Hints for Selected Exercises

Exercise 1.3.3 (p. 30): Take an ergodic circle rotation $R_\alpha : \mathbb{T} \to \mathbb{T}$ and choose a measurable set $A \subseteq \mathbb{T}$ for which the map $\phi : \mathbb{T} \to X$ defined by

$$\phi(x)_n = \mathbb{1}_A(R_\alpha^n(x))$$

gives a suitable measure $\mu = \phi^* m_{\mathbb{T}}$ with the required properties.

Exercise 1.3.7 (p. 31): It is enough to show that

$$h_\mu(T, \xi) \leqslant \frac{1}{|F|} H_\mu \left(\bigvee_{n \in F} T^{-n} \xi \right)$$

for a finite set $F = \{n_1, \dots, n_k\} \subseteq \mathbb{N}$, with $n_1 < n_2 < \dots < n_k$. We have

$$\begin{aligned} \frac{1}{|F|} H_\mu \left(\bigvee_{n \in F} T^{-n} \xi \right) &= \frac{1}{k} \left(H_\mu \left(T^{-n_k} \xi \right) + H_\mu \left(T^{-n_{k-1}} \xi \middle| T^{-n_k} \xi \right) \right. \\ &\qquad \left. + \dots + H_\mu \left(T^{-n_1} \xi \middle| T^{-n_2} \xi \vee \dots \vee T^{-n_k} \xi \right) \right) \\ &\geqslant H_\mu \left(T^{-n_i} \xi \middle| T^{-n_{i+1}} \xi \vee \dots \vee T^{-n_k} \xi \right) \\ &= H_\mu \left(\xi \middle| T^{n_i - n_{i+1}} \xi \vee \dots \vee T^{n_i - n_k} \xi \right) \end{aligned}$$

for some i. Now use Proposition 1.15.

Exercise 2.2.3 (p. 63): For a given $\varepsilon > 0$ approximate ξ by a finite partition $\eta \subseteq \sigma(\xi)$ such that $H(\xi \middle| \eta) < \varepsilon$ and apply Proposition 2.12.

Exercise 2.4.2 (p. 74): A finite partition ξ is measurable with respect to $\mathscr{P}(T)$ if and only if $h_\mu(T, \xi) = 0$. Use this characterization and treat the cases $n = -1$ and $n > 1$ separately.

Exercise 2.4.4 (p. 74): To obtain (a), phrase the claim in terms of a nested chain of Hilbert subspaces $H_n = L^2(\xi_n^\infty) \supseteq H_{n+1}$ with

$$\bigcap_{n=1}^{\infty} H_n = H_\infty = L^2(\mathscr{T}).$$

For (c), consider $T^{-n} A \in \xi_n^\infty \subseteq \xi_1^\infty$, and whether or not it intersects A non-trivially. For (d), let $B \subseteq \xi_1^\infty$ be the complement of 'all positive measure atoms' of ξ_1^∞. Show that $VL^\infty(\xi_1^\infty) \subseteq V$ and use partitions of B into ℓ sets of measure $\frac{\mu(B)}{\ell}$ to 'cut' an arbitrary $f \in V$ into at least $\lceil \frac{\mu(\{x|f(x)\neq 0\})}{\mu(B)}\ell \rceil$ many linearly independent factors.

Exercise 2.5.1 (p. 78): Let $\mathsf{Z}_{\mathscr{P}}$ be the Pinsker factor of

$$(X \times Y, \mathscr{B}_X \otimes \mathscr{B}_Y, T \times S, \rho)$$

and let $\pi_{\mathscr{P}} : X \times Y \to Z_{\mathscr{P}}$ denote the factor map. Consider the image of ρ under $\pi_X \times \pi_{\mathscr{P}}$, which gives a joining between X and $\mathsf{Z}_{\mathscr{P}}$. Now apply Theorem 2.31. In order to relate the conclusion of the theorem in this case to the desired conclusion, consider the measure

$$(\pi_X \times \pi_Y \times \pi_{\mathscr{P}})_* \rho = \int_{Z_{\mathscr{P}}} \rho_z^{\mathscr{P}} \,\mathrm{d}(\pi_{\mathscr{P}})_*\rho(z).$$

Exercise 2.5.2 (p. 78): Show that the system $(\mathbb{T}^r \times \mathbb{T}^r, \mathscr{B}_{\mathbb{T}^r \times \mathbb{T}^r}, \mu, T_A \times T_A)$ is measurably isomorphic to $(\mathbb{T}^r \times \mathbb{T}^r, \mathscr{B}_{\mathbb{T}^r \times \mathbb{T}^r}, m_{\mathbb{T}^r} \times \rho, T_A \times T_A)$.

Exercise 3.1.1 (p. 93): For (a) see Lemma 4.3. For (b), show that every element P in

$$\bigvee_{i=0}^{n-1} T_A^{-i}\xi$$

also belongs to

$$\bigvee_{j=0}^{m-1} T^{-j}\widetilde{\xi},$$

where

$$m = m(n, P) = \sum_{i=0}^{n-1} r_A(T_A^i(x)).$$

Apply Theorem 3.1 for almost every $x \in A$ both for T_A and the partition ξ and for T with the partition $\widetilde{\xi}$. Finally use Kac's theorem (Exercise A.1.2(a)) to calculate

$$\lim_{n\to\infty} \frac{m(n, P)}{n}.$$

For (c) use (b) together with Theorem 2.20.

Exercise 3.1.2 (p. 93): For (a) fix some $\delta > 0$, apply Theorem 3.1 to find some N and measurable set X_N of measure $> 1 - \frac{\varepsilon}{2}$ such that for all $n \geqslant N$ and $x \in X_N$ we have

$$e^{-(h_\mu(T,\xi)+\delta)n} \leqslant \mu([x]_{\xi_0^{n-1}}) \leqslant e^{-(h_\mu(T,\xi)-\delta)n}.$$

Any set of measure $> \varepsilon$ has to intersect X_N in a set of measure $> \frac{\varepsilon}{2}$. For (b) use instead Theorem 3.2 to show the limit equals $\operatorname{ess\,sup} h_{\mu_x^{\mathscr{E}}}(T, \xi)$.

Exercise 3.4.3 (p. 107): Fix an $\varepsilon \in (0, \frac{1}{4})$, and apply Theorem 3.12 to find a set $Y \subseteq X$ with μ-measure greater than $1 - \varepsilon$ and some $\delta_0 > 0$ for which

$$\mathrm{e}^{-(h_\mu(T)+\varepsilon)n} \leqslant \mu\left(D(y, n, \delta)\right) \leqslant \mathrm{e}^{-(h_\mu(T)-\varepsilon)n} \tag{4.8}$$

for all $n \geqslant N(\delta)$ and $\delta \in (0, \delta_0]$. The lower bound in (4.8) implies that Y can be covered with fewer than $\lfloor \mathrm{e}^{(h_\mu(T)+\varepsilon)n} \rfloor$ Bowen $(n, 2\delta)$-balls for $n \geqslant N(\delta)$ and $\delta \in (0, \delta_0)$. If, on the other hand, $\delta \in (0, \delta_0)$ and $Z \subseteq X$ has measure greater than $\frac{1}{2}$ and is covered by $M(n, \delta/2)$ Bowen $(n, \delta/2)$-balls, then

$$\mu(Y \cap Z) > \tfrac{1}{2} - \varepsilon > \tfrac{1}{4}$$

and the upper bound in (4.8) shows that $M(n, \delta/2) \geqslant \frac{1}{4} e^{(h_\mu(T)-\varepsilon)n}$. For (c), the right-hand side of the formula in (a) is given by

$$\operatorname*{ess\,sup}_{x \in X} h_{\mu_x^{\mathscr{E}}}(T).$$

The argument needed is a refinement of that given above.

Exercise 5.1.4 (p. 129): For the second claim show first that a partition ξ whose elements have small enough diameter still satisfies $h_\mu(T) = h_\mu(T, \xi)$ for all $\mu \in \mathscr{M}^T$. For this analyse first the atoms of $\xi_{-\infty}^{\infty}$. Assume μ is non-atomic and ergodic and let η be a countable partition such that for every $\varepsilon > 0$ there exists some $P \in \eta$ with $\mu(P) > 0$ and diameter less than ε. Show that $\xi \vee \eta$ is a generator for T_A and μ. Since $H_\mu(\eta)$ can be made arbitrarily small, the claim follows from Theorem 2.33.

Exercise 6.1.7 (p. 161): There are several ways to see this; one approach is given by Zhou [214].

Exercise 7.2.2 (p. 188): Using a continuous surjective map from $\{0,1\}^{\mathbb{N}}$ onto $[0,1]$ to construct a topological factor map from the shift map on $(\{0,1\}^{\mathbb{N}})^{\mathbb{Z}}$ onto the shift map on $[0,1]^{\mathbb{Z}}$.

Exercise 8.2.1 (p. 202): Use solenoids to construct the invertible extension in each case, and argue as in the proof of Theorem 8.7.

Exercise 8.3.3 (p. 207): By Theorem 2.27 and expansiveness, we have

$$\mathscr{P}_\mu(T_A) \underset{\mu}{=} \bigcap_{n=1}^{\infty} \bigvee_{k=n}^{\infty} T_A^{-k}\xi = \mathscr{Y}(\xi).$$

Prove that for some $N \in \mathscr{B}(V^-)$, all $x \in \mathbb{T}^r$ and all $v \in V^-$ we have

$$[x]_{\mathscr{Y}(\xi)} = [x+v]_{\mathscr{Y}(\xi)}.$$

Deduce that $B \smallsetminus N \in \mathscr{B}(V^-)$ for all $B \in \mathscr{Y}(\xi)$. For the reverse inclusion, show that for any $B \subseteq \mathscr{B}(V^-)$ we have $[x]_{\mathscr{Y}(\xi)} \subseteq B$ or $[x]_{\mathscr{Y}(\xi)} \subseteq X \smallsetminus B$. Deduce that $B \underset{\mu}{\in} \mathscr{Y}(\xi)$ and so $\mathscr{B}(V^-) \underset{\mu}{\subseteq} \mathscr{Y}(\xi)$.

Exercise 8.3.4 (p. 207): By Theorem 2.29,

$$\mathscr{P}_\mu(T_A) \underset{\mu}{=} \bigvee_{k \geqslant 1} \mathscr{Y}(\xi_k)$$

if $\xi_k \nearrow \mathscr{B}(\mathbb{T}^r)$. Choose ξ_k with $\max_{P \in \xi_k} \operatorname{diam}(P) < \frac{1}{k}$ and with the properties from Exercise 8.3.2. Now argue as in the first half of Exercise 8.3.3. For the second half of the argument, use Lemma 8.17.

Exercise 8.3.5 (p. 207): Argue as in Exercise 8.3.4.

Exercise 8.5.1 (p. 219): In the proof of Theorem 8.19 replace the centralizer by the group H whose Lie algebra is the direct sum of all generalized eigenspaces of Ad_a with absolute value one. Use disjointness of T with respect to m_X and irrational circle rotations to show that the set of generic points for T and m_X is again invariant under H.

Exercise 9.2.2 (p. 248): Express the given conditions in terms of properties of the Fourier transform $\widehat{\rho}$ of the unknown measure, and try to use these to show that $\widehat{\mu} = \widehat{m_{\bullet\bullet}}$.

Exercise 9.3.1 (p. 262): Part (a) is a very special case of the argument in Proposition 9.15. However, in part (b) ergodicity of the map T_3 is used, while the correct assumption is ergodicity of the $\mathbb{Z}^2$-action (and invariance of the function under T_2 is not easy to obtain). If we are willing to assume ergodicity for T_3 (but this unnatural assumption would prevent many applications, for example Exercises 9.3.2–9.3.3) then the outline works.

Exercise 9.3.3 (p. 262): Use results from Section 5.3.1 as well as Section 9.3.

Exercise 9.3.4 (p. 262): One way to see this is to show that the invariance forces the Fourier transform of the invariant measure to coincide with the Fourier transform of Lebesgue measure, if it is not the point measure at 0.

Exercise A.1.1 (p. 272): Consider the set $\bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} T^{-n}B$.

Exercise A.1.2 (p. 272): Consider the set $A_n = \{x \in A \mid r_A(x) = n\}$ and the sets $T^k A_n$ for $k = 0, \dots, n-1$ and $n = 1, 2 \dots$.

Exercise A.1.4 (p. 273): Use the characterization of weak-mixing in terms of eigenfunctions to show first that T is weak-mixing with respect to μ. Deduce that $T \times T$ on $X \times X$ is ergodic with respect to $\mu \times \mu$. Then define a sequence of measures μ_n on $X \times X$ by the property that

$$\int_{X \times X} f(x, y) \,\mathrm{d}\mu_n(x, y) = \int_X f(x, T^n x) \,\mathrm{d}\mu(x)$$

for all $f \in L^\infty(\mu \times \mu)$. Show that the only limit point of the sequence (μ_n) must be $\mu \times \mu$, and use the fact that the sequence must have a convergent subsequence to deduce that T is mixing.

Exercise A.4.3 (p. 286): Notice that a measure-preserving system $(X, \mathscr{B}, \mu, T)$ is weak-mixing if and only if

$$\frac{1}{N} \sum_{n=0}^{N-1} \left| \mu\left(T^{-n} A \cap B\right) - \mu(A)\mu(B) \right| \longrightarrow 0$$

as $N \to \infty$ (see [52, Def. 2.35, Th. 2.36]).

Exercise A.6.3 (p. 291): By assumption $\mathscr{B}$ is countably generated, so we may choose a sequence of sets $B_1, B_2, \dots$ so that together they generate $\mathscr{B}$. Assume that

$$\mu(B \triangle TB) = 0$$

for all $B \subseteq F$. Apply this to $B_i \cap F$ and collect the null sets.