$). Let B be the connected upper Borel subgroup of $G = \mathrm{SL}_2(\mathbb{R})$ as above. Let π_B be a unitary representation of $B < G = \mathrm{SL}_2(\mathbb{R})$ on some Hilbert space $\mathcal{H}_B$. The *unitary (*or *normalized) induced representation*

$$(\mathcal{H}_G, \pi_G) = \mathrm{Ind}_B^G(\mathcal{H}_B, \pi_B)$$

† Alternatively, we could also calculate the Radon–Nikodym derivative and use Proposition 1.5. The approach taken here has the advantage of revealing the connection to the regular representation and leads to the same representation.

is defined to be the left-regular representation on the space of those functions $f\colon G \to \mathcal{H}_B$ with the following properties:

(1) f is measurable,
(2) $f(gb) = \Delta_B(b)^{\frac{1}{2}}\pi_B(b)^{-1}f(g)$ for all $g \in G$ and $b \in B$, and
(3) $\|f|_K\|_{L^2(K)} < \infty$,

where $\Delta_B(a_t u_x) = \Delta_B(a_t)$ is the modular character of B, and

$$\|f|_K\|_{L^2(K)}^2 = \int_K \|f(k)\|_{\mathcal{H}_B}^2 \,\mathrm{d}m_K(k)$$

defines the norm on the Hilbert space $\mathcal{H}_G$. As usual, we identify any function of semi-norm zero with the zero function.

Instead of concentrating on the technical details of the representation constructed above in general, we will for now only discuss the following crucial special case.

We let $\mathbb{1}_B$ denote the trivial representation of B on $\mathbb{C}$, and define

$$(\mathcal{H}_0, \pi^0) = \mathrm{Ind}_B^G(\mathbb{C}, \mathbb{1}_B)$$

by setting $\mathcal{H}_0$ to be the space of functions $f\colon G \to \mathbb{C}$ with the properties

(1) f is measurable;
(2) $f\big(gb_0\big) = \Delta_B(b_0)^{\frac{1}{2}} f(g)$ for all $g \in G, b_0 \in B$; and
(3) $\|f\|_{\mathcal{H}_0} = \|f|_K\|_{L^2(K)} < \infty$.

We also write π^0 for the left-regular representation on $\mathcal{H}_0$, that is

$$\pi^0(g)(f)(x) = f(g^{-1}x)$$

for all $g, x \in G$, and write $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ for the inner product on $\mathcal{H}_0$ compatible with the norm $\|\cdot\|_{\mathcal{H}_0}$, as in (3).

We will see in Corollary 8.36 that $f \in \mathcal{H}_0$ and $g \in G$ implies $\pi^0(g)(f) \in \mathcal{H}_0$, and that π^0 is indeed a unitary representation. Using only the definition, we now show that

$$\mathcal{H}_0 \ni f \longmapsto f|_K \in L^2(K) \tag{8.25}$$

is an isomorphism of Hilbert spaces. To see that the map is onto, suppose that f_K is a square-integrable function on K and define $f\colon G \to \mathbb{C}$ by

$$f(kb) = f_K(k)\Delta_B(b)^{\frac{1}{2}}$$

for $kb \in G = KB$. We only have to verify (2), so let $g = kb \in G$ and $b_0 \in B$. Then

$$f(gb_0) = f(kbb_0) = f_K(k)\Delta_B(bb_0)^{\frac{1}{2}} = \Delta_B(b_0)^{\frac{1}{2}} f(g)$$

as required. Similarly, any $f \in \mathcal{H}_0$ is uniquely determined by $f|_K$, because of $G = KB$ and property (2). Finally, the map in (8.25) is a well-defined isometry by the definition of $\|\cdot\|_{\mathcal{H}_0}$ in (3).

We also note that two functions $f_1, f_2 \in \mathcal{H}_0$ satisfy $\|f_1 - f_2\|_{\mathcal{H}_0} = 0$ if and only if f_1 and f_2 agree m_G-almost everywhere on G. For this, recall that we may define the Haar measure m_G on $G = KB$ as the product measure of m_K and $m_B^{(\mathrm{r})}$ (see Lemma 8.13). Suppose now that $f_1, f_2 \in \mathcal{H}_0$. Then

$$\|f_1 - f_2\|_{\mathcal{H}_0} = 0$$

if and only if $f_1(k) = f_2(k)$ for m_K-almost every $k \in K$, which by (2) is equivalent to $f_1(kb) = f_2(kb)$ for m_K-almost every $k \in K$ and all $b \in B$, and hence also to $f_1(g) = f_2(g)$ for m_G-almost every $g \in G$.

We note that our implicit requirement in the discussion above was that f as in (1)–(3) be defined on all of G. As is common with other function spaces, it will be convenient to loosen this at times by permitting f to not be defined (or to be equal to ∞) on a right B-invariant null set $S = SB \subseteq G$. Setting

$$\widetilde{f}(g) = \begin{cases} f(g) & \text{for } g \in G \smallsetminus S, \\ 0 & \text{for } g \in S \end{cases}$$

then defines an equivalent function $\widetilde{f} \in \mathcal{H}_0$ (since $S = SB$ is a null set in G if and only if $S \cap K$ is a null set in K).

Exercise 8.34. (a) Show directly that $\pi^0(g)\mathcal{H}_0 = \mathcal{H}_0$, and that $\pi^0(g)$ is unitary.
(b) Show that $\pi^0_g(f)$ depends continuously on $g \in G$ for any fixed $f \in \mathcal{H}_0$. In other words, show that $(\mathcal{H}_0, \pi^0)$ is a unitary representation of G.

8.5.3 The Hertz Domination Principle

The link between the regular representation and the representation π^0 introduced above is revealed in the next fundamental result.

Proposition 8.35 (Hertz domination principle). *Let $G = \mathrm{SL}_2(\mathbb{R})$. To any $f \in L^2(G)$ we can associate an element $\breve{f} \in \mathcal{H}_0$ defined by*

$$\breve{f}(g) = \left(\int_B |f(gb)|^2 \, \mathrm{d}m_B^{(\mathrm{r})}(b) \right)^{\frac{1}{2}}$$

for all $g \in G$. Then

(1) $\breve{f} \in \mathcal{H}_0$,
(2) $\breve{cf} = |c| \breve{f}$ *for* $c \in \mathbb{C}$,
(3) $\|\breve{f}\|_{\mathcal{H}_0} = \|f\|_2$,

(4) $\overbrace{\lambda_g f} = \pi^0_g \breve{f}$ *for all* $g \in G$, *and finally*
(5) $|\langle \lambda_g f_1, f_2\rangle| \leqslant \langle \pi^0_g \breve{f_1}, \breve{f_2}\rangle$ *for all* $g \in G$.

Proof. Let $f \in L^2(G)$, define $\breve{f}\colon G \to [0,\infty]$ as in the proposition, and let $g \in G$ and $b_0 \in B$. Using the definition of $\breve{f}$ and $\Delta_B\colon B \to (0,\infty)$ we get from (8.24) that

$$\begin{aligned}\breve{f}(gb_0) &= \left(\int_B |f(gb_0b)|^2 \,\mathrm{d}m_B^{(\mathrm{r})}(b)\right)^{\frac12}\\ &= \left(\Delta_B(b_0)\int_B |f(gb)|^2 \,\mathrm{d}m_B^{(\mathrm{r})}(b)\right)^{\frac12} = \Delta_B(b_0)^{\frac12}\,\breve{f}(g)\end{aligned}$$

whenever at least one side is defined. Notice that $\breve{cf} = |c|\,\breve{f}$ for $c \in \mathbb{C}$ follows from the definition. Moreover,

$$\begin{aligned}\|\breve{f}\|^2_{\mathcal{H}_0} &= \|\breve{f}|_K\|^2_{L^2(K)}\\ &= \int_K\int_B |f(kb)|^2 \mathrm{d}m_B^{(\mathrm{r})}(b)\,\mathrm{d}m_K(k) = \int_G |f(g)|^2 \mathrm{d}m(g) = \|f\|_2^2 < \infty,\end{aligned}$$

since the Haar measure $\mathrm{d}m_G$ decomposes as $\mathrm{d}m_K\,\mathrm{d}m_B^{(\mathrm{r})}$ in the coordinates of the Iwasawa decomposition. In particular, we have $\breve{f}(k) < \infty$ for m_K-almost every $k \in K$, and $N = \{g \in G \mid \breve{f}(g) = \infty\}$ is a right B-invariant null set in G. Formally, we may modify $\breve{f}$ and set it equal to 0 on N, but instead we will simply ignore this null set. In that sense, $\breve{f} \in \mathcal{H}_0$ and we obtain the statements (1), (2), and (3).

Now fix some $g_0 \in G$. Just using the definitions, we see that

$$\begin{aligned}\overbrace{\lambda_{g_0} f}(g) &= \left(\int_B |(\lambda_{g_0} f)(gb)|^2 \,\mathrm{d}m_B^{(\mathrm{r})}(b)\right)^{\frac12}\\ &= \left(\int_B |f(g_0^{-1}gb)|^2 \,\mathrm{d}m_B^{(\mathrm{r})}(b)\right)^{\frac12}\\ &= \breve{f}(g_0^{-1}g) = (\pi^0_{g_0}\breve{f})(g)\end{aligned}\tag{8.26}$$

for all $g \in G$, which gives claim (4) in the proposition.

As we now show, the last statement of the proposition is a simple corollary of the decomposition of the Haar measure on G already used above and the Cauchy–Schwarz inequality applied in the right way. Indeed, let f_1, f_2 be functions in $L^2(G)$. Then

$$\begin{aligned}
\left|\langle\lambda_g(f_1),f_2\rangle\right| &= \left|\int_K\int_B f_1(g^{-1}kb)f_2(kb)\,\mathrm{d}m_B^{(\mathrm{r})}(b)\,\mathrm{d}m_K(k)\right| \\
&\leqslant \int_K\int_B |f_1(g^{-1}kb)f_2(kb)|\,\mathrm{d}m_B^{(\mathrm{r})}(b)\,\mathrm{d}m_K(k) \\
&\leqslant \int_K \left(\int_B |f_1(g^{-1}kb)|^2\,\mathrm{d}m_B^{(\mathrm{r})}(b)\right)^{\frac12} \\
&\qquad\qquad \left(\int_B |f_2(kb)|^2\,\mathrm{d}m_B^{(\mathrm{r})}(b)\right)^{\frac12}\,\mathrm{d}m_K(k) \\
&= \int_K \left(\pi_g^0\,\breve{f_1}\right)(k)\,\breve{f_2}(k)\,\mathrm{d}m_K(k) \\
&= \langle \pi_g^0\,\breve{f_1},\breve{f_2}\rangle_{\mathcal{H}_0}
\end{aligned}$$

as claimed. □

With the link between λ and π^0 provided by Proposition 8.35, it is now straightforward to check that π^0 is a unitary representation.

Corollary 8.36 (Unitarity of π^0). *π^0 defines a unitary representation of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}_0$.*

Proof. Let $f_0\in\mathcal{H}_0$. For $g_0,g\in G$ and $b\in B$ we then have

$$\pi_{g_0}^0(f_0)(gb)=f_0(g_0^{-1}gb)=\Delta(b)^{\frac12}f_0(g_0^{-1}g)=\Delta(b)^{\frac12}\pi_{g_0}^0(f_0)(g),$$

as is required for elements of $\mathcal{H}_0$. To see that $\pi_{g_0}^0(f_0)\in\mathcal{H}_0$, we need to show that it is square-integrable on K. In fact we will show that

$$\|\pi_{g_0}^0(f_0)|_K\|_{L^2(K)}=\|\pi_{g_0}^0f_0\|_{\mathcal{H}_0}=\|f_0\|_{\mathcal{H}_0}.$$

For this, we fix some measurable $S\subseteq B$ with $m_B^{(\mathrm{r})}(S)=1$. We now define a measurable function $f\colon G\to\mathbb{C}$ by

$$f(kb)=|f_0(k)|\mathbb{1}_S(b).$$

Since $f_0|_K\in L^2(K)$ by definition of $\mathcal{H}_0$, we see that $f\in L^2(G)$ (by the product structure of the Haar measure). Applying the definition in Proposition 8.35 we also get

$$\breve{f}(g)=|f_0(g)| \tag{8.27}$$

first for all $g=k\in K$, but since both $\breve{f}$ and f_0 belong to $\mathcal{H}_0$ the defining property of $\mathcal{H}_0$ extends this equality to all $g\in G$. In particular, we obtain $\|f_0\|_{\mathcal{H}_0}=\|f\|_2$. Now fix some $g\in G$ and use (8.27), Proposition 8.35(4), (3), unitarity of the regular representation λ on $L^2(G)$, and (8.27) again to obtain

$$\|\pi^0_g f_0\|_{\mathcal{H}_0} = \|\pi^0_g \breve{f}\,\|_{\mathcal{H}_0} = \|\,\overbrace{\lambda_g f}\,\|_{\mathcal{H}_0} = \|\lambda_g f\|_2$$
$$= \|f\|_2 = \|\,\breve{f}\,\|_{\mathcal{H}_0} = \|f_0\|_{\mathcal{H}_0}.$$

This shows that $\pi^0_g f_0 \in \mathcal{H}_0$, and that π^0_g is unitary.

To see that π^0 defines a unitary representation, we have to check the continuity requirement. By Lemma 1.9, it suffices to prove this for a dense set of vectors in $\mathcal{H}_0$ and at the identity. Now recall that $C(K)$ is dense in $L^2(K)$, and that every $f_K \in C(K)$ extends uniquely to an element $f \in \mathcal{H}_0 \cap C(G)$. However, in that case the continuity statement

$$\|\pi^0_g f - f\|^2_{\mathcal{H}_0} = \int_K |f(g^{-1}k) - f(k)|^2 \,\mathrm{d}m_K(k) \longrightarrow 0$$

as $g \to I$ follows from dominated convergence. □

We note that π^0 is not irreducible, since $M = \{\pm I\}$ does not act as a scalar. In fact, splitting $\mathcal{H}_0$ into even and odd components for M defines two unitary representations that will be studied more carefully in the next chapter. As we will see in Chapter 9, the even subspace is irreducible and the odd subspace is a sum of two irreducible subspaces.

Exercise 8.37 (Hertz domination for $L^2(\mathbb{R}^2)$). For $f \in L^2(\mathbb{R}^2)$ define

$$\breve{f}(g) = \Big(|f(rge_1)|^2 r \,\mathrm{d}r\Big)^{\frac{1}{2}}$$

for $g \in \mathrm{SL}_2(\mathbb{R})$. Consider the Koopman representation $\pi^{\mathbb{R}^2}$ of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{R}^2)$, and show properties (1)–(5) of Proposition 8.35 (replacing the regular representation λ by $\pi^{\mathbb{R}^2}$).

8.5.4 The Harish-Chandra Spherical Function

Definition 8.38 (Harish-Chandra spherical function for $\mathrm{SL}_2(\mathbb{R})$). After extending the constant function $\mathbb{1}_K$ in $L^2(K)$ to an element f_0 in $\mathcal{H}_0$ with $f_0|_K \equiv \mathbb{1}_K$, we define the Harish-Chandra spherical function (for $\mathrm{SL}_2(\mathbb{R})$) by

$$\varXi(g) = \left\langle \pi^0_g f_0, f_0 \right\rangle_{\mathcal{H}_0}$$

for $g \in \mathrm{SL}_2(\mathbb{R})$.

Proposition 8.39 (Estimate for HC-spherical function for $\mathrm{SL}_2(\mathbb{R})$). *The Harish-Chandra spherical function is continuous and bi-K-invariant, meaning that*

$$\varXi(kgk') = \varXi(g)$$

for all $g \in G = \mathrm{SL}_2(\mathbb{R})$ and $k, k' \in K$. Moreover,

$$\Xi(g) \ll |g|^{-1} \log |g| \ll_{\varepsilon} |g|^{-1+\varepsilon} \tag{8.28}$$

for all $g \in G$*, and* $\Xi \in L^{2+\varepsilon}(G)$ *for all* $\varepsilon > 0$.

Proof. First note that

$$f_0\big(k a_t u_x\big) = \Delta_B(a_t)^{\frac{1}{2}} = \mathrm{e}^{-t}$$

defines a continuous function f_0 on $k a_t u_x \in G$. For $g \in G$ and $k, k' \in K$ we have

$$\Xi(kgk') = \left\langle \pi^0_g \pi^0_{k'} f_0, \pi^0_{k^{-1}} f_0 \right\rangle_{\mathcal{H}_0} = \Xi(g)$$

since $\pi^0_{k^{-1}} f_0 = \pi^0_{k'} f_0 = f_0$.

It follows that it is enough to consider

$$g = a_t = \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix}$$

with $t \geqslant 0$ in the proof of the estimate (8.28). We now calculate

$$\begin{aligned} \Xi(a_t) &= \left\langle \pi^0_{a_t} f_0, f_0 \right\rangle_{\mathcal{H}_0} \\ &= \int_K f_0(a_t^{-1} k) \underbrace{f_0(k)}_{=1} \,\mathrm{d}m_K(k) = \frac{1}{2\pi} \int_0^{2\pi} f_0(a_t^{-1} k_\theta) \,\mathrm{d}\theta \end{aligned}$$

and arrive at the problem of writing $a_t^{-1} k_\theta$ in the form $k_\psi a_{t_0} u_{x_0}$. More precisely, we only need to calculate $t_0 = t_0(t, \theta)$, since we have

$$f_0(a_t^{-1} k_\theta) = f_0(k_\psi a_{t_0} u_{x_0}) = \Delta_B(a_{t_0})^{\frac{1}{2}} = \mathrm{e}^{-t_0}$$

by definition of $f_0 \in \mathcal{H}_0$. Notice that e^{t_0} is the length of the first column of $a_t^{-1} k_\theta$, since

$$a_t^{-1} k_\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = k_\psi a_{t_0} u_{x_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{e}^{t_0} k_\psi \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which gives

$$\mathrm{e}^{t_0} = \left\| \begin{pmatrix} \mathrm{e}^{-t} & \\ & \mathrm{e}^{t} \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\| = \sqrt{\mathrm{e}^{-2t} \cos^2\theta + \mathrm{e}^{2t} \sin^2\theta}.$$

Therefore

$$\Xi(a_t) = \frac{4}{2\pi} \int_0^{\frac{\pi}{2}} \left(\mathrm{e}^{-2t} \cos^2\theta + \mathrm{e}^{2t} \sin^2\theta \right)^{-\frac{1}{2}} \,\mathrm{d}\theta.$$

Next notice that

$$\mathrm{e}^{-2t}\cos^2\theta + \mathrm{e}^{2t}\sin^2\theta \asymp \max\left\{\mathrm{e}^{-2t}\cos^2\theta, \mathrm{e}^{2t}\sin^2\theta\right\}.$$

Moreover, the maximum is $\mathrm{e}^{2t}\sin^2\theta$ unless θ is close to 0, specifically unless $\tan^2\theta < \mathrm{e}^{-4t}$. This gives

$$\begin{aligned}\Xi(a_t) &\asymp \int_0^{\arctan \mathrm{e}^{-2t}} \underbrace{\left(\mathrm{e}^{-2t}\cos^2\theta\right)^{-\frac{1}{2}}}_{\asymp \mathrm{e}^t} \mathrm{d}\theta + \int_{\arctan \mathrm{e}^{-2t}}^{\frac{\pi}{2}} \underbrace{\left(\mathrm{e}^{2t}\sin^2\theta\right)^{-\frac{1}{2}}}_{\asymp \mathrm{e}^{-t}\frac{1}{\theta}} \mathrm{d}\theta \\ &\asymp \mathrm{e}^t \arctan \mathrm{e}^{-2t} + \mathrm{e}^{-t}|\log\arctan \mathrm{e}^{-2t}| \\ &\asymp \mathrm{e}^{-t} + \mathrm{e}^{-t}t \ll_\varepsilon \mathrm{e}^{-t(1-\varepsilon)}\end{aligned}$$

for all $\varepsilon > 0$. Also recall that $\|a_t\| \asymp \mathrm{e}^t$ for $t \geqslant 0$, which proves the estimate for Ξ in the proposition.

For the last claim, we need the description of the Haar measure on $\mathrm{SL}_2(\mathbb{R})$ in the coordinates of the Cartan decomposition from (8.12). Using the coordinates given by $g = k_\theta a_t k_\psi$ with $\theta \in [0, 2\pi)$, $t \in [0, \infty)$, and $\psi \in [0, \pi)$, we have

$$\mathrm{d}m_G \propto \sinh 2t \,\mathrm{d}\theta \,\mathrm{d}t \,\mathrm{d}\psi$$

by (8.12). Therefore,

$$\begin{aligned}\int_G (\Xi(g))^{2+\varepsilon}\, \mathrm{d}m_G(g) &\ll \int_0^\infty (\Xi(a_t))^{2+\varepsilon} \sinh 2t \,\mathrm{d}t \\ &\ll_\delta \int_0^\infty \left(\mathrm{e}^{-t(1-\delta)}\right)^{2+\varepsilon} \mathrm{e}^{2t}\,\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{t(-(2+\varepsilon)(1-\delta)+2)} < \infty\end{aligned}$$

if $\delta > 0$ is sufficiently small in comparison to ε. □

Proof of Theorem 8.31. Let $f_1, f_2 \in L^2(G)$ be K-eigenfunctions for the regular representation, and let $k_\theta \in K$. Then

$$\lambda_{k_\theta}(f_j) = \mathrm{e}^{2\pi \mathrm{i} n_j \theta} f_j,$$

and Proposition 8.35(4) imply that

$$\pi^0_{k_\theta}(\overbrace{f_j}) = \overbrace{\lambda_{k_\theta} f_j} = \overbrace{\mathrm{e}^{2\pi \mathrm{i} n_j\theta} f_j} = \overbrace{f_j},$$

or in other words, that $\overbrace{f_j}$ is invariant under K. We know that $\pi^0|_K$ is equal to the regular representation on $\mathcal{H}_0 \cong L^2(K)$. Therefore, there exists up to scalar multiples only one K-invariant function in $\mathcal{H}_0 \cong L^2(K)$. Since $\|f_0\|_{\mathcal{H}_0} = 1$ (by our choice that $m_K(K) = 1$), it follows that

$$\breve{f_j} = \|f_j\| f_0$$

for $j = 1, 2$. Therefore, Proposition 8.35(5) gives

$$|\left\langle \lambda_g f_1, f_2 \right\rangle| \leqslant \left\langle \pi_g^0 \|f_1\| f_0, \|f_2\| f_0 \right\rangle_{\mathcal{H}_0} = \varXi(g) \|f_1\| \|f_2\|.$$

The estimates for $\varXi(g)$ are given by Proposition 8.39. □

Exercise 8.40 (Decay of Koopman representation on $L^2(\mathbb{R}^2)$). Use Exercise 8.37 and the results above to prove the same decay estimates as in Theorem 8.31 for the Koopman representation of $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{R}^2)$.

8.6 Tempered Decay for $\mathrm{SL}_2(\mathbb{R})$

We continue with our study of the case $G = \mathrm{SL}_2(\mathbb{R})$, and present the companion result to Theorem 8.5, also due to Cowling, Haagerup and Howe [13], for this case. For a unitary representation π of a group K, we say that $v \in \mathcal{H}_\pi$ is K-finite if $\dim\langle \pi(K)v \rangle < \infty$. Below, we will again set

$$K = \mathrm{SO}_2(\mathbb{R}) < \mathrm{SL}_2(\mathbb{R}).$$

Theorem 8.41 (Tempered representations of $\mathrm{SL}_2(\mathbb{R})$). *For any unitary representation π of $\mathrm{SL}_2(\mathbb{R})$ the following are equivalent:*

(i) *π is tempered (that is, $\pi \prec \lambda$);*
(ii) *π is almost square integrable (that is, on a dense set of vectors the matrix coefficients belong to $L^{2+\varepsilon}$ for all $\varepsilon > 0$);*
(iii) *for any two K-finite vectors $v, w \in \mathcal{H}_\pi$ we have*

$$\left|\left\langle \pi_g v, w \right\rangle\right| \leqslant (\dim \langle \pi(K)v \rangle)^{\frac{1}{2}} (\dim \langle \pi(K)w \rangle)^{\frac{1}{2}} \|v\| \|w\| \varXi(g);$$

and
(iv) *for any two C^1-smooth vectors $v, w \in \mathcal{H}_\pi$ we have*

$$\left|\left\langle \pi_g v, w \right\rangle\right| \ll_\varepsilon \mathcal{S}(v)\, \mathcal{S}(w)\, |g|^{-1+\varepsilon},$$

where $\mathcal{S}(\cdot)$ denotes a degree-one Sobolev norm, and the implied constant depends on ε but not on the representation π.

Proof. The proof that (iii) implies (iv) follows from Proposition 7.25 and the estimate concerning the Harish-Chandra spherical function in Proposition 8.39.

Assume now that (iv) holds. Then the last step of the proof of Proposition 8.39 shows that the matrix coefficients of smooth vectors of $\mathcal{H}_\pi$ belong to $L^{2+\varepsilon}(G)$ for all $\varepsilon > 0$. This implies (ii) since the smooth vectors are dense by Proposition 7.5.

Next note that Theorem 8.5 shows that (ii) implies (i). Hence it remains to show that (i) implies (iii).

Suppose now that (i) holds. Let $\Omega = K\Omega K \subseteq G$ be a bi-K-invariant compact set, and choose some $\varepsilon > 0$. By our assumption (i) and condition $(\prec_{\mathrm{mc}})$ in Theorem 4.28, there exist, for every $v, w \in \mathcal{H}_\pi$, some $J \geqslant 1$ and function $f_{v,j}, f_{w,j} \in L^2(G)$ for $j = 1, \dots, J$ such that

$$\sum_{j=1}^{J} \|f_{v,j}\| \|f_{w,j}\| \leqslant \|v\| \|w\| \tag{8.29}$$

and

$$\left| \langle \pi_g v, w \rangle - \sum_{j=1}^{J} \langle \lambda_g f_{v,j}, f_{w,j} \rangle \right| < \varepsilon \tag{8.30}$$

for all $g \in \Omega$.

Assume now in addition that $v, w \in \mathcal{H}_\pi$ are K-eigenvectors, so that

$$\pi_{k_\theta} v = \mathrm{e}^{\mathrm{i}m\theta} v = \chi_m(k_\theta) v$$

and

$$\pi_{k_\theta} w = \mathrm{e}^{\mathrm{i}n\theta} w = \chi_n(k_\theta) w$$

for some $m, n \in \mathbb{Z}$ and all $k_\theta \in K$.

We wish to obtain from the above an approximation that only uses K-eigenfunctions. For this, we let $J \geqslant 1$, $f_{v,j}$, $f_{w,j} \in L^2(G)$ for $j = 1, \dots, J$ be as above, and define the projected function

$$\widetilde{f}_{v,j} = \bigl(\lambda|_K\bigr)_*(\overline{\chi_m}) f_{v,j} = \int_K \overline{\chi_m}(k_\theta) \lambda_{k_\theta} f_{v,j} \,\mathrm{d}m_K(k_\theta)$$

and

$$\widetilde{f}_{w,j} = \bigl(\lambda|_K\bigr)_*(\overline{\chi_n}) f_{w,j},$$

so that

$$\begin{aligned}\lambda_{k_\psi} \widetilde{f}_{v,j} &= \int_K \overline{\chi_m}(k_\theta) \lambda_{k_\psi k_\theta} f_{v,j} \,\mathrm{d}m_K(k_\theta) \\ &= \chi_m(k_\psi) \int_K \overline{\chi_m}(k_\psi k_\theta) \lambda_{k_\psi k_\theta} f_{v,j} \,\mathrm{d}m_K(k_\theta) = \chi_m(k_\psi) \widetilde{f}_{v,j},\end{aligned}$$

and similarly

$$\lambda_{k_\psi} \widetilde{f}_{w,j} = \chi_n(k_\psi) \widetilde{f}_{w,j}$$

for all $k_\psi \in K$. Moreover, we have

$$\begin{aligned}\|\widetilde{f}_{v,j}\| &\leqslant \|f_{v,j}\|,\\ \|\widetilde{f}_{w,j}\| &\leqslant \|f_{w,j}\|.\end{aligned} \tag{8.31}$$

Using the definition of $\widetilde{f}_{v,j}$ and $\widetilde{f}_{w,j}$, and the estimate (8.30) we obtain for

$$g \in \varOmega = K\varOmega K$$

that

$$\sum_{j=1}^{J}\bigl\langle \lambda_g \widetilde{f}_{v,j}, \widetilde{f}_{w,j}\bigr\rangle$$

equals

$$\int_K\int_K \overline{\chi_m}(k_\theta)\chi_n(k_\psi)\sum_j \underbrace{\Bigl\langle \lambda_{gk_\theta} f_{v,j}, \lambda_{k_\psi} f_{w,j}\Bigr\rangle}_{=\bigl\langle \lambda_{k_\psi^{-1}gk_\theta} f_{v,j}, f_{w,j}\bigr\rangle} \,\mathrm{d}m_K(k_\theta)\,\mathrm{d}m_K(k_\psi)$$

$$\begin{aligned}&= \int_K\int_K \overline{\chi_m}(k_\theta)\chi_n(k_\psi)\bigl\langle \pi(gk_\theta)v, \pi(k_\psi)w\bigr\rangle \,\mathrm{d}m_K(k_\theta)\,\mathrm{d}m_K(k_\psi) + \mathrm{O}(\varepsilon)\\ &= \bigl\langle \pi_g v, w\bigr\rangle + \mathrm{O}(\varepsilon).\end{aligned}$$

Using this, the fact that $\widetilde{f}_{v,j}, \widetilde{f}_{w,j}$ are K-eigenfunctions for $j = 1,\dots,J$, the decay properties of K-eigenfunctions for the regular representation in Theorem 8.31, (8.31), and (8.29), we finally obtain

$$\begin{aligned}\bigl|\bigl\langle \pi_g v, w\bigr\rangle\bigr| &\leqslant \sum_j \Bigl|\Bigl\langle \lambda_g \widetilde{f}_{v,j}, \widetilde{f}_{w,j}\Bigr\rangle\Bigr| + \mathrm{O}(\varepsilon)\\ &\leqslant \sum_j \|\widetilde{f}_{v,j}\|\|\widetilde{f}_{w,j}\|\varXi(g) + \mathrm{O}(\varepsilon)\\ &\leqslant \sum_j \|f_{v,j}\|\|f_{w,j}\|\varXi(g) + \mathrm{O}(\varepsilon) \leqslant \|v\|\|w\|\varXi(g) + \mathrm{O}(\varepsilon)\end{aligned}$$

for all $g \in \varOmega$. Varying the bi-K-invariant compact subset $\varOmega \subseteq K$ and $\varepsilon > 0$, we deduce that

$$\bigl|\bigl\langle \pi_g v, w\bigr\rangle\bigr| \leqslant \|v\|\|w\|\varXi(g)$$

for all K-eigenvectors $v, w \in \mathcal{H}_\pi$ and $g \in G$, which is property (iii) in the special case of K-eigenvectors.

Suppose now that $v, w \in \mathcal{H}_\pi$ are K-finite. Applying the projections $(\pi|_K)_*(\overline{\chi}_m)$, we can write v as a sum of K-eigenvectors $v_m \in \overline{\langle \pi(K)v\rangle}$ for $m \in \mathbb{Z}$. Since v is K-finite, it follows that

$$v = \sum_{m\in I} v_m,$$

and similarly

$$w = \sum_{n \in J} w_n$$

with $I, J \subseteq \mathbb{Z}$ satisfying $|I| = \dim(\langle \pi(K)v \rangle)$ and $|J| = \dim(\langle \pi(K)w \rangle)$. Then

$$\left|\left\langle \pi_g v, w \right\rangle\right| \leqslant \sum_{\substack{m \in I \\ n \in J}} \left|\left\langle \pi_g v_m, w_n \right\rangle\right| \leqslant \sum_{m \in I} \|v_m\| \sum_{n \in J} \|w_n\| \Xi(g).$$

On the other hand,

$$\sum_{m \in I} \|v_m\| \leqslant |I|^{\frac{1}{2}} \left(\sum_{m \in I} \|v_m\|^2 \right)^{\frac{1}{2}} = (\dim \langle \pi(K)v \rangle)^{\frac{1}{2}} \|v\|$$

by Cauchy–Schwarz and orthogonality of the decomposition of v into its different K-eigenvectors. Putting this together gives

$$\left|\left\langle \pi_g v, w \right\rangle\right| \leqslant (\dim \langle \pi(K)v \rangle)^{\frac{1}{2}} (\dim \langle \pi(K)w \rangle)^{\frac{1}{2}} \|v\| \|w\| \Xi(g)$$

for all $g \in G$ and K-finite vectors $v, w \in \mathcal{H}_\pi$, as required. □

8.7 (Non-Uniform) Integrability and Decay Exponents*

In the Howe–Moore theorem (Theorem 1.80), we have seen that matrix coefficients for unitary representations of $\mathrm{SL}_2(\mathbb{R})$ without fixed vectors decay at infinity. In Section 7.3 we went much further for $\mathrm{SL}_3(\mathbb{R})$, and showed the existence of a uniform decay exponent. This raises the question of whether $\mathrm{SL}_2(\mathbb{R})$ also has a uniform decay exponent. We show in this section that there is no uniform decay exponent as in Definition 7.22 for $\mathrm{SL}_2(\mathbb{R})$, which motivates the more detailed study of $\widehat{\mathrm{SL}_2(\mathbb{R})}$ in Chapter 9. However, we begin by defining and studying the notion of integrability exponents.

8.7.1 Integrability Exponents

The following definition is closely related to the notion of almost square integrability (Definition 8.4) and effective decay of matrix coefficients (Definition 7.21).

Definition 8.42 (Integrability exponents). Let G be a locally compact σ-compact metric group and π a unitary representation of G. For $p \in [1, \infty]$ we say that π is *p-integrable* or that p is an *integrability exponent* of π if there exists a dense set of vectors $\mathcal{V} \subseteq \left(\mathcal{H}_\pi^G\right)^\perp$ such that the matrix coefficients

$$G \ni g \longmapsto \varphi_{v,w}(g) = \langle \pi(g)v, w\rangle$$

lie in $L^p(G)$ for $v, w \in \mathcal{V}$. The *almost integrability exponent* $p_\pi \in [1,\infty]$ is defined by

$$p_\pi = \inf\{p \in [1,\infty] \mid \pi \text{ is } p\text{-integrable}\}.$$

We note that, in general, p_π could be infinite. As a first preliminary relationship between p_π and the parameter κ_π in Definition 7.21, we prove the following.

Lemma 8.43 (Decay implies integrability). *For a unitary representation π of $\mathrm{SL}_2(\mathbb{R})$ we have the inequality $p_\pi \leqslant \frac{2}{\kappa_\pi}$.*

Proof. We assume for simplicity of notation that π has no fixed vectors. Suppose that $\kappa_\pi > 0$ and $p > \frac{2}{\kappa_\pi}$ so that $\kappa = \frac{2}{p}$ satisfies $\kappa < \kappa_\pi$. Then

$$|\varphi_{v,w}(g)| = |\langle \pi_g v, w\rangle| \ll \lfloor g \rceil^{-\kappa}\, \mathcal{S}(v)\, \mathcal{S}(w)$$

for all C^r-smooth $v, w \in \mathcal{H}_\pi$ and some $r > 0$. Using (8.12) we now calculate

$$\begin{aligned}\|\varphi_{v,w}\|_p^p &= \int_{\mathrm{SL}_2(\mathbb{R})} |\varphi_{v,w}(g)|^p \,\mathrm{d}m(g)\\ &\ll_{v,w} \int_{KAK} \lfloor k_\varphi a_t k_\psi \rceil^{-\kappa p} \sinh 2t \,\mathrm{d}\varphi \,\mathrm{d}t \,\mathrm{d}\psi\\ &\ll_{v,w} \int_0^\infty \mathrm{e}^{-\kappa p t}\mathrm{e}^{2t}\,\mathrm{d}t < \infty.\end{aligned}$$

Therefore, and by Proposition 7.5, $p_\pi \leqslant p$. As $p > \frac{2}{\kappa_\pi}$ was arbitrary, we obtain $p_\pi \leqslant \frac{2}{\kappa_\pi}$ as claimed. □

The results of Chapter 9 will lead to the much more satisfactory statement that $\kappa_\pi = \frac{2}{p_\pi}$ for any unitary representation of $\mathrm{SL}_2(\mathbb{R})$ (see Section 9.6). For now we prove the following weaker claim.

Proposition 8.44 (Integrability implies decay). *Let π be a unitary representation of G with almost integrability exponent $p_\pi < \infty$, and let $d \in \mathbb{N}$ satisfy $\frac{p_\pi}{d} < 2$. If $\mathcal{H}_\pi$ has no invariant vectors, then $\pi^{\otimes d}$ is tempered. If $G = \mathrm{SL}_2(\mathbb{R})$, then π has exponential decay of matrix coefficients with decay exponent $\kappa = \frac{1}{d} - \varepsilon$ for all $\varepsilon > 0$. Moreover, it suffices to consider degree-one Sobolev norms, and the implicit constant in Definition* 7.21 *only depends on $\varepsilon > 0$.*

Proof. We suppose first that G is arbitrary (satisfying our standing assumptions) and that $\mathcal{H}_\pi$ has no invariant vectors. Let $\mathcal{V} \subseteq \mathcal{H}_\pi$ be a dense subset as in the definition of the integrability exponent, such that $\varphi_{v,w}^\pi \in L^p(G)$

for $v, w \in \mathcal{V}$ and $p = 2d > p_\pi$. For this, note that if $\mathcal{V}$ satisfies the integrability condition for $q \in [1, \infty)$ and $p > q$, then $\mathcal{V}$ also satisfies the condition for p since $L^q(G) \cap L^\infty(G) \subseteq L^p(G)$. By sesqui-linearity of matrix coefficients, we may also replace $\mathcal{V}$ by its linear hull, and assume as a result that $\mathcal{V}$ is a dense subspace.

We claim that the d-fold tensor product $\rho = \otimes^d \pi$ (defined inductively, using Section 5.2 as the inductive step) is 2-integrable. We note that the claim implies that ρ is tempered, by Theorem 8.5.

For the proof of the claim, we first show that $p_1, p_2 \in (0, \infty)$ implies that

$$L^{p_1}(G) L^{p_2}(G) \subseteq L^{\frac{p_1 p_2}{p_1 + p_2}}(G). \tag{8.32}$$

So let $f_j \in L^{p_j}(G)$ for $j \in \{1, 2\}$. This implies that

$$|f_1|^{\frac{p_1 p_2}{p_1 + p_2}} \in L^{\frac{p_1 + p_2}{p_2}}(G),$$

since

$$\int \left(|f_1|^{\frac{p_1 p_2}{p_1 + p_2}}\right)^{\frac{p_1 + p_2}{p_2}} \mathrm{d}m = \int |f_1|^{p_1} \, \mathrm{d}m < \infty,$$

and similarly

$$|f_2|^{\frac{p_1 p_2}{p_1 + p_2}} \in L^{\frac{p_1 + p_2}{p_1}}(G).$$

Using the relations $\frac{p_1 + p_2}{p_2} \geqslant 1$, $\frac{p_1 + p_2}{p_1} \geqslant 1$,

$$\frac{1}{\frac{p_1 + p_2}{p_2}} + \frac{1}{\frac{p_1 + p_2}{p_1}} = 1,$$

and Hölder's inequality, we deduce that

$$|f_1 f_2|^{\frac{p_1 p_2}{p_1 + p_2}} = |f_1|^{\frac{p_1 p_2}{p_1 + p_2}} |f_2|^{\frac{p_1 p_2}{p_1 + p_2}} \in L^1(G)$$

which proves (8.32). To iterate this, it is convenient to note that (8.32) is equivalent to

$$L^{\frac{1}{s_1}}(G) L^{\frac{1}{s_2}}(G) \subseteq L^{\frac{1}{s_1 + s_2}}(G)$$

for all $s_1, s_2 \in (0, \infty)$. By induction, this becomes

$$L^{\frac{1}{s_1}}(G) \cdots L^{\frac{1}{s_m}}(G) \subseteq L^{\frac{1}{s_1 + \cdots + s_m}}(G) \tag{8.33}$$

for all $m \in \mathbb{N}$ and $s_1, \ldots, s_m \in (0, \infty)$.

We return to the claim above. Using $p = 2d$ and (8.33) for $m = d$ and

$$s_1 = \cdots = s_d = \tfrac{1}{p},$$

we have

$$L^p(G) \cdots L^p(G) \subseteq L^{\frac{p}{d}}(G) = L^2(G).$$

For $v_1, \ldots, v_d, w_1, \ldots, w_d \in \mathcal{V}$, our assumption on $\mathcal{V}$ gives $\varphi^\pi_{v_j,w_j} \in L^p(G)$ for $j = 1, \ldots, d$. Using the defining properties of the tensor product representation extended to the d-fold tensor product $\rho = \otimes^d \pi$, we have

$$\begin{aligned}\varphi^\rho_{\widetilde{v},\widetilde{w}}(g) &= \bigl\langle \rho_g v_1 \otimes \cdots \otimes v_d, w_1 \otimes \cdots \otimes w_d \bigr\rangle_{\mathcal{H}_\rho} \\ &= \bigl\langle \pi_g v_1, w_1 \bigr\rangle_{\mathcal{H}_\pi} \cdots \bigl\langle \pi_g v_d, w_d \bigr\rangle_{\mathcal{H}_\pi} \\ &= \varphi^\pi_{v_1,w_1}(g) \cdots \varphi^\pi_{v_d w_d}(g)\end{aligned}$$

for $\widetilde{v} = v_1 \otimes \cdots \otimes v_d$ and $\widetilde{w} = w_1 \otimes \cdots \otimes w_d$ and $g \in G$. Together, we have $\varphi^\rho_{\widetilde{v},\widetilde{w}} \in L^2(G)$ for all pure tensors $\widetilde{v}$, $\widetilde{w}$ as above. Using sesqui-linearity of $\varphi^\rho_{\widetilde{v},\widetilde{w}}$ with respect to $\widetilde{v}$, $\widetilde{w}$, the property $\varphi^\rho_{\widetilde{v},\widetilde{w}} \in L^2(G)$ extends to all finite linear combinations of these pure tensors. However, the space of finite linear combinations is the tensor product $\bigotimes^d_{\mathrm{la}} \mathcal{V}$ of $\mathcal{V}$ in the sense of linear algebra and is a dense subspace of $\mathcal{H}_\rho = \bigotimes^d \mathcal{H}_\pi$ by the properties of tensor products in Section 5.2. However, this proves the claim, and so the first part of the corollary.

Assume now that $G = \mathrm{SL}_2(\mathbb{R})$. Comparing the definitions of decay exponents (Definition 7.21) and integrability exponents (Definition 8.42), we see that both only concern the restriction of π to $\bigl(\mathcal{H}^G_\pi\bigr)^\perp$. Hence it suffices to assume that π has no invariant vectors. Applying the first part of the corollary, we see that $\pi^{\otimes d}$ is tempered. If $v, w \in \mathcal{H}_\pi$ are K-eigenvectors, then so are the d-fold tensor products $\widetilde{v} = v \otimes \cdots \otimes v, \widetilde{w} = w \otimes \cdots \otimes w \in \mathcal{H}_\rho$. Applying Theorem 8.41(iii), we obtain

$$\Bigl|\bigl\langle \rho_g \widetilde{v}, \widetilde{w} \bigr\rangle_{\mathcal{H}_\rho}\Bigr| \ll \varXi(g) \|\widetilde{v}\|_{\mathcal{H}_\rho} \|\widetilde{w}\|_{\mathcal{H}_\rho}$$

for all $g \in \mathrm{SL}_2(\mathbb{R})$. By definition of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_\rho}$, this is an estimate for the nth power of the matrix coefficient of $v, w \in \mathcal{H}_\pi$. Together with the estimate concerning the Harish-Chandra spherical function in Proposition 8.39, we obtain

$$\bigl|\bigl\langle \pi_g v, w \bigr\rangle\bigr| \ll_\varepsilon |g|^{-\frac{1}{d}+\varepsilon} \|v\| \|w\|$$

for all $g \in \mathrm{SL}_2(\mathbb{R})$ and K-eigenvectors $v, w \in \mathcal{H}_\pi$. With Proposition 7.25, this concludes the proof. □

8.7.2 Dynamics on Quotients of $\mathbf{PSL_2}(\mathbb{R})$

Our construction below of unitary representations with non-uniform decay exponents is quite geometric. As explained in Section 8.3, the group

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$$

acts transitively via Möbius transformations on the hyperbolic plane $\mathbb{H}$. Moreover, after equipping $\mathbb{H}$ with the hyperbolic Riemannian metric, this extends to a simple transitive action of $\mathrm{PSL}_2(\mathbb{R})$ on the unit tangent bundle $\mathrm{T}^1\mathbb{H}$ of the hyperbolic surface. Choosing a reference vector in $\mathrm{T}^1\mathbb{H}$ (usually the upward pointing vector at the point $\mathrm{i} \in \mathbb{H}$) this gives an identification between $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{T}^1\mathbb{H}$.

For discrete subgroups $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ (the so-called *Fuchsian groups*[(13)]), we may use the discussions above to identify the quotient $X = \Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$ with the unit tangent bundle of the hyperbolic surface[†] $M = \Gamma\backslash\mathbb{H}$. Using a measurable fundamental domain $F_{\Gamma,\mathbb{H}}$ for the action of Γ on $\mathbb{H}$, one can define a measurable fundamental domain F_Γ in $\mathrm{PSL}_2(\mathbb{R})$ for the action of Γ (so that, in particular, $|\Gamma g \cap F_\Gamma| = 1$ for all $g \in \mathrm{PSL}_2(\mathbb{R})$). The discrete subgroup Γ is called a *lattice* if $m(F_\Gamma) < \infty$ (equivalently, if $\mathrm{vol}(F_{\Gamma,\mathbb{H}}) < \infty$). In this case one can use the Haar measure m restricted to F_Γ to induce a measure m_X on X, which will be finite and invariant under the action of $g \in G$ on $x = \Gamma h \in X$ defined by right multiplication,

$$g_\bullet x = g_\bullet(\Gamma h) = \Gamma h g^{-1}.$$

We will call m_X the Haar measure on X (see Exercise 8.45). Moreover, this gives rise to the Koopman representation defined by

$$\big(\pi_g^X f\big)(x) = f(xg)$$

for $g \in \mathrm{PSL}_2(\mathbb{R})$ (or $g \in \mathrm{SL}_2(\mathbb{R})$), $f \in L^2_{m_X}(X)$, and $x \in X$. Since we only need this for the construction of unitary representations with weak decay exponents (and this will become clearer in our examples), we have simply summarized the facts above, and refer any reader not familiar with these constructions to [20, Ch. 9].

Exercise 8.45 (Haar measure on X). Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be discrete, let

$$\pi\colon G \ni g \longmapsto \Gamma g \in X = \Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$$

denote the canonical quotient map, and let F_Γ be a measurable fundamental domain for Γ. Show that $m_X = \pi_*\big(m|_{F_\Gamma}\big)$ is a G-invariant measure independent of the choice of the fundamental domain F_Γ.

[†] If M is a manifold, then this is the unit tangent bundle in the usual sense. If Γ has torsion elements, then M is an orbifold with non-smooth points, and this should be viewed simply as a definition of the unit tangent bundle of M.

8.7.3 Non-uniform Decay for $\mathbf{SL_2}(\mathbb{R})$

Proposition 8.46 (A free lattice in $\mathrm{PSL}_2(\mathbb{R})$). *We let*

$$\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

considered in $\mathrm{PSL}_2(\mathbb{R})$*. The Sanov*[(14)] *subgroup* $\Gamma = \langle \alpha, \beta \rangle$ *generated by* α *and* β *is a free group on two generators, and is a lattice in* $\mathrm{PSL}_2(\mathbb{R})$*.*

As explained in Section 8.7.2, the lattice statement can be checked in the hyperbolic plane by exhibiting a fundamental domain of finite hyperbolic area for the action on $\mathbb{H}$. We will ignore the null sets arising from the boundaries of hyperbolic polygons in our discussions.

Proof of Proposition 8.46. We note that α acts on $z \in \mathbb{H}$ via $\alpha{\bullet}z = z+2$. This implies that

$$\left.\begin{aligned} \alpha{\bullet}(\mathbb{H}\smallsetminus D_\alpha^-) &\subseteq D_\alpha^+ \\ \alpha^{-1}{\bullet}(\mathbb{H}\smallsetminus D_\alpha^+) &\subseteq D_\alpha^- \end{aligned}\right\} \tag{8.34}$$

in the notation from Figure 8.1 (and, as promised, ignoring null sets). We wish to make a similar translation statement for β. However, β acts on $z \in \mathbb{H}$ via $\beta{\bullet}z = \frac{z}{2z+1}$, which may appear to be more complicated than the action of α. To understand the action of β better, we note that

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

acts on $z \in \mathbb{H}$ by $\sigma{\bullet}z = -\frac{1}{z}$, which moves everything from outside the unit circle (intersected with $\mathbb{H}$) into the unit circle (intersected with $\mathbb{H}$) and *vice versa*. Moreover, by looking at the boundary points $-1, \infty, 1$, we see that the two vertical lines in Figure 8.1 are moved by σ precisely to the two half circles and *vice versa*. In particular, $\sigma{\bullet}F = F$. Since

$$\sigma\alpha^{-1}\sigma^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \beta,$$

this explains that the action of β makes a similar move involving the half-circles as α does involving the vertical lines. More formally, we have

$$\left.\begin{aligned} \beta{\bullet}(\mathbb{H}\smallsetminus D_\beta^-) &\subseteq D_\beta^+, \\ \beta^{-1}{\bullet}(\mathbb{H}\smallsetminus D_\beta^+) &\subseteq D_\beta^- \end{aligned}\right\} \tag{8.35}$$

The subset $F \subseteq \mathbb{H}$ and the relations (8.34) and (8.35) allow the following simple 'ping-pong' argument to be made. Let $\gamma = \alpha^{m_1}\beta^{n_1}\alpha^{m_2}\beta^{n_2}\cdots\alpha^{m_k}\beta^{n_k}$ be a reduced word in the generators α and β in which we allow $m_1 n_k$ to be zero. Then we claim that

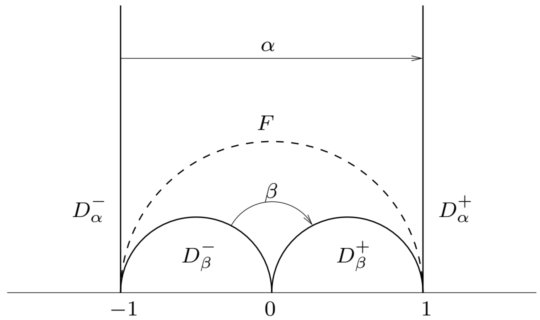

Fig. 8.1: A fundamental domain F for Γ in $\mathrm{SL}_2(\mathbb{R})$, with the unit circle shown in a dashed line. The sets D_α^+, D_α^-, D_β^+, and D_β^- together with F form a partition of $\mathbb{H}$, which can be used to show that α and β generate a free group.

- $\gamma\bullet F \subseteq D_\alpha^+$ if and only if $m_1 > 0$;
- $\gamma\bullet F \subseteq D_\alpha^-$ if and only if $m_1 < 0$;
- $\gamma\bullet F \subseteq D_\beta^+$ if and only if $m_1 = 0$ and $n_1 > 0$; and
- $\gamma\bullet F \subseteq D_\beta^-$ if and only if $m_1 = 0$ and $n_1 < 0$.

This is simply (8.34) and (8.35) if $\gamma = \alpha^{\pm 1}$ or $\gamma = \beta^{\pm 1}$ has length one. If we assume the claim for γ of a given length then (8.34) and (8.35) again show the same claim after increasing the length of γ (on the left) by one. It follows that the image of F under the Möbius action of γ determines the left-most group element $\alpha^{\pm 1}$ or $\beta^{\pm}$ in the reduced word. By induction, it follows that $\gamma\bullet F$ determines the powers $m_1, n_1, \ldots, m_k, n_k$ of the generators. We deduce that α and β generate a free group.

Next we claim that F is a fundamental domain. For this, we first note that the above also shows that F and $\gamma\bullet F$ are (apart from the boundaries) disjoint whenever $\gamma \in \Gamma$ is non-trivial. It remains to show that

$$D = \bigcup_{\gamma \in \Gamma} \gamma\bullet\overline{F} = \mathbb{H}.$$

We claim that there exists some uniform $\varepsilon > 0$ so that for all $z \in D$ we have $B_\varepsilon^{\mathbb{H}}(z) \subseteq D$. We note that this unusual 'uniform openness' implies that the set D is both open and closed, which by connectedness of $\mathbb{H}$ implies that $D = \mathbb{H}$.

For the claim, we first note that it is sufficient to consider $z \in F$ (since Γ acts by isometries on $\mathbb{H}$). For $z \in F$ belonging to the middle compact region in Figure 8.2 this is obvious, since D contains the four neighbouring copies $\alpha^{\pm 1}\bullet F$ and $\beta^{\pm 1}\bullet F$ of our region F. For $z \in F$ belonging to the top region in Figure 8.2, the two neighbouring copies may not be enough for this argument. In fact the ball $B_\varepsilon^{\mathbb{H}}(z)$ around such a point z may spread over many copies $\alpha^{\pm n}\bullet F$ with $n \in \mathbb{Z}$ of our set, but if $\varepsilon > 0$ is sufficiently small it will not reach further down than into the horizontal translates of the middle compact region. As explained before, applying σ exchanges 0 and ∞ and the generators α, β^{-1} of Γ, which can be used to reduce the claim for the region stretching towards the previous case. For the remaining two pieces near ± 1, we use

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

corresponding to the Möbius transformation $\tau\colon z \mapsto z+1$. We note that

$$\tau\alpha\tau^{-1} = \alpha,$$

$$\tau\beta\tau^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix},$$

and

$$\alpha\beta^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} = -\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

together show that conjugation by τ normalizes $\Gamma < \mathrm{PSL}_2(\mathbb{R})$. Therefore

$$\tau^{\pm 1}\cdot D = \bigcup_{\gamma \in \Gamma} \gamma\tau^{\pm 1}\cdot F = D$$

since $\tau^{\pm 1}\cdot F$ is partially contained in F and the other half belongs to $\alpha^{\pm 1}\cdot F$. Since $\tau^{\pm 1}$ acts isometrically and the claim already holds for the elements of the middle region near 0, we now also obtain the claim for the remaining two regions of F near ± 1.

To see that Γ is a lattice, divide F again, into the five regions shown in Figure 8.2. Clearly the compact medium shaded region has finite hyperbolic area. For the top lightly shaded region we calculate

$$\int_{y_0}^{\infty}\int_{-1}^{1} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = 2\int_{y_0}^{\infty} \frac{\mathrm{d}y}{y^2} < \infty.$$

The middle of the bottom three dark shaded pieces can be chosen as the image of the top piece under σ, and so also has finite area. Finally, the remaining two pieces near ± 1 can be put together using the Möbius transformation corresponding to τ to become congruent to the middle piece.

The fundamental region F_Γ for Γ in $\mathrm{PSL}_2(\mathbb{R})$ consists of all $g \in \mathrm{PSL}_2(\mathbb{R})$ with $g\cdot\mathrm{i} \in F$. With the Iwasawa decomposition $\mathrm{PSL}_2(\mathbb{R}) = B\,\mathrm{PSO}_2(\mathbb{R})$, this becomes

$$F_\Gamma \cong F\,\mathrm{PSO}_2(\mathbb{R}),$$

where we identify $b \in B$ with $b\cdot\mathrm{i} \in \mathbb{H}$ and hence F with a subset of B. Using Lemma 8.13 for the two subgroups B and $\mathrm{PSO}_2(\mathbb{R})$ of $\mathrm{PSL}_2(\mathbb{R})$, we obtain $m(F_\Gamma) < \infty$ as required. □

Since Γ is a free group, we may define a homomorphism

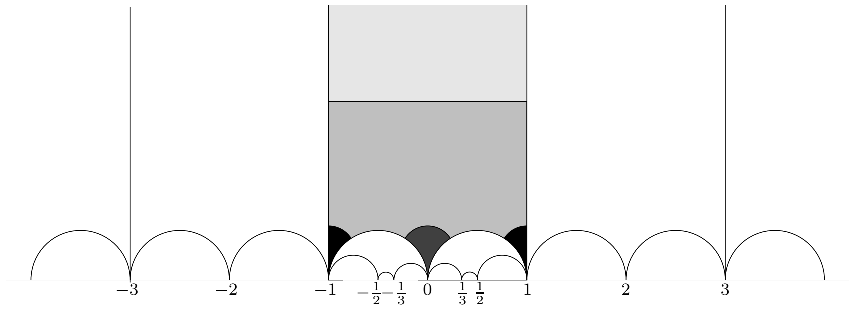

Fig. 8.2: The subgroup $\varGamma$ is a lattice.

$$\phi_n \colon \varGamma \longrightarrow \varGamma/[\varGamma,\varGamma] \cong \mathbb{Z}^2 \longrightarrow \mathbb{Z}/n\mathbb{Z}$$

for every $n \in \mathbb{N}$, by sending $\alpha \in \varGamma$ to the generator $1 \in \mathbb{Z}/n\mathbb{Z}$ and $\beta \in \varGamma$ to zero. Clearly the kernel $\varLambda_n = \ker \phi_n < \varGamma$ has index n in $\varGamma$, and so is again a lattice in $\mathrm{PSL}_2(\mathbb{R})$. We note that, geometrically, $X_n = \varLambda_n \backslash \mathrm{PSL}_2(\mathbb{R})$ is a $\mathbb{Z}/n\mathbb{Z}$-cover of $\varGamma\backslash\mathrm{PSL}_2(\mathbb{R})$. In fact, the fundamental domain $F_n = F_{\varLambda_n}$ of $\varLambda_n$ consists of n copies of the fundamental domain $F_\varGamma$ of $\varGamma$ that can be chosen to be adjacent, as in Figure 8.3. Note that $\alpha^n \in \varLambda_n$ will identify the left boundary of F_n with the right boundary of F_n, so that, roughly speaking and from far away, the geometries of $\varLambda_n\backslash\mathbb{H}$ or of $X_n = \varLambda_n\backslash\mathrm{PSL}_2(\mathbb{R})$ look like that of a circle. This is used to define, for each $n \in \mathbb{N}$, the unitary Koopman representation π^{X_n} on $L^2(X_n)$ and prove the following result.

Proposition 8.47 (Non-uniformity). *The decay exponent for the unitary representation π^{X_n} of $\mathrm{SL}_2(\mathbb{R})$ on $L^2\bigl(\varLambda_n\backslash\mathrm{PSL}_2(\mathbb{R})\bigr)$ goes to zero as $n \to \infty$. Equivalently, the integrability exponent of π^{X_n} goes to infinity as $n \to \infty$.*

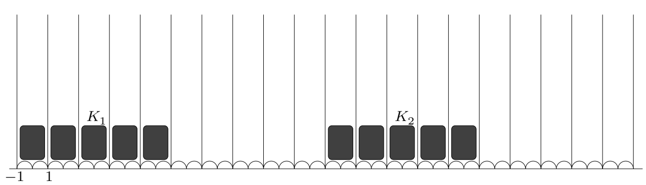

Fig. 8.3: We let $\ell = \lceil \frac{n}{4} \rceil$, define the compact subset K_1 using the first ℓ copies of F in F_n, and the compact subset K_2 using the third ℓ copies of F in F_n. This forces K_1 and K_2 to have a large distance between them once n is large.

Proof of Proposition 8.47. Let $\kappa_n = \kappa_{\pi^{X_n}} \geqslant 0$ be the decay exponent, and let $p_n = p_{\pi^{X_n}} \leqslant \infty$ be the integrability exponent for π^{X_n}. We note that by Lemma 8.43 and Proposition 8.44 the claims

$$\kappa_n \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

and

$$p_n \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

are equivalent. Let $d \in \mathbb{N}$ be arbitrary, and suppose that $p_n < 2d$. By Proposition 8.44 it follows that π^{X_n} has decay exponent $\frac{1}{d} - \frac{1}{2d} = \frac{1}{2d}$, and that it suffices to consider degree-one Sobolev norms and a uniform (that is, independent of n) implicit constant. We will show that this forces n to be bounded. As $d \in \mathbb{N}$ was arbitrary, this will prove that $p_n \to \infty$ and $\kappa_n \to 0$ as $n \to \infty$.

The proof that $p_n < 2d$ forces n to be bounded will consist of analysing the behaviour of the matrix coefficients for the characteristic functions

$$\mathbb{1}_{K_1}, \mathbb{1}_{K_2} \in L^2\bigl(\Lambda_n \backslash \mathrm{PSL}_2(\mathbb{R})\bigr),$$

where $K_1, K_2 \subseteq X_n = \Lambda_n \backslash \mathrm{PSL}_2(\mathbb{R})$ are defined using the map

$$\Lambda_n \backslash \mathrm{PSL}_2(\mathbb{R}) \ni \Lambda_n g \longmapsto \Lambda_n g \mathrm{PSO}_2(\mathbb{R}) \in \Lambda_n \backslash \mathbb{H} \cong F_n$$

and Figure 8.3. More formally, let $K_\Gamma \subseteq F_\Gamma$ be a fixed compact subset with non-empty interior, and define

$$\left.\begin{aligned} K_1 &= \Lambda_n \bigcup_{j=0}^{\ell-1} \alpha^j K_\Gamma, \\ K_2 &= \Lambda_n \bigcup_{j=0}^{\ell-1} \alpha^{2\ell+j} K_\Gamma. \end{aligned}\right\} \tag{8.36}$$

We note that

$$v_n = m_{X_n}(X_n) = \mathrm{vol}(F_n) = n\mathrm{vol}(F_\Gamma).$$

Let $c_0 = \frac{\mathrm{vol}(K_\Gamma)}{\mathrm{vol}(F_\Gamma)} > 0$ be the ratio of the volume of the compact region K_Γ within one copy of F_Γ and $\mathrm{vol}(F_\Gamma)$, which is independent of n. The construction of K_1 and K_2 (using $\ell = \lceil \frac{n}{4} \rceil$) then implies that

$$m_{X_n}(K_1) = m_{X_n}(K_2) = \ell c_0 \mathrm{vol}(F_\Gamma) \geqslant \frac{c_0}{4} v_n. \tag{8.37}$$

We now apply the assumption that π^{X_n} has decay exponent $\frac{1}{2d}$. Since the transitive action of $\mathrm{SL}_2(\mathbb{R})$ on X_n is ergodic (see Exercise 8.48), the only trivial representation appearing in $L^2_{m_{X_n}}(X_n)$ is $\mathbb{C}\mathbb{1}$. Hence on the orthogonal complement

$$L^2_0(X_n) = \left\{ f \in L^2_{m_{X_n}}(X_n) \mid \int_{X_n} f \,\mathrm{d}m_{X_n} = 0 \right\}$$

of the constants, we may apply the decay property. In order to apply the definition of decay exponents, we need to work with smooth functions. So we fix $\psi \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$ (independent of n) with $\psi \geqslant 0$ and $\int \psi \,\mathrm{d}m = 1$. We define

$$f_j = \pi_*^{X_n}(\psi)\mathbb{1}_{K_j} = \int \psi(h)\mathbb{1}_{K_j}(\cdot h)\,\mathrm{d}m(h)$$

with

$$\int f_j \,\mathrm{d}m_{X_n} = m_{X_n}(K_j),$$

and note that by Proposition 7.5 the degree-one Sobolev norm satisfies

$$\mathcal{S}(f_j) \ll_\psi \|\mathbb{1}_{K_j}\|_2 \leqslant \sqrt{m_{X_n}(X_n)} = \sqrt{v_n} \tag{8.38}$$

for $j = 1, 2$. Since

$$\widetilde{f}_j = f_j - v_n^{-1}\int f_j \,\mathrm{d}m_{X_n}\mathbb{1}_{X_n} = f_j - \frac{m_{X_n}(K_j)}{v_n}\mathbb{1}_{X_n} \in L_0^2(X_n) \tag{8.39}$$

for $j = 1, 2$, we obtain

$$\left|\left\langle \pi_g^{X_n}\widetilde{f}_1, \widetilde{f}_2\right\rangle\right| \ll c_1\,|g|^{-\frac{1}{2d}}\,\mathcal{S}(\widetilde{f}_1)\,\mathcal{S}(\widetilde{f}_2). \tag{8.40}$$

Using

$$m_{X_n}(K_j) = \int f_j \,\mathrm{d}m_{X_n} \leqslant v_n$$

and (8.38), we obtain

$$\mathcal{S}(\widetilde{f}_j) \leqslant \mathcal{S}(f_j) + v_n^{-1}\underbrace{\int f_j \,\mathrm{d}m_{X_n}}_{\leqslant v_n}\underbrace{\mathcal{S}(\mathbb{1}_{X_n})}_{=\|\mathbb{1}_{X_n}\|_2} \ll 2\sqrt{v_n}$$

for $j = 1, 2$. Putting this into (8.40) gives

$$|\langle \pi_g^{X_n}\widetilde{f}_1, \widetilde{f}_2\rangle| < c_1\,|g|^{-\frac{1}{2d}}\,v_n$$

for a constant c_1 (that depends on d and the choice of ψ, but not on n). With the definition of $\widetilde{f}_1$ and $\widetilde{f}_2$ in (8.39), this becomes

$$\begin{aligned}
\left\langle \pi_g^{X_n} f_1, f_2\right\rangle &= \left\langle \frac{m_{X_n}(K_1)}{v_n}\mathbb{1}_{X_n} + \pi_g^{X_n}\widetilde{f}_1, \frac{m_{X_n}(K_2)}{v_n}\mathbb{1}_{X_n} + \widetilde{f}_2\right\rangle \\
&> \frac{m_{X_n}(K_1)}{v_n}\frac{m_{X_n}(K_2)}{v_n}\left\langle \mathbb{1}_{X_n}, \mathbb{1}_{X_n}\right\rangle - c_1\,|g|^{-\frac{1}{2d}}\,v_n \\
&> \frac{c_0^2}{16}v_n - c_1\,|g|^{-\frac{1}{2d}}\,v_n.
\end{aligned}$$

For $\frac{c_0^2}{16} \geqslant c_1 |g|^{-\frac{1}{2d}}$, or, equivalently, for $|g| \geqslant \left(\frac{16c_1}{c_0^2}\right)^{2d}$, this implies that

$$\langle \pi_g^{X_n} f_1, f_2 \rangle \neq 0 \tag{8.41}$$

only assuming that n satisfies $p_n < 2d$.

So suppose now that $g \in \mathrm{SL}_2(\mathbb{R})$ satisfies (8.41). More specifically, this means

$$\begin{aligned} 0 &\neq \iint\limits_{G\,G} \langle \pi_{gh_1}^{X_n} \mathbb{1}_{K_1}, \pi_{h_2}^{X_n} \mathbb{1}_{K_2} \rangle \psi(h_1)\psi(h_2)\,\mathrm{d}m(h_1)\,\mathrm{d}m(h_2) \\ &= \iint\limits_{G\,G} \int\limits_{X_N} \mathbb{1}_{K_1 h_1^{-1}}(xg) \mathbb{1}_{K_2 h_2^{-1}}(x) \psi(h_1)\psi(h_2)\,\mathrm{d}m_{X_n}(x)\,\mathrm{d}m(h_1)\,\mathrm{d}m(h_2), \end{aligned}$$

and so there exist $h_1, h_2 \in \operatorname{supp}\psi$ with $(K_1 h_1^{-1} g^{-1}) \cap (K_2 h_2^{-1}) \neq \emptyset$. Our construction of K_1 and K_2 in Figure 8.3 now ensures that any g with this property must be large if n is large. In fact, using the formal definition in (8.36), we see that

$$(K_1 h_1^{-1} g^{-1}) \cap (K_2 h_2^{-1}) \neq \emptyset$$

implies that there exist $\lambda \in \Lambda_n$, $k_1, k_2 \in K_\Gamma$, $j_1, j_2 \in \{0, \dots, \ell - 1\}$ with

$$\lambda \alpha^{j_1} k_1 h_1^{-1} g^{-1} = \alpha^{2\ell + j_2} k_2 h_2^{-1}$$

or, equivalently, that

$$\gamma = \alpha^{-2\ell - j_2} \lambda \alpha^{j_1} = k_2 h_2^{-1} g h_1 k_1^{-1} \in \Gamma \cap \left(K(\operatorname{supp}\psi)^{-1} g (\operatorname{supp}\psi) K^{-1}\right).$$

Since $\lambda \in \Lambda_n = \langle \alpha^n, \beta \rangle$, $\ell = \lceil \frac{n}{4} \rceil$, $j_1, j_2 \in \{0, \dots, \ell - 1\}$, and $\Gamma = \langle \alpha, \beta \rangle$ is the free group with generators α, β, we see from the left-hand side that γ is a non-trivial word in α and β. In fact, the smallest length of γ is achieved if $\lambda = \alpha^n$, in which case γ still has length at least $n - 3\ell \geqslant \frac{1}{4}n - 3$. On the other hand, the right-hand side gives $|\gamma| \ll |g|$. We define

$$a_n = \min\left\{ |\gamma| \mid \gamma \in \Gamma \text{ has length at least } \tfrac{1}{4}n - 3 \right\},$$

and note that $a_n \to \infty$ as $n \to \infty$ since $\Gamma < G$ is a discrete subgroup. Using this sequence, we can summarize the discussion above by saying that (8.41) implies that $a_n \leqslant c_2 |g|$ for some constant c_2 independent of n.

Together with the argument leading to (8.41), we arrive at the inequality

$$c_2 \left(\frac{16c_1}{c_0^2}\right)^{2d} \geqslant a_n$$

for any $n \in \mathbb{N}$ with $p_n < 2d$. Since $a_n \to \infty$ as $n \to \infty$, this gives an upper bound on such n. As $d \in \mathbb{N}$ was arbitrary, we conclude that $p_n \to \infty$ and $\kappa_n \to 0$ as $n \to \infty$. □

Exercise 8.48. Prove that the measure m_{X_n} is ergodic for the action of $\mathrm{SL}_2(\mathbb{R})$ on X_n for any $n \geqslant 1$.

We note that the Koopman representation on a quotient $X = \Gamma\backslash\mathrm{PSL}_2(\mathbb{R})$ by a lattice $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ always has a positive decay exponent, equivalently a finite integrability exponent, or spectral gap. Moreover for the so-called 'congruence lattices' one even has a uniform decay exponent. Unfortunately we will not reach these results in this volume.

Exercise 8.49. Use the argument of Proposition 8.46 to show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by an element of order 4 and an element of order 3. Deduce that a character on $\mathrm{SL}_2(\mathbb{Z})$ must take values in the 12th roots of unity.

In contrast to Exercise 1.79, one can write down(15) explicit non-trivial finite-dimensional characters of $\mathrm{SL}_2(\mathbb{Z})$. We will have other instances where the structure and behaviour of unitary representations for a discrete group are quite different to the unitary representation theory of its continuous counterpart.

Exercise 8.50. Show that the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \mathrm{e}^{\frac{\pi \mathrm{i}}{6}\left((1-c^2)(bd+3(c-1)d+c+3)+c(a+d-3)\right)}$$

is a character on $\mathrm{SL}_2(\mathbb{Z})$ with image equal to $\{z \in \mathbb{C} \mid z^{12} = 1\}$.

8.8 A Special Case of the Kunze–Stein Phenomenon*

We again let $G = \mathrm{SL}_2(\mathbb{R})$, and will explain a special case of a surprising result that was first discovered by Kunze and Stein [43] and extended to semi-simple groups by Cowling [14] (after other results in the same spirit; we refer to Cowling's work for references).

Recall that any function $\psi \in L^1(G)$ acts as a convolution operator $\pi_*(\psi)$ on $\mathcal{H}_\pi$ for any unitary representation π, so that we have the trivial operator bound

$$\|\pi_*(\psi)\|_{\mathrm{op}} \leqslant \|\psi\|_1.$$

In the context of ergodic theory, one often sets $\psi = \frac{1}{m(S)}\mathbb{1}_S$ for some measurable set $S \subseteq G$ of large finite measure. Unfortunately, the trivial bound is, in this context, quite useless. However, the operator bound in the following theorem and its corollary can be powerful in ergodic theory and its applications to number theory (see the monograph of Gorodnik and Nevo [30] for more on this).

Theorem 8.51 (Spherical Kunze–Stein for $\mathrm{SL}_2(\mathbb{R})$). *Let $G = \mathrm{SL}_2(\mathbb{R})$ and $K = \mathrm{SO}_2(\mathbb{R}) < G$. Let $\varepsilon > 0$ and let $\psi \in L^1(G) \cap L^{2-\varepsilon}(G)$ be bi-K-invariant in the sense that $\psi(kgk') = \psi(g)$ for all $k, k' \in K$ and $g \in G$. Then we have*

$$\|\pi_*(\psi)\|_{\text{op}} \ll_\varepsilon \|\psi\|_{2-\varepsilon}$$

for any tempered representation π of G.

We note that the restriction to bi-K-invariant functions is not needed, but it does dramatically simplify the proof of the theorem. The general case is a corollary of the Plancherel formula for $\mathrm{SL}_2(\mathbb{R})$, which we will not discuss in this volume.

Proof of Theorem 8.51. Let π be a unitary representation of G, and let ψ be a function in $L^1(G)$ satisfying $\psi(kgk') = \psi(g)$ for all $k, k' \in K$ and $g \in G$. Let $P_K \colon \mathcal{H}_\pi \to \mathcal{H}_\pi^K$ denote the orthogonal projection onto the subspace of K-invariant vectors in $\mathcal{H}_\pi$. Note that $P_K = \bigl(\pi|_K\bigr)_*(\mathbb{1})$.

We claim that our assumption on ψ implies that $\pi_*(\psi) = P_K \pi_*(\psi) P_K$. Indeed, using abbreviated and slightly informal notation, we have

$$\begin{aligned}\pi_*(\psi) &= \int_G \underbrace{\psi(g)}_{=\psi(k_1^{-1}gk_2^{-1})} \pi_g \,\mathrm{d}m(g)\\ &= \int_G \psi(h)\pi_{k_1}\pi_h\pi_{k_2}\,\mathrm{d}m(h)\\ &= \pi_{k_1}\circ\pi_*(\psi)\circ\pi_{k_2}\\ &= (\pi|_K)_*(\mathbb{1})\circ\pi_*(\psi)\circ(\pi|_K)_*(\mathbb{1}) = P_K\pi_*(\psi)P_K\end{aligned}$$

by using the substitution $h = k_1^{-1}gk_2^{-1}$, and then integrating with respect to the normalized Haar measure over $k_1, k_2 \in K$.

We now specialize to the regular representation. For $f_1, f_2 \in L^2(G)$ we then obtain

$$\begin{aligned}\bigl|\langle\lambda_*(\psi)f_1, f_2\rangle\bigr| &= \bigl|\langle\lambda_*(\psi)P_K f_1, P_K f_2\rangle\bigr|\\ &= \left|\int_G \psi(g)\,\langle\lambda_g P_K f_1, P_K f_2\rangle\,\mathrm{d}m(g)\right|\\ &\leqslant \int_G |\psi(g)|\underbrace{\bigl|\langle\lambda_g P_K f_1, P_K f_2\rangle\bigr|}_{\leqslant\left\langle\pi_g^0\,\overbrace{P_K f_1},\overbrace{P_K f_2}\right\rangle}\,\mathrm{d}m(g)\\ &\leqslant \int_G |\psi(g)|\left\langle\pi_g^0\,\overbrace{P_K f_1},\overbrace{P_K f_2}\right\rangle\,\mathrm{d}m(g)\end{aligned}$$

by the Hertz domination principle in Proposition 8.35. Using the other properties in Proposition 8.35, more can be said. In fact, since $P_K f_j$ is K-invariant and $\overbrace{\lambda_g f} = \pi^0(g)\overbrace{f}$ for all $g \in G$ and $f \in L^2(G)$, we see that $\overbrace{P_K f_j}$ is K-invariant also. However, in $\mathcal{H}_0 \cong L^2(K)$, there is only one K-invariant function up to scalars. It follows that

$$\overbrace{P_K f_j} = \|\,\overbrace{P_K f_j}\,\|_{\mathcal{H}_0} f_0$$

with

$$\| \overbrace{P_K f_j} \|_{\mathcal{H}_0} = \| \overbrace{P_K f_j} \|_{L^2(K)} \leqslant \|f_j\|_2$$

for $j = 1, 2$.

Recalling the definition of the Harish-Chandra spherical function

$$\Xi(g) = \langle \pi_g^0 f_0, f_0 \rangle_{\mathcal{H}_0},$$

and putting this into the above estimate, we obtain

$$\begin{aligned} |\langle \lambda_*(\psi) f_1, f_2 \rangle| &\leqslant \int_G |\psi(g)| \Xi(g) \,\mathrm{d}m(g) \|f_1\|_2 \|f_2\|_2 \\ &\leqslant \|\psi\|_{2-\varepsilon} \|\Xi\|_q \|f_1\|_2 \|f_2\|_2 \end{aligned}$$

by assuming $\psi \in L^{2-\varepsilon}(G)$ for some $\varepsilon > 0$ and letting $q \in (2, \infty)$ be the Hölder conjugate of $p = 2 - \varepsilon$. This shows that

$$\|\lambda_*(\psi)\|_{\mathrm{op}} \leqslant \underbrace{\|\Xi\|_q}_{<\infty} \|\psi\|_{2-\varepsilon}$$

by the integrability properties of Ξ in Proposition 8.39.

Finally, let π be a tempered representation. Then we also have

$$\|\pi_*(\psi)\| \leqslant \|\lambda_*(\psi)\|_{\mathrm{op}} \leqslant \|\Xi\|_q \|\psi\|_{2-\varepsilon}$$

by the characterization ($\prec_{\mathrm{op}}$) of weak containment in Theorem 4.28. □

8.8.1 The Kunze–Stein Phenomenon as an Ergodic Theorem

We now suppose that $G = \mathrm{SL}_2(\mathbb{R})$ acts continuously on X, preserving a locally finite measure μ. We write π for the induced Koopman representation on $L^2_\mu(X)$ (see Proposition 1.3). Recall that in this case we have

$$\pi_*(\psi) f(x) = \int_G \psi(g) f(g^{-1} \cdot x) \,\mathrm{d}m(g)$$

for almost every $x \in X$. Moreover, if $\psi = \frac{1}{m(S)} \mathbb{1}_S$ for a measurable set $S \subseteq G$ with finite measure, then

$$\pi_*(\psi) f(x) = \frac{1}{m(S)} \int_S f(g^{-1} \cdot x) \,\mathrm{d}m(g)$$

is called an *ergodic average*. We recall that the action is called *ergodic* if any G-invariant measurable subset $B_X \subseteq X$ satisfies $\mu(B_X) = 0$ or $\mu(X \smallsetminus B_X) = 0$.

Corollary 8.52 (Kunze–Stein in ergodic theory). *Let π be a Koopman unitary representation of $G = \mathrm{SL}_2(\mathbb{R})$ as above. Assume in addition that π has integrability exponent $p_\pi < \infty$. Let $\ell \in \mathbb{N}$ satisfy $\frac{p_\pi}{2\ell} < 2$. Then we have*

$$\left\| \pi_*(\psi)|_{(\mathcal{H}_\pi^G)^\perp} \right\|_{\mathrm{op}} \ll_\varepsilon \|\psi\|_{2-\varepsilon}^{\frac{1}{2\ell}}.$$

for any $\varepsilon > 0$ and $\psi \in L^1(G) \cap L^{2-\varepsilon}(G)$ with

- $\psi \geqslant 0$;
- $\int \psi \, \mathrm{d}m = 1$;
- $\psi(kgk') = \psi(g)$ *for all* $k, k' \in K$ *and* $g \in G$.

Moreover, for a measurable set $S = KSK \subseteq G$ with $m(S) < \infty$, we have

$$\left\| \frac{1}{m(S)} \int_S f(g^{-1} \bullet x) \, \mathrm{d}m(g) - I_f \right\|_2 \ll_\varepsilon m(S)^{-\frac{1}{4\ell}+\varepsilon} \|f\|_2$$

for all $\varepsilon > 0$. Here $I_f \in L^2_\mu(X)$ is given by

$$I_f = \begin{cases} \left(\int f \, \mathrm{d}\mu \right) \mathbb{1}_X & \text{if } \mu(X) = 1, \\ 0 & \text{if } \mu(X) = \infty. \end{cases}$$

Proof. We let $d = 2\ell$. Proposition 8.44 shows that $\pi|^{\otimes d}_{(\mathcal{H}_\pi^G)^\perp}$ is tempered. Let $f_1, f_2 \in (\mathcal{H}_\pi^G)^\perp$ be real-valued. Then

$$\begin{aligned} |\langle \pi_*(\psi) f_1, f_2 \rangle|^d &= \left| \int_G \left\langle \pi_g f_1, f_2 \right\rangle \psi(g) \, \mathrm{d}m(g) \right|^d \\ &\leqslant \left| \int_G \underbrace{\left\langle \pi_g f_1, f_2 \right\rangle^d}_{= \langle \pi_g^{\otimes d} f_1^{\otimes d}, f_2^{\otimes d} \rangle} \psi(g) \, \mathrm{d}m(g) \right| \end{aligned}$$

by Jensen's inequality for the real-valued function $g \mapsto \langle \pi_g f_1, f_2 \rangle$, the convexity of the function $\mathbb{R} \ni t \mapsto t^d$ (for $d = 2\ell$ even), and the probability measure $\psi \, \mathrm{d}m$ on G. Moreover, Theorem 8.51 gives

$$\left| \left\langle \pi_*^{\otimes d}(\psi) f_1^{\otimes d}, f_2^{\otimes d} \right\rangle \right| \ll_p \|\psi\|_{2-\varepsilon} \|f_1^{\otimes d}\| \|f_2^{\otimes d}\|.$$

Taking the dth root, and combining the two estimates, we obtain

$$|\langle \pi_*(\psi) f_1, f_2 \rangle| \ll_p \|\psi\|_{2-\varepsilon}^{\frac{1}{d}} \|f_1\|_2 \|f_2\|_2.$$

Using linearity, this extends to complex-valued functions, and implies the first part of the corollary.

If $\mu(X)=\infty$ and the action is ergodic, then $\mathcal{H}_\pi^G=\{0\}$ and hence we can apply the above to $\psi=\frac{1}{m(S)}\mathbb{1}_S$. Since

$$\|\psi\|_{2-\varepsilon}^{\frac{1}{2\ell}}=\Big(m(S)^{-(2-\varepsilon)}m(S)\Big)^{\frac{1}{(2-\varepsilon)2\ell}}=m(S)^{\frac{-1+\varepsilon}{(2-\varepsilon)2\ell}}$$

and $\varepsilon>0$ is arbitrary, this implies the infinite volume case. If $\mu(X)=1$, then $\mathcal{H}_\pi^G=\mathbb{C}\mathbb{1}_X$ and we can apply the above to

$$f-\Big(\int f\,\mathrm{d}\mu\Big)\mathbb{1}_X\in\big(\mathcal{H}_\pi^G\big)^\perp.$$

□

We refer to the monograph of Gorodnik and Nevo [30] for a more thorough discussion of the applications of the Kunze–Stein phenomenon in ergodic theory, and its connection to number theory.

Exercise 8.53 (Counting lattice points). Suppose $\Gamma<\mathrm{SL}_2(\mathbb{R})$ is a lattice, and let $X=\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$. Suppose also that the unitary representation π^X on $L^2(X)$ has decay exponent $\kappa>0$. Use Corollary 8.52 to prove an asymptotic counting result for $\{\gamma\in\Gamma\mid \|g\|\leqslant R\}$ with a concrete error term depending on κ.