Chapter 6
Lie Algebras, $SU(2,\mathbb{R})$, and Smooth Vectors

We continue the discussion of unitary representations and develop some general tools for unitary representations of Lie groups. In particular this will allow us to describe the representation theory of the compact group $SU_2(\mathbb{R})$, which represents an important example. For this we will sometimes use the following notational conventions in addition to the standing assumptions and notations of Section 1.1.

- If G is a real not necessarily connected Lie group, then we write $\mathfrak{g} = \operatorname{Lie} G$ for its Lie algebra and write $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathfrak{g}$ for its elements. Recall that there is a smooth map $\exp\colon \mathfrak{g} \to G$ with a local inverse $\log\colon B_\delta^G(I) \to \mathfrak{g}$ defined on some neighbourhood $B_\delta^G(I)$ of the identity $I \in G$ with $\delta > 0$.
- We will use the letters s, t to denote real numbers.

6.1 Finite-dimensional Representation Theory of SL(2)

For the classification of simple (and semi-simple) Lie groups and their finite-dimensional representations the most important Lie group to understand is $SL_2(\mathbb{R})$. As we will see later (starting in Chapter 8), this remains true for the theory of unitary representations. As a warm-up for this discussion as well as because of its independent interest, we study in this chapter the twin sibling $SU_2(\mathbb{R})$ of $SL_2(\mathbb{R})$. In fact, as we will explain shortly these two groups are strongly related: $SU_2(\mathbb{R})$ is the *compact real form* and $SL_2(\mathbb{R})$ is the *split real form* of $SL_2(\mathbb{C})$, and they have identical descriptions of their finite-dimensional representations (which can be made unitary for $SU_2(\mathbb{R})$ but are not unitary except in trivial cases for $SL_2(\mathbb{R})$ — see Exercise 1.66 and 1.69). As before, we will always study representations on complex vector spaces.

6.1.1 The Lie Groups $\mathbf{SU(2,\mathbb{R})}$ and $\mathbf{SL(2,\mathbb{R})}$

We recall the definition of the real Lie group

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{22}(\mathbb{R}) \mid \det g = ad - bc = 1 \right\},$$

called the *special linear group in* 2 *dimensions*, and its Lie algebra

$$\mathfrak{sl}_2(\mathbb{R}) = \{\mathbf{m} \in \mathrm{Mat}_{22}(\mathbb{R}) \mid \mathrm{tr}\,\mathbf{m} = 0\},$$

which contains all elements of the form

$$\mathbf{m} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$. As in any Lie group, the Lie algebra has the property that $\exp\bigl(\mathfrak{sl}_2(\mathbb{R})\bigr)$ is an open neighbourhood of the identity $I \in \mathrm{SL}_2(\mathbb{R})$ and so can be used to describe $\mathrm{SL}_2(\mathbb{R})$ locally.

This, and many other facts concerning the connection between Lie groups and their Lie algebras, are easy to verify for closed linear Lie groups $G < \mathrm{GL}_n(\mathbb{R})$ using only the properties of the exponential map

$$\begin{aligned} \exp\colon \mathrm{Mat}_{n,n}(\mathbb{R}) &\longrightarrow \mathrm{GL}_n(\mathbb{R}) \\ m &\longmapsto \exp(m) = \sum_{k=0}^{\infty} \tfrac{1}{k!} m^k. \end{aligned}$$

We note that these cases will suffice for all of our discussions.

We will frequently use the basis of $\mathfrak{sl}_2(\mathbb{R})$ given by

$$\mathbf{a} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{6.1}$$

which we will refer to as the $\mathfrak{sl}_2$-triple. These three elements satisfy the following relations

$$\left.\begin{aligned} [\mathbf{a},\mathbf{e}] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\mathbf{e} \\ [\mathbf{a},\mathbf{f}] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\mathbf{f} \\ [\mathbf{e},\mathbf{f}] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{a}. \end{aligned}\right\} \tag{6.2}$$

In other words, with respect to the map $\mathrm{ad}_{\mathbf{a}} = [a, \cdot]\colon \mathfrak{sl}_2(\mathbb{R}) \to \mathfrak{sl}_2(\mathbb{R})$, $\mathbf{e}$ is an eigenvector with eigenvalue 2, $\mathbf{f}$ is an eigenvector with eigenvalue -2, $\mathbf{a}$ is an eigenvector with eigenvalue 0, and $\mathbf{e}$ and $\mathbf{f}$ together generate $\mathbf{a}$. As a consequence, the number 2 will be quite prevalent in the representation theory of $\mathrm{SL}_2(\mathbb{R})$. Much of what we wish to discuss here will use these simple relations.

We will also use the complex Lie group

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{22}(\mathbb{C}) \mid \det g = ad - bc = 1 \right\},$$

and, more specifically, its complex Lie algebra

$$\mathfrak{sl}_2(\mathbb{C}) = \{\mathbf{m} \in \mathrm{Mat}_{22}(\mathbb{C}) \mid \mathrm{tr}\, \mathbf{m} = 0\},$$

which also has the Lie algebra elements $\mathbf{a}$, $\mathbf{e}$, and $\mathbf{f}$ as a basis over $\mathbb{C}$. A more formal way of stating this is to say that there is an isomorphism

$$\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}, \tag{6.3}$$

where $\mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ is the complex Lie algebra obtained by bilinearly extending the Lie bracket map from $\mathbb{R}$ to $\mathbb{C}$. Because of the isomorphism in (6.3) we also say that $\mathfrak{sl}_2(\mathbb{R})$ is a *real form* or, more specifically, the *split real form* of $\mathfrak{sl}_2$. We now turn our attention to the only other real form (up to isomorphism).

The *special unitary group*

$$\mathrm{SU}_2(\mathbb{R}) = \{g \in \mathrm{Mat}_{22}(\mathbb{C}) \mid g^* g = I, \det g = 1\},$$

is a real Lie group with Lie algebra

$$\mathfrak{su}_2(\mathbb{R}) = \{\mathbf{m} \in \mathrm{Mat}_{22}(\mathbb{C}) \mid \mathbf{m}^* + \mathbf{m} = 0, \mathrm{tr}\, \mathbf{m} = 0\}.$$

We note that these are not a complex Lie group and Lie algebra since the adjoint operation is semi-linear over $\mathbb{C}$. The elements of $\mathfrak{su}_2(\mathbb{R})$ have the form

$$\mathbf{m} = \begin{pmatrix} a\mathrm{i} & b\mathrm{i} - c \\ b\mathrm{i} + c & -a\mathrm{i} \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$ and we may use the basis

$$\mathbf{b}_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \ \mathbf{b}_2 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \mathbf{b}_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{6.4}$$

with the relations

$$\left.\begin{aligned}
[\mathbf{b}_1, \mathbf{b}_2] &= \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} - \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = 2\mathbf{b}_3 \\
[\mathbf{b}_2, \mathbf{b}_3] &= \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = 2\mathbf{b}_1 \\
[\mathbf{b}_3, \mathbf{b}_1] &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} - \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2\mathbf{b}_2.
\end{aligned}\right\} \qquad (6.5)$$

Since the elements in (6.4) are also a basis of $\mathfrak{sl}_2(\mathbb{C})$ over $\mathbb{C}$, we obtain once more an isomorphism

$$\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

so that $\mathfrak{su}_2(\mathbb{R})$ is a real form of $\mathfrak{sl}_2(\mathbb{C})$ which is called the *compact real form* since $\mathrm{SU}_2(\mathbb{R})$ is compact.

To better understand $\mathfrak{su}_2(\mathbb{R})$ and $\mathrm{SU}_2(\mathbb{R})$, we may also present these in equivalent ways as in the next lemma.

Lemma 6.1 (Two isomorphisms). (a) *The vector space* $\mathbb{R}^3$ *equipped with the cross product is a Lie algebra isomorphic to* $\mathfrak{su}_2(\mathbb{R})$.
(b) *The sphere* $\mathbb{S}^3 \subseteq \mathbb{H}$ *inside the four-dimensional Hamiltonian quaternions* $\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ *forms a real Lie group with respect to multiplication so that* $\mathbb{S}^3 \cong \mathrm{SU}_2(\mathbb{R})$ *as real Lie groups.*

Proof of Lemma 6.1(a). The cross product map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} y\gamma - z\beta \\ z\alpha - x\gamma \\ x\beta - y\alpha \end{pmatrix}$$

for

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathbb{R}^3$$

is bilinear and antisymmetric, just as a Lie bracket is. Also note that we have the relations $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$, and $e_3 \times e_1 = e_2$. We define a linear map $\varphi\colon \mathbb{R}^3 \to \mathfrak{su}_2(\mathbb{R})$ by

$$\begin{aligned}
\varphi(e_1) &= \tfrac{1}{2}\mathbf{b}_1, \\
\varphi(e_2) &= \tfrac{1}{2}\mathbf{b}_2, \\
\varphi(e_3) &= \tfrac{1}{2}\mathbf{b}_3.
\end{aligned}$$

Then φ is a linear isomorphism which satisfies $\varphi(a \times b) = [\varphi(a), \varphi(b)]$. In fact this follows first for a and b being any two basis vectors in positive order by dividing the relations in (6.5) by 4. By antisymmetry of the cross product and the Lie bracket this extends to any two basis vectors. Finally, by bi-linearity of the cross product and the Lie bracket this extends to all $a, b \in \mathbb{R}^3$. As $\mathfrak{su}_2(\mathbb{R})$

is a Lie algebra, the same therefore holds for $\mathbb{R}^3$ with the cross product, proving (a). □

Proof of Lemma 6.1(b). We recall first that the Hamiltonian quaternions are defined by

$$\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are formal symbols satisfying the relations[†]

$$\begin{aligned} \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -1, \\ \mathbf{ij} = -\mathbf{ji} &= \mathbf{k}, \\ \mathbf{jk} = -\mathbf{kj} &= \mathbf{i}, \end{aligned}$$

and

$$\mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

We may identify $\mathbb{H}$ with

$$\left\{ \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C} \right\} \subseteq \mathrm{Mat}_{22}(\mathbb{C})$$

because the matrices $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ in (6.4) satisfy these relations so that

$$\varphi\colon \mathbb{H} \ni x + a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \longmapsto xI + a\mathbf{b}_1 + b\mathbf{b}_2 + c\mathbf{b}_3 = \begin{pmatrix} x + a\mathrm{i} & b\mathrm{i} - c \\ b\mathrm{i} + c & x - a\mathrm{i} \end{pmatrix} \in \mathrm{Mat}_{2,2}(\mathbb{C})$$

is an algebra isomorphism to a subalgebra of $\mathrm{Mat}_{2,2}(\mathbb{C})$.

The norm operator defined by

$$N(x + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = x^2 + a^2 + b^2 + c^2 = \det \varphi(x + a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

for $x, a, b, c \in \mathbb{R}$ satisfies $N(\mathbf{gh}) = N(\mathbf{g})N(\mathbf{h})$ for all $\mathbf{g}, \mathbf{h} \in \mathbb{H}$. It follows that

$$\mathbb{S}^3 = \{\mathbf{g} \in \mathbb{H} \mid N(\mathbf{g}) = 1\}$$

is a Lie group under multiplication which is mapped under φ to

$$\varphi(\mathbb{S}^3) = \left\{ g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \mid z, w \in \mathbb{C}, \det g = |z|^2 + |w|^2 = 1 \right\}.$$

Note that

$$g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} \in \varphi(\mathbb{S}^3)$$

satisfies

[†] These relations are easy to reconstruct because of their symmetry under cyclic permutations.

$$g^*g = \begin{pmatrix} \overline{z} & -\overline{w} \\ w & z \end{pmatrix} \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix} = \begin{pmatrix} |z|^2 + |w|^2 & 0 \\ 0 & |z|^2 + |w|^2 \end{pmatrix} = I$$

and so $\varphi(\mathbb{S}^3) \subseteq \mathrm{SU}_2(\mathbb{R})$. Conversely, if

$$g = \begin{pmatrix} z & z_1 \\ w & w_1 \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R}),$$

then $\left\| \begin{pmatrix} z \\ w \end{pmatrix} \right\| = 1$ and

$$0 = \left\langle \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} -\overline{w} \\ \overline{z} \end{pmatrix} \right\rangle$$

implies that $\begin{pmatrix} z_1 \\ w_1 \end{pmatrix} = \alpha \begin{pmatrix} -\overline{w} \\ \overline{z} \end{pmatrix}$ for some $\alpha \in \mathbb{C}$. Taking the determinant this gives

$$1 = \det g = \det \begin{pmatrix} z & -\alpha\overline{w} \\ w & \alpha\overline{z} \end{pmatrix} = \alpha|z|^2 + \alpha|w|^2 = \alpha.$$

This proves that $\varphi(\mathbb{S}^3) = \mathrm{SU}_2(\mathbb{R})$, and hence part (b) of the lemma. □

The following lemma describes the above Lie groups from a topological point of view.

Lemma 6.2 (Topological properties). (a) *The groups* $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SU}_2(\mathbb{R})$, *and* $\mathrm{SL}_2(\mathbb{C})$ *are connected.*

(b) *The Lie groups* $\mathrm{SU}_2(\mathbb{R})$ *and* $\mathrm{SL}_2(\mathbb{C})$ *are simply connected.*

(c) *The universal cover* $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ *of* $\mathrm{SL}_2(\mathbb{R})$ *is a* $\mathbb{Z}$*-cover of* $\mathrm{SL}_2(\mathbb{R})$.

Proof. For $\mathrm{SU}_2(\mathbb{R})$ both (a) and (b) follow easily from Lemma 6.1(b) since $\mathrm{SU}_2(\mathbb{R}) \cong \mathbb{S}^3$ is connected and simply connected.

The facts concerning $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ follow from the Iwasawa decomposition of these groups, as we will now show.

For $\mathrm{SL}_2(\mathbb{R})$, let

$$A = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mid t > 0 \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$$

and $K = \mathrm{SO}_2(\mathbb{R})$. Then every $g \in \mathrm{SL}_2(\mathbb{R})$ has a unique decomposition $g = kan$ with $k \in K$, $a \in A$, and $n \in N$ by the Iwasawa decomposition (alternatively, by the Gram–Schmidt procedure). This gives a homeomorphism $\mathrm{SL}_2(\mathbb{R}) \cong KAN \cong \mathbb{S}^1 \times \mathbb{R}^2$ which proves the claims in (a) and (c) for $\mathrm{SL}_2(\mathbb{R})$.

For $\mathrm{SL}_2(\mathbb{C})$ we have A as before,

$$N = \left\{ \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} \mid \alpha \in \mathbb{C} \right\},$$

and $K = \mathrm{SU}_2(\mathbb{R}) \cong \mathbb{S}^3$, which gives a homeomorphism

$$\mathrm{SL}_2(\mathbb{C}) = KAN \cong \mathbb{S}^3 \times \mathbb{R}^3,$$

and hence the remaining claims in the lemma. □

6.1.2 A Quick Review of Lie Algebra Representations

Let us start by recalling that for a representation $\rho\colon G \to \mathrm{GL}(W)$ on a finite-dimensional vector space W of a (real or complex) Lie group G we can take the derivative of ρ at the identity to obtain a representation of the Lie algebra $\mathfrak{g}$ of G. We will denote this map by

$$\mathrm{D}\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(W) = \mathrm{Hom}(W)$$

and show that $\mathrm{D}\rho$ is a Lie algebra homomorphism, meaning that it satisfies

$$\mathrm{D}\rho\big([\mathbf{b}, \mathbf{c}]\big) = [\mathrm{D}\rho(\mathbf{b}), \mathrm{D}\rho(\mathbf{c})] \tag{6.6}$$

for all $\mathbf{b}, \mathbf{c} \in \mathfrak{g}$. For this, consider first some $g \in G$ and $\mathbf{c} \in \mathfrak{g}$. Then, by definition, the adjoint representation applied to g and $\mathbf{c}$ is given by

$$\mathrm{Ad}_g(\mathbf{c}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g \exp(t\mathbf{c}) g^{-1}.$$

For the homomorphism ρ and its derivative $\mathrm{D}\rho$, this gives

$$\begin{aligned} \mathrm{Ad}_{\rho_g}(\mathrm{D}\rho(\mathbf{c})) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_g \exp(t\,\mathrm{D}\rho(\mathbf{c}))\rho_g^{-1} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho_g \rho(\exp(t\mathbf{c}))\rho_g^{-1} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho\big(g \exp(t\mathbf{c}) g^{-1}\big) = \mathrm{D}\rho\,\mathrm{Ad}_g(\mathbf{c}), \end{aligned}$$

where, after the definition, we used in turn the categorical property

$$\rho \circ \exp = \exp \circ \mathrm{D}\rho$$

of the exponential map, the homomorphism property of ρ, the chain rule for differentiation, and once again the definition of $\mathrm{Ad}_g(\mathbf{c})$. If now $\mathbf{b}, \mathbf{c} \in \mathfrak{g}$ then we can set $g = \exp(t\mathbf{b})$ and use the defining property

$$[\mathbf{b},\mathbf{c}] = \mathrm{ad}_{\mathbf{b}}(\mathbf{c}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\exp(t\mathbf{b})}(\mathbf{c})$$

in the same way. In fact we have

$$\begin{aligned}[][\mathrm{D}\rho(\mathbf{b}), \mathrm{D}\rho(\mathbf{c})] &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\exp(t\,\mathrm{D}\rho(\mathbf{b}))}\big(\mathrm{D}\rho(\mathbf{c})\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Ad}_{\rho(\exp(t\mathbf{b}))}\big(\mathrm{D}\rho(\mathbf{c})\big) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{D}\rho\,\mathrm{Ad}_{\exp(t\mathbf{b})}(\mathbf{c}) = \mathrm{D}\rho[\mathbf{b},\mathbf{c}].\end{aligned}$$

Hence the study of finite-dimensional representations of G leads to the study of representations of $\mathfrak{g}$. Below we will denote the representation of $\mathfrak{g}$ induced from a representation ρ of G also by ρ.

Let $V \subseteq W$ be a subspace of a finite-dimensional vector space W carrying a representation ρ of a Lie group G. If V is invariant under $\rho(G)$, then it is also invariant under $\rho(\mathfrak{g})$. If G is connected, the reverse also holds. This follows, since $\rho(c)V \subseteq V$ for $c \in \mathfrak{g}$ implies that

$$\rho\big(\exp(c)\big)V = \exp\big(\rho(c)\big)V \subseteq V$$

for all $c \in \mathfrak{g}$, which then extends to $\rho(g)V \subseteq V$ for all $g \in G^o = G$. Hence for the groups $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$, and $\mathrm{SU}_2(\mathbb{R})$ the notions of irreducibility for finite-dimensional representations of the Lie group or of the Lie algebra coincide.

As discussed above, any finite-dimensional representation of G gives rise to a representation of its Lie algebra $\mathfrak{g}$. However, the converse to this only states that every finite-dimensional representation of $\mathfrak{g}$ gives rise to a finite-dimensional representation of the universal cover $\widetilde{G}$ of G. In the case of $G = \mathrm{SL}_2(\mathbb{C})$ and $G = \mathrm{SU}_2(\mathbb{R})$ this and Lemma 6.2 explains the correspondence between the irreducible representations of G and its Lie algebra in Theorem 6.3 below. However, for $G = \mathrm{SL}_2(\mathbb{R})$ this correspondence is a special feature.

6.1.3 Irreducible Representations of the Lie Algebra

We now specialize to the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and classify its irreducible representations in the next theorem, which is crucial for the representation theory of all semi-simple Lie groups and the classification of semi-simple Lie groups.

Theorem 6.3 (Irreducible representations of $\mathfrak{sl}_2$). *The irreducible finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ (respectively of $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{su}_2(\mathbb{R})$) are in a natural one-to-one correspondence with the elements of $\mathbb{N}_0$. In fact for every $n \in \mathbb{N}_0$ the representations of $\mathrm{SL}_2(\mathbb{C})$ on the* symmetric tensor product

$$\mathrm{Sym}^n(\mathbb{C}^2) = \Big\{\sum_{k=0}^{n} \alpha_k e_1^{\odot k} \odot e_2^{\odot(n-k)} \mid \alpha_0, \dots, \alpha_n \in \mathbb{C}\Big\}$$

gives rise to an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$*. By restriction, we also obtain irreducible representations of* $\mathfrak{sl}_2(\mathbb{R})$*, and of* $\mathfrak{su}_2(\mathbb{R})$*. Any irreducible finite-dimensional representation of* $\mathfrak{sl}_2(\mathbb{C})$*, of* $\mathfrak{sl}_2(\mathbb{R})$*, or of* $\mathfrak{su}_2(\mathbb{R})$ *is isomorphic to one of these.*

Before we begin the proof of the theorem we first wish to describe the representations $\mathrm{Sym}^n(\mathbb{C}^2)$ in more detail.

Given a finite-dimensional vector space W and a representation ρ of a group G on it, one can define the symmetric tensor product

$$\mathrm{Sym}^n(W) = \langle w_1 \odot w_2 \odot \cdots \odot w_n \mid w_1, \dots, w_n \in W\rangle$$

as the linear hull of all formal commuting products of n vectors in W with the product map

$$W^n \ni (w_1, \dots, w_n) \longmapsto w_1 \odot \cdots \odot w_n \in \mathrm{Sym}^n(W)$$

being multilinear. In fact

$$\mathrm{Sym}^n(W) = \bigotimes^n W / \langle w_1 \otimes \cdots \otimes w_n - w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)} \mid \sigma \in \mathrm{S}_n\rangle,$$

where S_n again denotes the symmetric group of $\{1, \dots, n\}$. For $n = 0$ we define $\mathrm{Sym}^0(W) = \mathbb{C}$. We refer to Hungerford [26] for more details.

Moreover, any linear map $A \in \mathrm{Hom}(W, W)$ can be used to induce a linear map $\mathrm{Sym}^n(A) \in \mathrm{Hom}\big(\mathrm{Sym}^n(W), \mathrm{Sym}^n(W)\big)$ with

$$\mathrm{Sym}^n(A)(w_1 \odot \cdots \odot w_n) = (Aw_1) \odot \cdots \odot (Aw_n).$$

Returning to the setting of Theorem 6.3, we note that $\mathbb{C}^2$ carries the standard representation ρ defined by the linear action $\rho_g \colon \mathbb{C}^2 \ni v \mapsto gv \in \mathbb{C}^2$ of $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$, or $\mathrm{SU}_2(\mathbb{R})$. This defines the representation on $\mathrm{Sym}^n(\mathbb{C}^2)$ for all $n \in \mathbb{N}$, where $n = 1$ corresponds to the standard representation. In the special case $n = 0$ we use the trivial representation on $\mathrm{Sym}^n(\mathbb{C}^2) = \mathbb{C}$.

If we instead let W be the vector space of linear maps on $\mathbb{C}^2$, then $\mathrm{Sym}^n(W)$ becomes the space of homogeneous polynomials of degree no more than n on $\mathbb{C}^2$. The isomorphism $\mathrm{Sym}^n(\mathbb{C}^2) \cong \mathrm{Sym}^n(W)$ would also follows from the proof of the theorem below, but let us indicate briefly where it comes from. In fact, we will show that the standard representation on $\mathbb{C}^2$ and the representation ρ on W are isomorphic. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and then send the linear map f defined by $f(X,Y) = \alpha X + \beta Y$ to

$$f \circ g^{-1}\colon \begin{pmatrix} X \\ Y \end{pmatrix} \longmapsto g^{-1}\begin{pmatrix} X \\ Y \end{pmatrix} \longmapsto (\alpha,\beta) g^{-1}\begin{pmatrix} X \\ Y \end{pmatrix}.$$

In terms of the basis X, Y (dual to the standard basis of $\mathbb{C}^2$), this corresponds to the map

$$\begin{aligned} \rho(g) = \left(g^{-1}\right)^{\mathrm{t}} &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}^{\mathrm{t}} \\ &= \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \end{aligned}$$

or $\rho(g) = kgk^{-1}$ for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, where $k = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

This shows that the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ and its dual on W are isomorphic, which is a special property of $\mathrm{SL}_2(\mathbb{C})$ due to the special rule for calculating g^{-1}. This isomorphism can be used to find the isomorphism between $\mathrm{Sym}^n(\mathbb{C}^2)$ and $\mathrm{Pol}_n(\mathbb{C}^2)$.

Proof of irreducibility of $\mathrm{Sym}^n(\mathbb{C}^2)$. Let us now work with the representation $\mathrm{Sym}^n(\mathbb{C}^2)$ of $G = \mathrm{SL}_2(\mathbb{C})$ or $G = \mathrm{SL}_2(\mathbb{R})$. The trivial representation corresponding to $n = 0$ is clearly irreducible.

We now assume that $n \geqslant 1$. For $\mathbf{a} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $t \in \mathbb{R}$ we have

$$\exp(t\mathbf{a}) = \begin{pmatrix} \mathrm{e}^t & \\ & \mathrm{e}^{-t} \end{pmatrix}$$

and $\rho\big(\exp(t\mathbf{a})\big)e_1 = \mathrm{e}^t e_1$ and $\rho\big(\exp(t\mathbf{a})\big)e_2 = \mathrm{e}^{-t} e_2$ for the representation on $\mathbb{C}^2$ and hence

$$\begin{aligned} \mathrm{Sym}^n\big(\exp(t\mathbf{a})\big)\big(e_1^{\odot k} \odot e_2^{\odot(n-k)}\big) &= \big(\mathrm{e}^t e_1\big)^{\odot k} \odot \big(\mathrm{e}^{-t} e_2\big)^{\odot(n-k)} \\ &= \mathrm{e}^{kt-(n-k)t} e_1^{\odot k} \odot e_2^{\odot(n-k)} \end{aligned}$$

for $k = 0, \ldots, n$. This shows that $\exp(t\mathbf{a})$ acts diagonally on $\mathrm{Pol}_n(\mathbb{C}^2)$ with eigenvalues $\mathrm{e}^{-n}, \mathrm{e}^{-n+2}, \ldots, \mathrm{e}^{n-2}, \mathrm{e}^n$. Taking the derivative we also obtain that the Lie algebra element $\mathbf{a}$ acts diagonally via

$$\mathrm{Sym}^n(\mathbf{a}) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathrm{Sym}^n(\exp(t\mathbf{a})),$$

with eigenvalues $-n, -n+2, \ldots, n-2, n$.

For $\mathbf{e} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ and $t \in \mathbb{R}$ we have $\exp(t\mathbf{e}) = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ and $\rho(\exp(t\mathbf{e}))e_1 = e_1$ and $\rho(\exp(t\mathbf{e}))e_2 = te_1 + e_2$ for the representation on $\mathbb{C}^2$ and hence

$$\begin{aligned}\operatorname{Sym}^n\big(\rho(\exp(t\mathbf{e}))\big)\big(e_1^{\odot k}\odot e_2^{\odot(n-k)}\big) &= e_1^{\odot k}(te_1+e_2)^{\odot(n-k)}\\ &= e_1^{\odot k}\Big(e_2^{\odot(n-k)}+te_1\binom{n-k}{1}e_2^{\odot(n-k-1)}+\cdots\Big),\end{aligned}$$

where the dots indicate the terms of order two and higher with respect to the variable t. Taking the derivative at $t=0$, this gives

$$\operatorname{Sym}^n(\mathbf{e})(e_1^{\odot k}\odot e_2^{\odot(n-k)}) = (n-k)e_1^{\odot(k+1)}\odot e_2^{\odot(n-k-1)} \tag{6.7}$$

and similarly

$$\operatorname{Sym}^n(\mathbf{f})(e_1^{\odot k}\odot e_2^{\odot(n-k)}) = ke_1^{\odot(k-1)}\odot e_2^{\odot(n-k+1)} \tag{6.8}$$

for all $k=0,\dots,n$.

Suppose now that $V\subseteq \operatorname{Sym}^n(\mathbb{C}^2)$ is non-trivial and invariant under Sym^n. Since $\operatorname{Sym}^n(\mathbf{a})$ is diagonal with $n+1$ different eigenvalues, it follows that

$$e_1^{\odot k}\odot e_2^{\odot(n-k)}\in V$$

for some $k\in\{0,1,\dots,n\}$. However, using (6.7) and (6.8) we also see that

$$e_1^{\odot(k+1)}\odot e_2^{\odot(n-k-1)}\in V$$

if $k<n$, and

$$e_1^{\odot(k-1)}e_2^{\odot(n-k+1)}\in V$$

if $k>0$. Iterating this shows that $V=\operatorname{Sym}^n(\mathbb{C}^2)$ contains all basis vectors, and irreducibility of $\operatorname{Sym}^n(\mathbb{C}^2)$ follows. □

As already visible in the proof above the eigenvectors of $\rho(\mathbf{a})$ and their eigenvalues play an important role in the theory. Hence they deserve a special name: the eigenvalues of $\rho(\mathbf{a})$ are called *weights* and the corresponding eigenvectors are called *weight vectors*. Moreover, the numbers 2 and -2 are called the *roots* and the vectors $\mathbf{e},\mathbf{f}\in\mathfrak{sl}_2(\mathbb{C})$ the *root vectors*.

Proof of completeness. We now will show that for $G=\mathrm{SL}_2(\mathbb{C})$ and for $G=\mathrm{SL}_2(\mathbb{R})$ the list of irreducible finite-dimensional representations above is complete. For this we let W carry an arbitrary finite-dimensional representation ρ of the Lie algebra $\mathfrak{g}$ of G. Later we will assume that W is irreducible, but the initial part of the construction is more general. Since W is finite-dimensional, $\rho(\mathbf{a})\in\operatorname{Hom}(W)$ must have at least one weight, that is an eigenvalue of $\rho(\mathbf{a})$. Let us assume that $\lambda_0\in\mathbb{C}$ is a weight with the property that $\Re\lambda_0$ is maximal in the set $\{\Re\lambda\mid\lambda\text{ is a weight}\}$. Let $w_0\in W$ be a weight vector for weight λ_0. We will be using $\rho(\mathbf{f})$ and the following more general claim to find more weight vectors with weights λ_0-2, λ_0-4,... which will lead to a complete classification of W.

Fundamental calculation. If $v \in W$ is a weight vector for weight λ, then we claim that $\rho(\mathbf{e})v$ and $\rho(\mathbf{f})v$ are weight vectors for weight $\lambda+2$ and $\lambda-2$ respectively (but might be zero).

The proof of the claim is rather simple. Indeed, using the defining property (6.6) of a Lie algebra homomorphism and the defining properties of the $\mathfrak{sl}_2$-triple in (6.2), we have

$$\begin{aligned}\rho(\mathbf{a})\rho(\mathbf{e})v &= \bigl(\rho(\mathbf{a})\rho(\mathbf{e})-\rho(\mathbf{e})\rho(\mathbf{a})\bigr)v+\rho(\mathbf{e})\rho(\mathbf{a})v\\ &= \bigl[\rho(\mathbf{a}),\rho(\mathbf{e})\bigr]v+\rho(\mathbf{e})\lambda v\\ &= \rho\bigl([\mathbf{a},\mathbf{e}]\bigr)v+\lambda\rho(\mathbf{e})v\\ &= \rho(2\mathbf{e})v+\lambda\rho(\mathbf{e})v=(\lambda+2)\rho(\mathbf{e})v.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\rho(\mathbf{a})\rho(\mathbf{f})v &= [\rho(\mathbf{a}),\rho(\mathbf{f})]v+\rho(\mathbf{f})\rho(\mathbf{a})v\\ &= \rho\bigl([\mathbf{a},\mathbf{f}]\bigr)v+\lambda\rho(\mathbf{f})v=(\lambda-2)\rho(\mathbf{f})v.\end{aligned}$$

Construction of eigenvectors. Let $w_0 \in W$ be a weight vector with maximal real part of its weight as above. By the fundamental calculation $\rho(\mathbf{e})w_0$ has weight $\lambda+2$, which by maximality of $\Re\lambda$ implies that

$$\rho(\mathbf{e})w_0=0. \tag{6.9}$$

On the other hand we can define

$$w_1=\rho(\mathbf{f})w_0, w_2=\rho(\mathbf{f})w_1,\ldots \tag{6.10}$$

and obtain weight vectors for weights $\lambda-2,\lambda-4,\ldots$. As eigenvectors for different eigenvalues are always linearly independent and $\dim W<\infty$ it follows that there must exist some $n \geqslant 0$ with $w_n=\rho(\mathbf{f})^n w_0 \neq 0$ but

$$\rho(\mathbf{f})w_n=0. \tag{6.11}$$

An invariant subspace. We let $V=\langle w_0,\ldots,w_n\rangle \subseteq W$ and claim that V is invariant under ρ. Since V is generated by eigenvectors for $\rho(\mathbf{a})$, V is clearly invariant under $\rho(\mathbf{a})$. Moreover, by the construction in (6.10) and the property (6.11) we have $\rho(\mathbf{f})w_k \in V$ for all $k=0,\ldots,n$ and hence $\rho(\mathbf{f})V \subseteq V$. It remains to study $\rho(\mathbf{e})$, where we will prove by induction that

$$\rho(\mathbf{e})w_k \begin{cases}=0 & \text{for } k=0;\\ \in \mathbb{C}w_{k-1} & \text{for } k>0\end{cases} \tag{6.12}$$

for $k=0,\ldots,n$. Indeed, we know this for $k=0$ by (6.9). If now (6.12) is already known for some $k \in \{0,\ldots,n-1\}$, then by construction

$$w_{k+1} = \rho(\mathbf{f})w_k$$

and

$$\begin{aligned}\rho(\mathbf{e})w_{k+1} &= \rho(\mathbf{e})\rho(\mathbf{f})w_k \\ &= [\rho(\mathbf{e}), \rho(\mathbf{f})]w_k + \rho(\mathbf{f})\rho(\mathbf{e})w_k \\ &= \rho(\mathbf{a})w_k + \rho(\mathbf{f})\rho(\mathbf{e})w_k \in \mathbb{C}w_k + \rho(\mathbf{f})\mathbb{C}w_{k-1} = \mathbb{C}w_k,\end{aligned}$$

by the invariance assumption. This shows the inductive step.

Assuming irreducibility. Suppose now in addition that W is irreducible. Since $V \subseteq W$ is non-trivial and invariant under ρ, it follows that $V = W$ has the basis $w_0, \dots, w_n$ consisting of weight vectors for $\rho(\mathbf{a})$ for the weights

$$\lambda, \lambda - 2, \dots, \lambda - n.$$

In particular, $\rho(\mathbf{a})$ is diagonalizable with these eigenvalues, and the trace of $\rho(\mathbf{a})$ is

$$\sum_{k=0}^{n}(\lambda - 2k) = (n+1)\lambda - 2\sum_{k=0}^{n} k = (n+1)\lambda - (n+1)n.$$

Since $\rho(\mathbf{a}) = \rho\big([\mathbf{e}, \mathbf{f}]\big) = [\rho(\mathbf{e}), \rho(\mathbf{f})]$, we also know that $\operatorname{tr}\rho(\mathbf{a}) = 0$, and so $\lambda = n \in \mathbb{N}_0$.

An isomorphism. We now combine the arguments above with our discussion of $\operatorname{Sym}^n(\mathbb{C}^2)$ by constructing the graph of an isomorphism within

$$\widetilde{W} = \operatorname{Sym}^n(\mathbb{C}^2) \oplus W.$$

In fact the vector $v_0 = (e_1^n, w_0) \in \widetilde{W}$ is a weight vector of weight n satisfying $\operatorname{Sym}^n(\mathbf{e}) \oplus \rho(\mathbf{e})v_0 = 0$ just as in the case of our original vector $w_0 \in W$. Applying the same argument as before we produce weight vectors $v_0, v_1, \dots, v_n \in \overline{W}$ such that $V = \langle v_0, v_1, \dots, v_n\rangle$ is invariant. Moreover, $\dim V = n + 1 = \dim \operatorname{Sym}^n(\mathbb{C}^2) = \dim W$ and the projections of V onto $\operatorname{Sym}^n(\mathbb{C}^2)$ respectively onto W are surjective, which follows from (6.8) for $\operatorname{Sym}^n(\mathbb{C}^2)$ and since $w_0, w_1, \dots, w_n$ is a basis of W. This shows that $V = \operatorname{Graph}(\phi)$ for some linear isomorphism $\phi\colon \operatorname{Sym}^n(\mathbb{C}^2) \to W$ and invariance of V implies that ϕ is an isomorphism of the representations. □

The maximal weight as it appeared in the proof above is called the *highest weight* and the corresponding weight vectors are called *highest weight vectors*.

Exercise 6.4. Use the arguments from the proof of Theorem 6.3 to show that for any finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$ a highest weight vector for highest weight λ always generates an irreducible subrepresentation of dimension $\lambda + 1$.

We refer to Fulton and Harris [20] for an accessible treatment of the theory of highest weight vectors for more general semi-simple groups. We will see similar mechanisms for creating more eigenvectors out of an initial eigenvector also for unitary representations later.

To summarize, we have proved Theorem 6.3 in the two cases $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, where we did not see any difference in the arguments as the $\mathfrak{sl}_2$-triple belonged to $\mathfrak{sl}_2(\mathbb{R})$ and all subspaces of the representation space are assumed to be complex subspaces. As we will now show, the extension to $\mathfrak{su}_2(\mathbb{R})$ does not require much except for the right insight.

Proposition 6.5 (Complexification). *Any finite-dimensional (and, as always, complex) representation*

$$\rho\colon \mathfrak{su}_2(\mathbb{R}) \longrightarrow \mathrm{Hom}(W)$$

(or $\rho\colon \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{Hom}(W)$) can be extended in a unique way to a representation $\rho_{\mathbb{C}}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Hom}(W)$. A subspace $V \subseteq W$ is invariant under $\rho(\mathfrak{su}_2(\mathbb{R}))$ (respectively $\rho_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C}))$) if and only if it is invariant under $\rho_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C}))$. In particular, they have the same list of irreducible finite-dimensional representations.

Proof. Since $\mathrm{Hom}(W)$ is a complex vector space, $\mathfrak{su}_2(\mathbb{R})$ is a real vector space, and $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{su}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ by the discussion in Section 6.1.1, every $\mathbb{R}$-linear map $\rho\colon \mathfrak{su}_2(\mathbb{R}) \to \mathrm{Hom}(W)$ has a uniquely defined $\mathbb{C}$-linear extension $\rho_{\mathbb{C}}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{Hom}(W)$ satisfying

$$\rho_{\mathbb{C}}(b_1 + \mathrm{i} b_2) = \rho(b_1) + \mathrm{i}\rho(b_2)$$

for all $b_1, b_2 \in \mathfrak{su}_2(\mathbb{R})$. Moreover, if

$$\rho\big([b_1, b_2]\big) = [\rho(b_1), \rho(b_2)]$$

for all $b_1, b_2 \in \mathfrak{su}_2(\mathbb{R})$ then this property extends by bilinearity of $[\cdot,\cdot]$ and linearity of $\rho_{\mathbb{C}}$ from $\mathfrak{su}_2(\mathbb{R})$ to all $b_1, b_2 \in \mathfrak{sl}_2(\mathbb{C})$. □

We have shown that $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}_2(\mathbb{R})$, and $\mathfrak{su}_2(\mathbb{R})$ as well as $\mathrm{SL}_2(\mathbb{C})$, $\mathrm{SL}_2(\mathbb{R})$, and $\mathrm{SU}_2(\mathbb{R})$ have the same list of irreducible finite-dimensional representations, which concludes the proof of Theorem 6.3. In this discussion, the Lie algebra $\mathfrak{su}_2(\mathbb{R})$ and its Lie group $\mathrm{SU}_2(\mathbb{R})$ were the odd ones out as they required extra effort (because $\mathfrak{su}_2(\mathbb{R})$ does not contain an $\mathfrak{sl}_2$-triple). However, in the next section the special properties of $\mathrm{SU}_2(\mathbb{R})$ are used to prove an important property of $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{R})$.

Exercise 6.6. Show that $\mathfrak{so}_n(\mathbb{R})$ and $\mathfrak{so}_{n-k,k}(\mathbb{R})$ with $k \in \{1, \ldots, n-2\}$ are real forms of $\mathfrak{so}_n(\mathbb{C})$, and conclude that Proposition 6.5 holds in the same way for these Lie algebras.

6.1.4 The Weyl Unitary Trick

Proposition 6.7 (Semi-simplicity of representations of $\mathfrak{su}_2(\mathbb{R})$). *Assume that W is a finite-dimensional representation of $\mathfrak{su}_2(\mathbb{R})$. If a complex subspace V of W is invariant under $\mathfrak{su}_2(\mathbb{R})$, then there exists an invariant complementary subspace V' so that $W = V \oplus V'$.*

Proof. As explained after SectionsectionquickreviewLieAlgebraRepresentationsinUnitary, W is also a representation space for $\mathrm{SU}_2(\mathbb{R})$ and V is invariant under $\mathrm{SU}_2(\mathbb{R})$. Fix some inner product on W and apply Proposition 5.8. Hence we may assume that W is a unitary representation of $\mathrm{SU}_2(\mathbb{R})$ and we may define $V' = V^\perp$ with respect to this inner product. □

Theorem 6.8 (Finite-dimensional representations of $\mathfrak{sl}_2$). *For finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{sl}_2(R)$, and $\mathfrak{su}_2(\mathbb{R})$ we have the following properties.*

(a) *(Semi-simplicity) Any invariant subspace has an invariant complement.*
(b) *(Description) The representation is a finite direct sum of irreducible representations as described in Theorem* 6.3.

Proof. For $\mathfrak{su}_2(\mathbb{R})$ part (a) is precisely the statement in Proposition 6.7. Part (b) follows from this by induction on the dimension. For $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{R})$ we combine Proposition 6.5 with the above. □

The argument above using compactness of $\mathrm{SU}_2(\mathbb{R}) \subseteq \mathrm{SL}_2(\mathbb{C})$ can in fact be used for all semi-simple real and complex Lie groups since there always exists a compact form(10) that can take the role of $\mathrm{SU}_2(\mathbb{R})$.

6.2 Harmonic Analysis of $\mathrm{SU}(2,\mathbb{R})$ and Quotients

6.2.1 Peter–Weyl Theorem for $\mathrm{SU}(2,\mathbb{R})$

†We now start the in-depth discussion of the harmonic analysis on $\mathrm{SU}_2(\mathbb{R})$ and related spaces. Schur orthogonality and the Peter–Weyl theorem (Theorems 5.11 and 5.15) give a complete description of $L^2(G)$ for a compact group G, assuming a complete description of $\widehat{G}$. In Theorem 6.3 we have obtained the description of all irreducible finite-dimensional representations of $\mathrm{SU}_2(\mathbb{R})$. Combining these two (and calculating the inner product

† This section discusses the first non-trivial and quite important example of a compact simple group. In order to be completely explicit, the discussion is quite heavy on concrete formulas. These are, however, not important for most of the subsequent discussions.

on $\mathrm{Sym}^n(\mathbb{C}^2)$) gives the following result. For this, we will be using the coordinate system $(z,w)\in\mathbb{C}^2$ with $|z|^2+|w|^2=1$ for the elements

$$g=\begin{pmatrix} z & -\overline{w}\\ w & \overline{z}\end{pmatrix}\in \mathrm{SU}_2(\mathbb{R}). \tag{6.13}$$

Corollary 6.9 (Peter–Weyl for $\mathrm{SU}_2(\mathbb{R})$). *The functions*

$$g\longmapsto \sqrt{n+1}\,\pi^{(n)}_{k,\ell}(g)$$

defined by

$$\pi^{(n)}_{k,\ell}(g)=\sqrt{k!(n-k)!\ell!(n-\ell)!}\\ \times\sum_{\substack{i\in\{0,\dots,n-k\}\\ j\in\{0,\dots,k\}\\ i+j=\ell}}\frac{(-1)^{k-j}}{i!j!(n-k-i)!(k-j)!}z^{n-k-i}w^i\overline{z}^j\overline{w}^{k-j}$$

for $k,\ell\in\{0,\dots,n\}$ are the normalized matrix coefficients associated to the irreducible representation on the $(n+1)$-dimensional space $\mathrm{Sym}^n(\mathbb{C}^2)$ for n in $\mathbb{N}_0$ (and using a convenient choice of orthonormal basis of $\mathrm{Sym}^n(\mathbb{C}^2)$). By also varying n in $\mathbb{N}_0$ we obtain an orthonormal basis of $L^2(\mathrm{SU}_2(\mathbb{R}))$.

Proof. As already hinted at before the corollary, we have done all the work required for the corollary apart from determining the inner product on $\mathrm{Sym}^n(\mathbb{C}^2)$ for $n\in\mathbb{N}_0$. We note that the inner product is uniquely determined up to a positive scalar by the irreducibility of the representations and Schur's lemma (Theorem 1.24).

The case $n=0$ corresponds to the trivial representation, and setting in addition k and ℓ to be 0 gives $\sqrt{1}\pi^{(0)}_{0,0}=1$. The case $n=1$ corresponds to the standard representation on $\mathbb{C}^2$ and the standard inner product on $\mathbb{C}^2$ makes the action unitary (by definition of $\mathrm{SU}_2(\mathbb{R})$). In this case we use the standard basis

$$w_0=e_1=\begin{pmatrix}1\\0\end{pmatrix},\ \ w_1=e_2=\begin{pmatrix}0\\1\end{pmatrix}$$

and obtain $\pi^{(1)}_{0,0}(g)=z$, $\pi^{(1)}_{0,1}(g)=w$, $\pi^{(1)}_{1,0}(g)=-\overline{w}$, $\pi^{(1)}_{1,1}(g)=\overline{z}$. Setting $n=1$ and using $k,\ell\in\{0,1\}$ in the formula in the corollary, we obtain the same four functions on $\mathrm{SU}_2(\mathbb{R})$. By Theorem 5.11, the vectors $\sqrt{2}\pi^{(1)}_{k,\ell}$ are then orthonormal for $k,\ell\in\{0,1\}$.

So suppose now that $n\geqslant 2$. Then the vectors

$$\widetilde{w_k}=e_1^{\odot(n-k)}\odot e_2^{\odot k}$$

are eigenvectors for the elements

$$\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R})$$

for all $z \in \mathbb{S}^1$ for eigenvalues $z^{n-k}\overline{z}^k = z^{n-2k}$ for $k = 0, \dots, n$. As the eigenvalues are distinct, they must be pairwise orthogonal with respect to the desired inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{Sym}^n(\mathbb{C}^2)$. We claim that the inner product can be chosen so that the vectors

$$w_k = \binom{n}{k}^{\frac{1}{2}} e_1^{\odot(n-k)} \odot e_2^{\odot k} \tag{6.14}$$

are an orthonormal basis of $\mathrm{Sym}^n(\mathbb{C}^2)$. This could be checked directly (for example, by showing that the representation of $\mathfrak{su}_2(\mathbb{R})$ only takes on anti-Hermitian matrices with respect to that basis). However, we will give a more conceptual argument for this.

We consider $\bigotimes^n(\mathbb{C}^2)$ and apply Proposition 4.63 (inductively extended to n factors) to define the unitary inner tensor product representation ρ of $\mathrm{SU}_2(\mathbb{R})$ on $\bigotimes^n(\mathbb{C}^2)$ so that

$$\rho(g)(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = (gu_1) \otimes (gu_2) \otimes \cdots \otimes (gu_n) \tag{6.15}$$

for all $u_1, \dots, u_n \in \mathbb{C}^2$. Next note that there is a canonical equivariant map

$$\mathrm{Com}\colon \bigotimes^n(\mathbb{C}^2) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2)$$

that sends any tensor product to its commutative counterpart, as

$$\bigotimes^n(\mathbb{C}^2) \ni u_1 \otimes u_2 \otimes \cdots \otimes u_n \longmapsto u_1 \odot u_2 \odot \cdots \odot u_n \in \mathrm{Sym}^n(\mathbb{C}^2).$$

In fact, as was already mentioned, $\mathrm{Sym}^n(\mathbb{C}^2)$ is defined as the quotient of $\bigotimes^n(\mathbb{C}^2)$ by the subspace generated by

$$u_1 \otimes u_2 \otimes \cdots \otimes u_n - u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)}$$

for all $u_1, u_2, \dots, u_n \in \mathbb{C}^2$ and permutations $\sigma \in \mathrm{S}_n$.

Also note that the permutation group S_n acts unitarily on $\bigotimes^n(\mathbb{C}^2)$ by setting

$$\lambda_\sigma(u_1 \otimes u_2 \otimes \cdots \otimes u_n) = u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(n)} \tag{6.16}$$

for all $\sigma \in \mathrm{S}_n$ and $u_1, \dots, u_n \in \mathbb{C}^2$ and then linearly extending this action. Moreover, using (6.15) and (6.16) it is easy to see that $\lambda_\sigma \rho_g = \rho_g \lambda_\sigma$ for all $\sigma \in \mathrm{S}_n$ and $g \in \mathrm{SU}_2(\mathbb{R})$. Therefore

$$V = \left\{ v \in \bigotimes^n(\mathbb{C}^2) \mid \lambda_\sigma(v) = v \text{ for all } \sigma \in \mathrm{S}_n \right\}$$

is invariant under ρ, and is non-trivial since $e_1 \otimes e_1 \otimes \cdots \otimes e_1 \in V$. Recall that $\bigotimes^n(\mathbb{C}^2)$ has

$$e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} \tag{6.17}$$

for $j_1, \ldots, j_n \in \{1, 2\}$ as an orthonormal basis, and note that λ_σ for $\sigma \in \mathrm{S}_n$ maps any such basis vector to another such basis vector. It follows from this that V is generated by the vectors

$$v_k = \sum_{\substack{j_1,\ldots,j_n \in \{1,2\} \\ n-k \text{ times } 1, \\ k \text{ times } 2}} e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n} \tag{6.18}$$

for $k = 0, \ldots, n$. In fact, if one of the basis vectors $e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_n}$ with $|\{\ell \mid j_\ell = 1\}| = n - k$ appears with a coefficient c in the expansion of some $v \in V$ with respect to the basis in (6.17), then all other basis vectors appearing in the sum (6.18) are images of the original basis vector under some $\sigma \in \mathrm{S}_n$. Hence these all appear with the same coefficient c in the vector v. Using this argument for all $k = 0, \ldots, n$ we deduce that v is a linear combination of $v_0, v_1, \ldots, v_n$.

Now note that the vector v_k in (6.18) has $\binom{n}{k}$ summands which are mutually orthogonal unit vectors in $\bigotimes^n(\mathbb{C}^2)$. Hence the vectors

$$\widetilde{v}_k = \binom{n}{k}^{-\frac{1}{2}} v_k$$

for $k = 0, \ldots, n$ are an orthonormal basis of V. Applying the equivariant map $\mathrm{Com}|_V \colon V \to \mathrm{Sym}^n(\mathbb{C}^2)$ we see that

$$\mathrm{Com}\,\widetilde{v}_k = \binom{n}{k}^{-\frac{1}{2}} \mathrm{Com}\,v_k = \binom{n}{k}^{-\frac{1}{2}} \binom{n}{k} e_1^{\odot(n-k)} \odot e_2^{\odot k} = w_k$$

for $k = 0, \ldots, n$. This proves the claim that the basis in (6.14) is an orthonormal basis with respect to an inner product on $\mathrm{Sym}^n(\mathbb{C}^2)$ that makes ρ a unitary representation π of $\mathrm{SU}_2(\mathbb{R})$.

We now calculate the matrix coefficients $\pi^{(n)}_{k,\ell} = \varphi_{w_k, w_\ell}$ for the basis vectors w_k, w_ℓ and $k, \ell = 0, \ldots, n$. Using the notation (6.13) once again, we have $ge_1 = ze_1 + we_2$ and $ge_2 = -\overline{w}e_1 + \overline{z}e_2$ and hence

$$
\begin{aligned}
\pi_g^{(n)}(w_k) &= \binom{n}{k}^{\frac{1}{2}} (ze_1 + we_2)^{\odot(n-k)} \odot (-\overline{w}e_1 + \overline{z}e_2)^{\odot k} \\
&= \binom{n}{k}^{\frac{1}{2}} \left(\sum_{i=0}^{n-k} \binom{n-k}{i} z^{n-k-i} w^i e_1^{\odot(n-k-i)} \odot e_2^{\odot i} \right) \\
&\qquad \odot \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \overline{w}^{k-j} \overline{z}^j e_1^{\odot(k-j)} \odot e_2^{\odot j} \right) \\
&= \binom{n}{k}^{\frac{1}{2}} \sum_{i=0}^{n-k} \sum_{j=0}^{k} \binom{n-k}{i} \binom{k}{j} (-1)^{k-j} z^{n-k-i} w^i \overline{z}^j \overline{w}^{k-j} e_1^{\odot(n-i-j)} \odot e_2^{\odot(i+j)}.
\end{aligned}
$$

Taking the inner product with

$$
w_\ell = \binom{n}{\ell}^{\frac{1}{2}} e_1^{\odot(n-\ell)} e_2^{\odot \ell}
$$

selects those terms in the sum above with $i + j = \ell$. Multiplying and dividing by

$$
\binom{n}{\ell}^{-\frac{1}{2}}
$$

and using $\|w_\ell\| = 1$ leads to

$$
\begin{aligned}
\pi_{k,\ell}^{(n)} &= \langle \pi_g(w_k), w_\ell \rangle \\
&= \binom{n}{k}^{\frac{1}{2}} \binom{n}{\ell}^{-\frac{1}{2}} \sum_{\substack{i \in \{0,\dots,n-k\} \\ j \in \{0,\dots,k\} \\ i+j=\ell}} \binom{n-k}{i} \binom{k}{j} (-1)^{k-j} z^{n-k-i} w^i \overline{z}^j \overline{w}^{k-j} \\
&= \sqrt{k!(n-k)!\ell!(n-\ell)!} \sum_{\substack{i \in \{0,\dots,n-k\} \\ j \in \{0,\dots,k\} \\ i+j=\ell}} \frac{(-1)^{k-j}}{i!j!(n-k-i)!(k-j)!} z^{n-k-i} w^i \overline{z}^j \overline{w}^{k-j}.
\end{aligned}
$$

Together with Theorem 5.15, this concludes the proof. □

6.2.2 Peter–Weyl Theorem for SO(3, ℝ)

We briefly explain in this section how Corollary 6.9 also gives rise to a Peter–Weyl theorem for the group $\mathrm{SO}_3(\mathbb{R})$. We will not, however, give the orthonormal basis explicitly in terms of the coordinates of $\mathrm{SO}_3(\mathbb{R})$.

The connection between $\mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SO}_3(\mathbb{R})$ is given by the following lemma. We recall that an isogeny between two semi-simple Lie groups is a finite-to-one surjection.

Lemma 6.10 (Isogeny for $\mathrm{SU}_2(\mathbb{R})$). *We have*

$$\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{R})/C,$$

where $C = \{\pm I\}$ is the centre of $\mathrm{SU}_2(\mathbb{R})$.

Proof. We recall from Lemma 6.1 that $\mathrm{SU}_2(\mathbb{R}) \cong \mathbb{S}^3 \subseteq \mathbb{H}$, and so in particular $\mathrm{SU}_2(\mathbb{R})$ is simply connected (see also Lemma 6.2). We claim now that $\mathrm{SO}_3(\mathbb{R})$ is also a connected three-dimensional Lie group. To see that it is connected, we let $g \in \mathrm{SO}_3(\mathbb{R})$. Then g has a real eigenvector v with eigenvalue 1. In fact, if all eigenvalues are real, then $g = I$ or the eigenvalues must equal -1, -1, and 1. If there is a non-real complex eigenvalue λ then the eigenvalues must be λ, $\overline{\lambda}$, and 1. Hence in either case $g = I$ or g can be viewed as a rotation about some axis in $\mathbb{R}^3$. It follows that g belongs to a one-parameter subgroup and hence to the connected component of $I \in \mathrm{SO}_3(\mathbb{R})$. As $g \in \mathrm{SO}_3(\mathbb{R})$ was arbitrary, we deduce that $\mathrm{SO}_3(\mathbb{R})$ is connected.

The Lie algebra of

$$\mathrm{SO}_3(\mathbb{R}) = \{g \in \mathrm{SL}_3(\mathbb{R}) \mid g^* g = I\}$$

is given by

$$\mathfrak{so}_3(\mathbb{R}) = \{m \in \mathfrak{sl}_3(\mathbb{R}) \mid m^* + m = 0\},$$

and so consists of all matrices of the form

$$m = \begin{pmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & -\gamma \\ \beta & \gamma & 0 \end{pmatrix}$$

for $\alpha, \beta, \gamma \in \mathbb{R}$. It follows that $\mathrm{SO}_3(\mathbb{R})$ is a three-dimensional connected Lie group as claimed.

To define the homomorphism $\mathrm{SU}_2(\mathbb{R}) \to \mathrm{SO}_3(\mathbb{R})$, we identify the vector

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$$

with the Lie algebra element

$$m = \begin{pmatrix} a\mathrm{i} & b\mathrm{i} - c \\ b\mathrm{i} + c & -a\mathrm{i} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R}), \tag{6.19}$$

and define

$$\rho(g)(m) = gmg^{-1} = gmg^*$$

for all $g \in \mathrm{SU}_2(\mathbb{R})$ and $m \in \mathfrak{su}_2(\mathbb{R})$. Since

$$\mathfrak{su}_2(\mathbb{R}) = \{m \in \mathfrak{gl}_2(\mathbb{C}) \mid m^* = -m, \operatorname{tr} m = 0\},$$

it follows that $\rho(g)(\mathfrak{su}_2(\mathbb{R})) \subseteq \mathfrak{su}_2(\mathbb{R})$. Moreover, for m as in (6.19), we have

$$\det m = a^2 - (b\mathrm{i} - c)(b\mathrm{i} + c) = a^2 + b^2 + c^2$$

and $\det(\rho(g)m) = \det m$ for all $g \in \mathrm{SU}_2(\mathbb{R})$. This shows that the adjoint representation ρ defines a homomorphism $\rho\colon \mathrm{SU}_2(\mathbb{R}) \to \mathrm{SO}_3(\mathbb{R})$.

Suppose now that $\rho(g) = I$, so $gmg^{-1} = m$ for all $m \in \mathfrak{su}_2(\mathbb{R})$. Since $\mathfrak{su}_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{su}_2(\mathbb{C})$ we see that $gm = mg$ for all $m \in \mathfrak{sl}_2(\mathbb{C})$. For

$$g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

and

$$m = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we deduce that

$$gm = \begin{pmatrix} z & \overline{w} \\ w & -\overline{z} \end{pmatrix}, \quad mg = \begin{pmatrix} z & -\overline{w} \\ -w & -\overline{z} \end{pmatrix}, \tag{6.20}$$

so $w = 0$. Similarly, by using

$$m = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

we obtain

$$gm = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad mg = \begin{pmatrix} 0 & \overline{z} \\ 0 & 0 \end{pmatrix}$$

and hence $z = \overline{z} \in \mathbb{R}$. To summarize, if $g \in \ker \rho$ then $g = I$ or $g = -I$. The converse of this statement is clear. The lemma follows from this: Since both $\mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SO}_3(\mathbb{R})$ are three-dimensional and ρ has finite kernel, $\rho\big(\mathrm{SU}_2(\mathbb{R})\big)$ is also three-dimensional, which together with connectedness of $\mathrm{SO}_3(\mathbb{R})$ implies that $\rho\big(\mathrm{SU}_2(\mathbb{R})\big) = \mathrm{SO}_3(\mathbb{R})$. □

Corollary 6.11 (Peter–Weyl for $\mathrm{SO}_3(\mathbb{R})$). *The irreducible representation $\pi^{(n)}$ on $\mathrm{Sym}^n(\mathbb{C}^2)$ gives rise to a unitary representation of*

$$\mathrm{SO}_3(\mathbb{R}) = \mathrm{SU}_2(\mathbb{R})/C$$

if and only if n is even. In particular, the normalized matrix coefficients

$$\sqrt{n+1}\,\pi^{(n)}_{k,\ell}$$

for $k, \ell \in \{0, \ldots, n\}$ *and* $n \in 2\mathbb{N}_0$ *give rise to an orthonormal basis of* $L^2(\mathrm{SO}_3(\mathbb{R}))$.

Proof. If π is an irreducible unitary representation of $\mathrm{SO}_3(\mathbb{R})$, then the isomorphism $\mathrm{SO}_3(\mathbb{R}) \cong \mathrm{SU}_2(\mathbb{R})/C$ can be used to consider π also as an irreducible unitary representation of $\mathrm{SU}_2(\mathbb{R})$, which we again denote by π. By Theorem 6.3, π is isomorphic to the representation $\pi^{(n)}$ on $\mathrm{Sym}^n(\mathbb{C}^2)$ for some $n \in \mathbb{N}_0$. By construction, we have $\pi^{(n)}(-I) = (-I)^n = I$, which implies that n is even.

On the other hand, if $n \in 2\mathbb{N}$ then $\pi^{(n)}(-I) = I$, which shows that $\pi^{(n)}$ descends to a unitary representation of $\mathrm{SO}_3(\mathbb{R})$.

The final claim follows from the Peter–Weyl theorem (Theorem 5.15) as in the proof of Corollary 6.9. □

6.2.3 The Unitary Representation on $L^2(\mathbb{S}^2)$

By the discussion in the previous section we know that $\mathrm{SU}_2(\mathbb{R})$ acts naturally on the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. We equip $\mathbb{S}^2$ with the natural surface area measure and obtain a measure-preserving action of $\mathrm{SU}_2(\mathbb{R})$ on $\mathbb{S}^2$. This in turn gives rise to a unitary representation of $\mathrm{SU}_2(\mathbb{R})$ on $L^2(\mathbb{S}^2)$ as in Proposition 1.3. We wish to use the description of irreducible representations of $\mathrm{SU}_2(\mathbb{R})$ in Theorem 6.3 to describe how $L^2(\mathbb{S}^2)$ splits into irreducible components.

Corollary 6.12 (Decomposition of $L^2(\mathbb{S}^2)$). *Using the unitary representation of* $\mathrm{SU}_2(\mathbb{R})$ *on* $L^2(\mathbb{S}^2)$ *as described above, we have*

$$L^2(\mathbb{S}^2) = \bigoplus_{n \in 2\mathbb{N}} \mathrm{Sym}^n(\mathbb{C}^2).$$

In other words, every irreducible unitary representation of $\mathrm{SU}_2(\mathbb{R})$ *of even highest weight (equivalently, every irreducible representation of* $\mathrm{SO}_3(\mathbb{R})$*) appears in* $L^2(\mathbb{S}^2)$ *with multiplicity one.*

Proof. We will combine the description of $L^2(\mathrm{SU}_2(\mathbb{R}))$ in Corollary 6.9 with the isomorphism

$$\mathbb{S}^2 \cong \mathrm{SU}_2(\mathbb{R})/T \tag{6.21}$$

where

$$T = \left\{ \begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} \mid \alpha \in \mathbb{S}^1 \right\}$$

is the diagonal subgroup in $\mathrm{SU}_2(\mathbb{R})$.

For the proof of (6.21) we recall from Section 6.2.2 that $\mathrm{SU}_2(\mathbb{R})$ factors onto $\mathrm{SO}_3(\mathbb{R})$ and acts on $\mathfrak{su}_2(\mathbb{R}) \cong \mathbb{R}^3$ while preserving the quadratic form det. Let

$$v_0 = \begin{pmatrix} \mathrm{i} & \\ & -\mathrm{i} \end{pmatrix}$$

and note that $gv_0g^{-1} = v_0$ if and only if $g \in T$ (see the argument in (6.20)). Moreover, as $\mathrm{SO}_3(\mathbb{R})$ acts transitively on $\mathbb{S}^2$ and $\mathrm{SU}_2(\mathbb{R})$ factors onto $\mathrm{SO}_3(\mathbb{R})$ by this action, we obtain $\mathbb{S}^2 = \rho(\mathrm{SU}_2(\mathbb{R}))v_0$, and (6.21) follows. More precisely, $gT \in \mathrm{SU}_2(\mathbb{R})/T$ corresponds to $gv_0g^{-1} \in \mathbb{S}^2$ under the isomorphism and the action of $\mathrm{SU}_2(\mathbb{R})$ corresponds to left multiplication on $\mathrm{SU}_2(\mathbb{R})$.

Also recall that the Haar measure on a homogeneous space is unique up to positive proportionality, which implies that the surface area measure described above the corollary agrees up to a positive multiple with the push-forward of the Haar measure on $\mathrm{SU}_2(\mathbb{R})$ onto $\mathrm{SU}_2(\mathbb{R})/T$.

To summarise, we may consider $L^2(\mathbb{S}^2)$ as the subspace $\mathcal{V}$ of $L^2(\mathrm{SU}_2(\mathbb{R}))$ consisting of all functions on $\mathrm{SU}_2(\mathbb{R})$ that are right-invariant under T. In other words,

$$L^2(\mathbb{S}^2) \cong \mathcal{V} = \{f \in L^2(\mathrm{SU}_2(\mathbb{R})) \mid f \text{ has weight } 0 \text{ for } T\}.$$

Here we say that $f \in L^2(\mathrm{SU}_2(\mathbb{R}))$ has weight $m \in \mathbb{Z}$ for T if $f(gt) = \alpha^m f(g)$ for all diagonal elements

$$t = \begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} \in T$$

and almost every $g \in \mathrm{SU}_2(\mathbb{R})$.

We now use the notation

$$g = \begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

again for elements of $\mathrm{SU}_2(\mathbb{R})$, and the orthonormal basis comprising the functions $\sqrt{n+1}\pi_{k,\ell}^{(n)}$ for $n \in \mathbb{N}_0$ and $k, \ell \in \{0, \dots, n\}$ of $L^2(\mathrm{SU}_2(\mathbb{R}))$ from Corollary 6.9. Using the concrete formula for $\pi_{k,\ell}^{(n)}$ and the notation for g and t above, we see that

$$gt = \begin{pmatrix} \alpha z & -\overline{\alpha w} \\ \alpha w & \overline{\alpha z} \end{pmatrix}$$

and

$$\begin{aligned} \pi_{k,\ell}^{(n)}(gt) &= (k!(n-k)!\ell!(n-\ell)!)^{\frac{1}{2}} \\ &\quad \times \sum_{\substack{i \in \{0,\dots,n-k\} \\ j \in \{0,\dots,k\} \\ i+j=\ell}} \frac{(-1)^{k-j}}{i!j!(n-k-i)!(k-j)!} (\alpha z)^{n-k-i} (\alpha w)^i (\overline{\alpha z})^j (\overline{\alpha w})^{k-j} \\ &= \alpha^{n-2k} \pi_{k,\ell}^{(n)}(g). \end{aligned}$$

In other words, the orthonormal basis consists of eigenvectors for the right-regular representation restricted to $T \cong \mathbb{S}^1$. Hence we can obtain an orthonormal basis of $\mathcal{V}$ by using only those normalized matrix coefficients $\sqrt{n+1}\pi_{k,\ell}^{(n)}$ with weight $n-2k=0$. This gives the corollary, and we have shown that $\mathcal{V}$ has the orthonormal basis consisting of the functions $\sqrt{2k+1}\pi_{k,\ell}^{(2k)}$ with $k \in \mathbb{N}_0$ and $\ell \in \{0,\dots,2k\}$, and the left-regular representation of $\mathrm{SU}_2(\mathbb{R})$ on

$$\mathcal{V}_{2k} = \left\langle \pi_{k,\ell}^{(2k)} \mid \ell \in \{0,\dots,2k\} \right\rangle$$

is isomorphic to $\mathrm{Sym}^{2k}(\mathbb{C}^2)$ by the argument used in the last part of the proof of Theorem 5.15). □

6.2.4 Conjugacy Classes and Characters of SU(2, ℝ)

We wish to finish the discussion of the compact group $\mathrm{SU}_2(\mathbb{R})$ by describing $\mathrm{SU}_2(\mathbb{R})^\sharp$ and calculating the characters of $\mathrm{SU}_2(\mathbb{R})$.

Proposition 6.13 (Conjugacy classes of $\mathrm{SU}_2(\mathbb{R})$ and the Sato–Tate measure). *The trace* $\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R}) \to [-2,2]$ *descends to a homeomorphism*

$$\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R})^\sharp \longrightarrow [-2,2].$$

The push-forward of the Haar measure is given by

$$\mathrm{tr}_* \, m_{\mathrm{SU}_2(\mathbb{R})}\big((a,b)\big) = \frac{1}{\pi}\int_a^b \sqrt{1-\tfrac{t^2}{4}}\,\mathrm{d}t.$$

Proof. For $A,B \in \mathrm{Mat}_{d,d}(\mathbb{C})$ recall that $\mathrm{tr}(AB)=\mathrm{tr}(BA)$, since

$$\mathrm{tr}(AB) = \sum_{i=1}^{d}(AB)_{i,i} = \sum_{i,j=1}^{d} A_{i,j}B_{j,i} = \sum_{j,i=1}^{d} B_{i,j}A_{j,i}.$$

In particular, the map $\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R})^\sharp \to \mathbb{R}$ defined by $\mathrm{tr}([g]) = \mathrm{tr}(g)$ is a well-defined continuous map. As the eigenvalues of any $g \in \mathrm{SU}_2(\mathbb{R})$ are of the form $\alpha, \overline{\alpha}$ for some $\alpha \in \mathbb{S}^1$, we see that $\mathrm{tr}(g)$ lies in $[-2,2]$. Varying $\alpha \in \mathbb{S}^1$ and using

$$t = \begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R}),$$

we also see that $\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R})^\sharp \to [-2,2]$ is surjective.

We claim that the trace map is also injective on $\mathrm{SU}_2(\mathbb{R})^\sharp$. So suppose that $\mathrm{tr}([g_1]) = \mathrm{tr}([g_2])$ for some $[g_1],[g_2] \in \mathrm{SU}_2(\mathbb{R})^\sharp$. Then the characteristic polynomials of g_1 and g_2 agree (since these are determined for 2×2 matrices by the trace and determinant). It follows that the eigenvalues $\alpha, \overline{\alpha}$ of g_1 and g_2 are equal. Since $g_1 \in \mathrm{SU}_2(\mathbb{R})$ there exists an orthonormal basis $u_1, u_2 \in \mathbb{C}^2$

consisting of eigenvectors for α, resp. $\overline{\alpha}$. Multiplying u_2 by a scalar of absolute value one if necessary, we may assume $\det h = 1$ where $h = (u_1, u_2)$. It follows that $h \in \mathrm{SU}_2(\mathbb{R})$ and

$$g_1 = h \begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} h^{-1}.$$

In other words,

$$[g_1] = \left[\begin{pmatrix} \alpha & \\ & \overline{\alpha} \end{pmatrix} \right],$$

and by symmetry between g_1, g_2 also $[g_1] = [g_2]$, as required.

As both $\mathrm{SU}_2(\mathbb{R})^\sharp$ and $[-2,2]$ are compact and tr is continuous, it follows that $\mathrm{tr}\colon \mathrm{SU}_2(\mathbb{R})^\sharp \to [-2,2]$ is a homeomorphism.

It remains to prove the explicit description of the image of Haar measure. For this, we again identify $\mathrm{SU}_2(\mathbb{R})$ with $\mathbb{S}^3 \subseteq \mathbb{H} \cong \mathbb{R}^4$. With this the Haar measure can be defined using the four-dimensional Lebesgue measure $m_{\mathbb{R}^4}$. In fact for $B \subseteq \mathbb{S}^3$ we define

$$m_{\mathbb{S}^3}(B) = m_{\mathbb{R}^4}\big(\{rv \mid r \in [0,1], v \in B\}\big) \tag{6.22}$$

and, since $\mathbb{S}^3$ acts linearly as a unimodular transformation on $\mathbb{R}^4$, it follows that $m_{\mathbb{S}^3}$ defines a measure on $\mathbb{S}^3$ with total measure $m_{\mathbb{S}^3}(\mathbb{S}^3) = m_{\mathbb{R}^4}\big(B_1^{\mathbb{R}^4}\big)$. This turns the description of $\mathrm{tr}_* m_{\mathbb{S}^3}$ into an exercise in multi-dimensional calculus.

In fact we will use four-dimensional spherical coordinates defined by

$$S\colon \begin{pmatrix} r \\ \theta \\ \phi \\ \psi \end{pmatrix} \longmapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta\cos\phi \\ r\sin\theta\sin\phi\cos\psi \\ r\sin\theta\sin\phi\sin\psi \end{pmatrix}$$

with total derivative

$$\begin{pmatrix} \cos\theta & -r\sin\theta & 0 & 0 \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi & 0 \\ \sin\theta\sin\phi\cos\psi & r\cos\theta\sin\phi\cos\psi & r\sin\theta\cos\phi\cos\psi & -r\sin\theta\sin\phi\sin\psi \\ \sin\theta\sin\phi\sin\psi & r\cos\theta\sin\phi\sin\psi & r\sin\theta\cos\phi\sin\psi & r\sin\theta\sin\phi\cos\psi \end{pmatrix}$$

and Jacobian determinant

$$r^3\sin^2\theta\sin\phi\det\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi & 0 \\ \sin\theta\sin\phi\cos\psi & \cos\theta\sin\phi\cos\psi & \cos\phi\cos\psi & -\sin\psi \\ \sin\theta\sin\phi\sin\psi & \cos\theta\sin\phi\sin\psi & \cos\phi\sin\psi & \cos\psi \end{pmatrix}.$$

Expanding the remaining determinant along the first row, we see that it is given by

$$\varDelta\cos^2\theta + \varDelta\sin^2\theta = \varDelta$$

where

$$\begin{aligned}\varDelta &= \det\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi\cos\psi & \cos\phi\cos\psi & -\sin\psi\\ \sin\phi\sin\psi & \cos\phi\sin\psi & \cos\psi\end{pmatrix}\\ &= \cos^2\phi\det\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix} + \sin^2\phi\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix} = 1.\end{aligned}$$

A convenient domain for the spherical coordinates is

$$U = (0,\infty)\times(0,\pi)\times(0,\pi)\times(0,2\pi),$$

and spherical coordinates define a diffeomorphism S from U to a full measure open set $V \subseteq \mathbb{R}^4$.

Now let $a < b$ be in $[-2,2]$, and define $\theta_a = \arccos\frac{a}{2}$ and $\theta_b = \arccos\frac{b}{2}$ so that

$$\begin{aligned}\mathrm{tr}^{-1}(a,b) &= \left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{S}^3 \mid \frac{a}{2} < x_1 < \frac{b}{2}\right\}\\ &= \left\{\begin{pmatrix}\cos\theta\\ \sin\theta\cos\phi\\ \sin\theta\sin\phi\cos\psi\\ \sin\theta\sin\phi\sin\psi\end{pmatrix} \mid \theta\in(\theta_b,\theta_a), \phi\in(0,\pi), \psi\in(0,2\pi)\right\}.\end{aligned}$$

Using the description of the Haar measure $m_{\mathbb{S}^3}$ in (6.22), this leads to

$$\begin{aligned}\mathrm{tr}_* m_{\mathbb{S}^3}\big((a,b)\big) &= \int_0^1 r^3\,\mathrm{d}r\int_{\theta_b}^{\theta_a}\sin^2\theta\,\mathrm{d}\theta\int_0^\pi\sin\phi\,\mathrm{d}\phi\int_0^{2\pi}\mathrm{d}\psi\\ &= \tfrac14\cdot 2\cdot 2\pi\int_{\theta_b}^{\theta_a}\sin^2\theta\,\mathrm{d}\theta.\end{aligned}$$

Instead of calculating the latter integral, we wish to rewrite it as an integral over $t\in[a,b]$ using $t = 2\cos\theta$ and $\mathrm{d}t = -2\sin\theta\,\mathrm{d}\theta$. This gives

$$\mathrm{tr}_* m_{\mathbb{S}^3}\big((a,b)\big) = \frac{\pi}{2}\int_a^b\sqrt{1-\tfrac{t^2}{4}}\,\mathrm{d}t.$$

Normalizing the measure to be a probability gives the proposition. □

Using the identification of $\mathrm{SU}_2(\mathbb{R})^\sharp$ and the interval $[-2,2]$ we now describe the characters of $\mathrm{SU}_2(\mathbb{R})$. By Theorem 6.3 the irreducible representations of $\mathrm{SU}_2(\mathbb{R})$ are given by the nth symmetric tensor products $\mathrm{Sym}^n(\mathbb{C}^2)$ of the standard representation for all $n\in\mathbb{N}_0$. For $t = z+\overline{z}\in[-2,2]$ with $z\in\mathbb{S}^1$ the eigenvalues for

$$\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix} \in \mathrm{SU}_2(\mathbb{R})$$

and the basis vector $e_1^{\odot(n-k)} \odot e_2^{\odot k}$ is $z^{n-k}\overline{z}^k = z^{n-2k}$ for $k \in \{0, \dots, n\}$ and $n \in \mathbb{N}_0$. Hence the character χ associated to $\mathrm{Sym}^n(\mathbb{C}^2)$ is given by

$$\chi_n(t) = \chi_n\left(\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix}\right) = \sum_{k=0}^{n} z^{n-2k}$$

for every $n \in \mathbb{N}_0$. Using the variable $t \in [-2, 2] \cong \mathrm{SU}_2(\mathbb{R})^\sharp$, the first few are given by

$$\begin{aligned}
\chi_0(t) &= 1,\\
\chi_1(t) &= t,\\
\chi_2(t) &= z^2 + 1 + z^{-2} = (z + \overline{z})^2 - 1 = t^2 - 1,
\end{aligned}$$

and

$$\chi_3(t) = z^3 + z + z^{-1} + z^{-2} = (z + \overline{z})^3 - 2(z + \overline{z}) = t^3 - 2t.$$

Using the notation $z = \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{S}^1$ with $\theta \in [0, 2\pi)$, the eigenvalues of $\pi^{(n)}$ are given by

$$z^{n-2k} = \mathrm{e}^{\mathrm{i}(n-2k)\theta}$$

for $k = 0, \dots, n$, and so

$$\begin{aligned}
\chi_n\left(\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix}\right) &= \mathrm{e}^{\mathrm{i}n\theta} + \mathrm{e}^{\mathrm{i}(n-2)\theta} + \cdots + \mathrm{e}^{-\mathrm{i}n\theta}\\
&= \mathrm{e}^{-\mathrm{i}n\theta}\frac{\mathrm{e}^{\mathrm{i}2(n+1)\theta} - 1}{\mathrm{e}^{\mathrm{i}2\theta} - 1}\\
&= \frac{\mathrm{e}^{\mathrm{i}(n+1)\theta} - \mathrm{e}^{-\mathrm{i}(n+1)\theta}}{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}} = \frac{\sin\bigl((n+1)\theta\bigr)}{\sin\theta}
\end{aligned}$$

by the geometric series summation formula (using $q = \mathrm{e}^{\mathrm{i}2\theta}$). Expressing this in terms of

$$t = \mathrm{tr}\begin{pmatrix} z & \\ & \overline{z} \end{pmatrix} = z + \overline{z} = 2\cos\theta$$

gives a formula for the character $\chi_n(t)$. Using instead the variable

$$T = \tfrac{t}{2} = \cos\theta,$$

this would give rise to the Chebyshev polynomials of the second kind.

Exercise 6.14. Describe $\mathrm{SO}_3(\mathbb{R})^\sharp$ and the characters of $\mathrm{SO}_3(\mathbb{R})$.

6.2.5 The Isometry Group of 3-Space

(Possible addition, to be decided later)

6.3 Smooth Vectors and Derivative Representations

6.3.1 Differential Operators and Smooth Vectors

Throughout this section we assume that G is a Lie group with Lie algebra $\mathfrak{g}$.

Definition 6.15. Let π be a unitary representation of the Lie group G. A vector $v \in \mathcal{H}_\pi$ has a *partial derivative* $\pi_\partial(\mathbf{a})v \in \mathcal{H}_\pi$ *in the direction* $\mathbf{a} \in \mathfrak{g}$ if

$$\pi_\partial(\mathbf{a})v = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\bigl(\pi(\exp(t\mathbf{a}))\bigr)v = \lim_{t\to 0}\frac{1}{t}\bigl(\pi(\exp(t\mathbf{a}))v - v\bigr)$$

exists in $\mathcal{H}_\pi$. We say that v is C^1*-smooth* if $\pi_\partial(\mathbf{a})v$ exists for all $\mathbf{a} \in \mathfrak{g}$, is C^r*-smooth* for some $r \geqslant 1$ if $\pi_\partial(\mathbf{a}_1)\cdots\pi_\partial(\mathbf{a}_r)v$ exists for all $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathfrak{g}$, and is *smooth* if v is C^r-smooth for all $r \geqslant 1$.

These notions will become more familiar after we see an example and establish some standard properties of derivatives and integrals in this context.

Example 6.16 (Smooth vectors for unitary representations of $\mathrm{SO}_2(\mathbb{R})$*).* Let

$$G = \mathrm{SO}_2(\mathbb{R}) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

and let

$$\mathbf{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g} = \mathfrak{so}(\mathbb{R}).$$

Furthermore, let π be a unitary representation of $\mathrm{SO}_2(\mathbb{R})$. Suppose first that $v \in \mathcal{H}_\pi$ is an eigenvector of weight $n \in \mathbb{Z}$ (that is, $\pi_{k_\theta}v = \mathrm{e}^{\mathrm{i}n\theta}v$ for all $k_\theta \in \mathrm{SO}_2(\mathbb{R})$). Then v is smooth, since

$$\pi_\partial(\mathbf{w})v = \lim_{t\to 0}\frac{1}{t}\bigl(\pi(\exp(t\mathbf{w}))v - v\bigr) = \lim_{t\to 0}\frac{1}{t}\bigl(\mathrm{e}^{\mathrm{i}nt} - 1\bigr) = \mathrm{i}nv.$$

More generally, $v \in \mathcal{H}_\pi$ with eigenvector decomposition $v = \sum_{n\in\mathbb{Z}} v_n$ has a partial derivative $\pi_\partial(\mathbf{w})v$ if and only if $\sum_{n\in\mathbb{Z}} n^2\|v_n\|^2 < \infty$, and in this case

$$\pi_\partial(\mathbf{w})v = \sum_{n\in\mathbb{Z}} \mathrm{i}nv_n.$$

Indeed, suppose first that $\sum_{n\in\mathbb{Z}} n^2\|v_n\|^2 < \infty$, which gives that

$$\pi_\partial(\mathbf{w})v = \lim_{t\to\infty}\sum_{n\in\mathbb{Z}}\underbrace{\frac{1}{t}\left(\mathrm{e}^{\mathrm{i}nt}-1\right)}_{|\cdot|\leqslant n} v_n = \sum_{n\in\mathbb{Z}}\mathrm{i}nv_n$$

by (a trivial form of) dominated convergence. Suppose now that $\pi_\partial(\mathbf{w})v = \widetilde{v}$ exists, and assume that $\widetilde{v} = \sum_{n\in\mathbb{Z}}\widetilde{v}_n$ is the eigenvalue decomposition. For any $n \in \mathbb{Z}$ and $u \in \mathcal{H}_\pi$ with eigenvector decomposition $u = \sum_{m\in\mathbb{Z}} u_m$, we then have

$$\begin{aligned}\langle\widetilde{v}_n, u\rangle = \langle\widetilde{v}_n, u_n\rangle = \langle\widetilde{v}, u_n\rangle &= \lim_{t\to 0}\frac{1}{t}\langle\pi(\exp(t\mathbf{w}))v - v, u_n\rangle\\ &= \lim_{t\to 0}\frac{1}{t}\big(\langle\pi(\exp(t\mathbf{w}))v, u_n\rangle - \langle v, u_n\rangle\big)\\ &= \lim_{t\to 0}\big(\langle v, \pi(\exp(-t\mathbf{w}))u_n\rangle - \langle v, u_n\rangle\big)\\ &= \lim_{t\to 0}\frac{1}{t}\big(\mathrm{e}^{\mathrm{i}nt}-1\big)\langle v_n, u_n\rangle = \langle\mathrm{i}nv_n, u\rangle.\end{aligned}$$

As this holds for all $u \in \mathcal{H}_\pi$, we see that $\widetilde{v}_n = \mathrm{i}nv_n$ and hence

$$\|\widetilde{v}\| = \sum_{n\in\mathbb{Z}} n^2\|v_n\|^2 < \infty$$

as claimed.

Lemma 6.17 (Linearity, Integral of Derivative). *Let $\mathbf{b}_1, \ldots, \mathbf{b}_{\dim\mathfrak{g}}$ be a basis of the Lie algebra $\mathfrak{g}$ of the Lie group G. Let π be a unitary representation of G, and suppose that $v \in \mathcal{H}_\pi$ has the property that $\pi_\partial(\mathbf{b}_j)v$ exists for all $j = 1, \ldots, \dim\mathfrak{g}$. Then $\pi_\partial(\mathbf{a})v$ exists for all $\mathbf{a} \in \mathfrak{g}$, depends linearly on $\mathbf{a}$, and satisfies*

$$\pi\left(\exp(t\mathbf{a})\right)v - v = \int_0^t \pi\left(\exp(s\mathbf{a})\right)\pi_\partial(\mathbf{a})v\,\mathrm{d}s \tag{6.23}$$

for all $t \in \mathbb{R}$ (with the usual sign conventions for Riemann integrals).

Proof. We start by proving (6.23) for the vector v as in the lemma and a direction $\mathbf{a} \in \mathfrak{g}$ for which $\pi_\partial(\mathbf{a})$ is already known to exist. For this, notice that $\pi(\exp(s\mathbf{a}))\pi_\partial(\mathbf{a})v$ depends continuously on s, which implies that the $\mathcal{H}_\pi$-valued weak integral on the right-hand side of (6.23) exists. Now fix some vector $w \in \mathcal{H}_\pi$ and notice that the derivative of the map $s \mapsto \langle\pi(\exp(s\mathbf{a}))v, w\rangle$ is given by

$$\lim_{t\to 0}\frac{\langle \pi\left(\exp((s+t)\mathbf{a})\right)v, w\rangle - \langle \pi\left(\exp(s\mathbf{a})\right)v, w\rangle}{t}$$
$$= \lim_{t\to 0}\frac{1}{t}\left\langle \pi\left(\exp(t\mathbf{a})\right)v - v, \pi\left(\exp(-s\mathbf{a})\right)w\right\rangle$$
$$= \langle \pi_\partial(\mathbf{a})v, \pi\left(\exp(-s\mathbf{a})\right)w\rangle$$
$$= \langle \pi\left(\exp(s\mathbf{a})\right)\pi_\partial(\mathbf{a})v, w\rangle,$$

and so is also continuous in s. Hence, by the fundamental theorem of calculus (for $\mathbb{C}$-valued functions),

$$\langle \pi\left(\exp(t\mathbf{a})\right)v, w\rangle - \langle v, w\rangle = \int_0^t \langle \pi\left(\exp(s\mathbf{a})\right)\pi_\partial(\mathbf{a})v, w\rangle \,\mathrm{d}s$$
$$= \left\langle \int_0^t \pi\left(\exp(s\mathbf{a})\right)\pi_\partial(\mathbf{a})v\,\mathrm{d}s, w\right\rangle.$$

As this holds for all $w \in \mathcal{H}_\pi$, we see that (6.23) holds for $\mathbf{a}$ (assuming only that $\pi_\partial(\mathbf{a})v$ exists).

By assumption, $\pi_\partial(\mathbf{b}_j)v$ exists for $j = 1, \dots, d = \dim\mathfrak{g}$ so that (6.23) holds, in particular, already for $\mathbf{a} = \mathbf{b}_j$. For the proof of the first part of the lemma, we will combine (6.23) with the coordinate system of the second kind defined by

$$\Psi\colon \mathbb{R}^d \ni (t_1, t_2, \dots, t_d) \longmapsto \exp(t_1\mathbf{b}_1)\exp(t_2\mathbf{b}_2)\cdots\exp(t_d\mathbf{b}_d) \in G.$$

Since the derivative of Ψ at 0 is the map $(s_1, \dots, s_d) \mapsto s_1\mathbf{b}_1 + \dots + s_d\mathbf{b}_d \in \mathfrak{g}$ and so is invertible, Ψ indeed defines a local diffeomorphism. For some $\mathbf{a} \in \mathfrak{g}$ we define smooth functions $t_j(t)$ for t close to 0 and $j = 1, \dots, d$ by

$$(t_1(t), t_2(t), \dots, t_d(t)) = \Psi^{-1}\left(\exp(t\mathbf{a})\right),$$

or equivalently by

$$\exp(t\mathbf{a}) = \exp(t_1(t)\mathbf{b}_1)\cdots\exp(t_d(t)\mathbf{b}_d).$$

Since the derivative of $t \mapsto \exp(t\mathbf{a})$ at $t = 0$ is $\mathbf{a} \in \mathfrak{g}$, we see that the derivative of $\Psi^{-1}(\exp(t\mathbf{a}))$ at $t = 0$ is equal to $(s_1, \dots, s_d) \in \mathbb{R}^d$ with

$$s_1\mathbf{b}_1 + \dots + s_d\mathbf{b}_d = \mathbf{a}.$$

Equivalently, we have

$$\lim_{t\to 0}\frac{t_j(t)}{t} = s_j \tag{6.24}$$

for $j = 1, \dots, d$.

We now express $\pi(\exp(t\mathbf{a}))v - v$ as the telescoping sum

$$\sum_{j=1}^{d}\Big(\pi\left(\exp(t_1\mathbf{b}_1)\cdots\exp(t_j\mathbf{b}_j)\right)v-\pi\left(\exp(t_1\mathbf{b}_1)\cdots\exp(t_{j-1}\mathbf{b}_{j-1})\right)v\Big)$$
$$=\sum_{j=1}^{d}\pi\Big(\exp(t_1\mathbf{b}_1)\cdots\exp(t_{j-1}\mathbf{b}_{j-1})\Big)\Big(\pi\left(\exp(t_j\mathbf{b}_j)\right)v-v\Big)$$

and apply (6.23) for the directions $\mathbf{b}_1,\dots,\mathbf{b}_d$. This shows that $\pi(\exp(t\mathbf{a}))v-v$ equals

$$\sum_{j=1}^{d}\pi\Big(\exp(t_1\mathbf{b}_1)\cdots\exp(t_{j-1}\mathbf{b}_{j-1})\Big)\int_0^{t_j}\pi\left(\exp(s\mathbf{b}_j)\right)\pi_\partial(\mathbf{b}_j)v\,\mathrm{d}s$$
$$=\sum_{j=1}^{d}\int_0^{t_j}\pi\Big(\exp(t_1\mathbf{b}_1)\cdots\exp(t_{j-1}\mathbf{b}_{j-1})\exp(s\mathbf{b}_j)\Big)\pi_\partial(\mathbf{b}_j)v\,\mathrm{d}s.$$

We now divide by t and use (6.24) together with continuity of the representation to obtain

$$\lim_{t\to 0}\frac{1}{t}\left(\pi(\exp(t\mathbf{a}))v-v\right)=\sum_{j=1}^{d}s_j\pi_\partial(\mathbf{b}_j)v.$$

This proves that

$$\pi_\partial(\mathbf{a})v=s_1\pi_\partial(\mathbf{b}_1)v+\cdots+s_d\pi_\partial(\mathbf{b}_d)v$$

exists and depends linearly on $\mathbf{a}\in\mathfrak{g}$. Together with the first part of the proof, the lemma follows. □

Exercise 6.18. Suppose that π is a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$. Suppose $v\in\mathcal{H}_\pi$ has $v_{\mathbf{a}}\in\mathcal{H}_\pi$ as a weak derivative in the direction $\mathbf{a}\in\mathfrak{g}$ in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\langle\pi\left(\exp(t\mathbf{a})\right)v,w\rangle=\langle v_{\mathbf{a}},w\rangle$$

for all $w\in\mathcal{H}_\pi$ (or just for a dense set of vectors). Show in this case that $v_{\mathbf{a}}=\pi_\partial(\mathbf{a})v$ is in fact the derivative of v in the sense of Definition 6.15.

Proposition 6.19. *Let π be a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$, $v\in\mathcal{H}_\pi$, and $\psi\in C_c^\infty(G)$. Then $\pi_*(\psi)v$ is smooth, and*

$$\pi_\partial(\mathbf{a})\pi_*(\psi)v=-\pi_*(\partial_{\mathbf{a}}\psi)v$$

for any $\mathbf{a}\in\mathfrak{g}$, where

$$\partial_{\mathbf{a}}\psi(g)=\frac{\partial}{\partial t}\Big|_{t=0}\psi\left(\exp(t\mathbf{a})g\right).$$

In particular, the smooth vectors in $\mathcal{H}_\pi$ are dense.

Proof. Using the definition of the convolution operator we see that

$$\begin{aligned}\pi\left(\exp(t\mathbf{a})\right)\pi_*(\psi)v &= \int_G \pi\left(\exp(t\mathbf{a})\right)\psi(h)\pi_h v\,\mathrm{d}m(h)\\ &= \int_G \psi\left(\exp(-t\mathbf{a})g\right)\pi_g v\,\mathrm{d}m(g)\end{aligned}$$

by using the substitution $g = \exp(t\mathbf{a})h$. This gives

$$\begin{aligned}\frac{1}{t}\big(\pi(\exp(t\mathbf{a}))\pi_*(\psi)v - \pi_*(\psi)v\big) &= \frac{1}{t}\int_G \big(\psi\left(\exp(-t\mathbf{a})g\right) - \psi(g)\big)\pi_g v\,\mathrm{d}m(g)\\ &= \int_G \frac{\psi(\exp(-t\mathbf{a})g) - \psi(g)}{t}\pi_g v\,\mathrm{d}m(g)\end{aligned}$$

for all $t \in \mathbb{R}\smallsetminus\{0\}$. As $\psi \in C_c^\infty(G)$, we know that

$$\frac{\psi(\exp(-t\mathbf{a})g) - \psi(g)}{t} \longrightarrow -\partial_a\psi(g) \tag{6.25}$$

as $t \to 0$, that this convergence is uniform in g, and that this convergence takes place inside a compact subset of G in the sense that the left-hand side vanishes for all $t \in [-1,1]\smallsetminus\{0\}$ outside the compact subset $\exp\left([-1,1]\mathbf{a}\right)\operatorname{supp}(\psi) \subseteq G$. In particular, the convergence also takes place in $L^1(G)$, and it follows that $\pi_\partial(\mathbf{a})\pi_*(\psi)v = \pi_*(-\partial_{\mathbf{a}}\psi)v$ exists. Applying this inductively to expressions of the form $\pi_\partial(\mathbf{a}_n)\cdots\pi_\partial(\mathbf{a}_1)\pi_*(\psi)v$ for $n \geqslant 1$ gives that $\pi_*(\psi)v$ is smooth.

Using an approximate identity in $L^1(G) \cap C_c^\infty(G)$ the proposition follows from Proposition 1.39. □

Definition 6.20. Let π be a unitary representation of G, let $\mathbf{b}_1, \dots, \mathbf{b}_{\dim\mathfrak{g}}$ be a basis of $\mathfrak{g} = \operatorname{Lie} G$, and let $r \geqslant 0$ be an integer. The *degree r Sobolev norm* of a C^r-smooth vector $v \in \mathcal{H}_\pi$ (with respect to the fixed basis) is defined by

$$\mathcal{S}(v)^2 = \mathcal{S}_r(v)^2 = \sum_{s=0}^{r}\sum_{j_1,\dots,j_s=1}^{\dim\mathfrak{g}} \|\pi_\partial(\mathbf{b}_{j_1})\cdots\pi_\partial(\mathbf{b}_{j_s})v\|^2.$$

Essential Exercise 6.21. Let π be a unitary representation of the Lie group G and v a C^1-smooth vector. Show that $\|\pi_{\exp a}v - v\| \leqslant \|a\|\,\mathcal{S}(v)$, where $\mathcal{S}$ is a degree-one Sobolev norm defined by an orthonormal basis of $\mathfrak{g}$.

Exercise 6.22. Extend Proposition 6.19, and show that $\mathcal{S}_r(\pi_*(\psi)v) \ll_\psi \|v\|$ (and express the implicit constant in terms of ψ).

Example 6.23 (Derivatives and smooth vectors for $\mathbb{R}^d$). We let $G = \mathbb{R}^d$ for some $d \in \mathbb{N}$, and will use the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ of its Lie algebra $\mathfrak{g}$ (which is also $\mathbb{R}^d$). Let π be a unitary representation. Applying the spectral theorem (Corollary 2.11), we assume that $\mathcal{H}_\pi = L^2_\mu(X)$ for a finite (or σ-finite) measure μ on $X = \mathbb{R}^d \times \mathbb{N}$ and that π is defined by the multiplication representation[†]

$$\big(\pi_x v\big)(t,n) = \mathrm{e}^{2\pi\mathrm{i}(x\cdot t)} v(t,n)$$

for all $x \in \mathbb{R}^d$, $v \in L^2_\mu(X)$, and $(t,n) \in \mathbb{R}^d \times \mathbb{N}$. In this case we obtain, for a C^1-smooth vector v, that

$$\pi_\partial(\mathbf{e}_j)v = \lim_{s\to 0} \underbrace{\frac{\mathrm{e}^{2\pi\mathrm{i} s t_j} - 1}{s}}_{|\cdot|\leqslant 2\pi|t_j|} v(t,n) = 2\pi\mathrm{i} t_j v(t,n) \tag{6.26}$$

for $j = 1, \dots, d$, so that

$$\mathcal{S}_1(v)^2 = \|v\|^2 + \sum_{j=1}^d \|M_{2\pi\mathrm{i}t_j} v\| \tag{6.27}$$

where $M_{2\pi\mathrm{i}t_j}$ is the multiplication operator on $L^2_\mu(X)$ defined by

$$\big(M_{2\pi\mathrm{i}t_j} v\big)(t,n) = 2\pi\mathrm{i} t_j v(t,n)$$

for all $v \in L^2_\mu(X)$, $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, and $n \in \mathbb{N}$. Moreover, if $v \in L^2_\mu(X)$ and $j \in \{1, \dots, d\}$ have the property that $M_{2\pi\mathrm{i}t_j}(v)$ belongs to $L^2_\mu(X)$, then, by applying dominated convergence in (6.26), we see that $\pi_\partial(\mathbf{e}_j)(v)$ exists. If this holds for all $j \in \{1, \dots, d\}$, then v is C^1-smooth and the Sobolev norm of degree one is given by (6.27).

Now let $r \in \mathbb{N}$. Applying the above recursively to the partial derivatives, we see that $v \in L^2(\mathbb{R}^d)$ is C^r-smooth if and only if $pv \in L^2_\mu(X)$ where p is any polynomial in $\mathbb{C}[t_1, \dots, t_d]$ of degree at most r.

Finally, we wish to apply this to the regular representation λ of $\mathbb{R}^d$ on $L^2(\mathbb{R}^d)$, which will reveal the connection to Sobolev spaces (see, for example, [16, Ch. 5]). By the Plancherel formula (Theorem 2.13), the regular representation is isomorphic to the multiplication representation as above for the Lebesgue measure $\mu = m_{\mathbb{R}^d}$. Applying the above, we see that $v \in L^2(\mathbb{R}^d)$ is smooth for the regular representation if and only if $\check{v} \in L^2(\mathbb{R}^d)$ satisfies $p\check{v} \in L^2(\mathbb{R}^d)$ for any polynomial $p \in \mathbb{C}[t_1, \dots, t_d]$. Using the polynomial

$$p(t) = \prod_{j=1}^d (t_j^2 + 1)$$

[†] In Corollary 2.11 we used a simplified notation and wrote M_g for the multiplication operator defined by the function $\widehat{G} \ni t \mapsto \langle g, t\rangle \in \mathbb{S}^1$. In the case of $x \in \mathbb{R}^d$ and $t \in \widehat{\mathbb{R}^d} \cong \mathbb{R}^d$ this function corresponds to $\mathbb{R}^d \ni t \mapsto \mathrm{e}^{2\pi\mathrm{i}(x\cdot t)}$ by Exercise 2.6 or Proposition 2.34).

which has $\frac{1}{p} \in L^2(\mathbb{R}^d)$, it follows that $\check{v} = \frac{1}{p}(p\check{v}) \in L^1(\mathbb{R}^d)$ and hence by Corollaries 2.11 and 2.5 that $v = \widehat{(\check{v})} \in C_0(\mathbb{R}^d)$. Now fix some $j \in \{1, \dots, d\}$ and note that the above also applies to $\lambda_\partial(\mathbf{e}_j)v \in C_0(\mathbb{R}^d)$. Here

$$\lambda_\partial(\mathbf{e}_j)v = \lim_{s\to 0} \frac{1}{s}\bigl(\lambda_{s\mathbf{e}_j} v - v\bigr),$$

which, for the isometric Fourier transform on L^2, becomes as above

$$M_\partial(\mathbf{e}_j)\check{v}(t) = \lim_{s\to 0} \frac{1}{s}\bigl(\mathrm{e}^{2\pi \mathrm{i} s t_j} - 1\bigr)\check{v}(t) = 2\pi \mathrm{i} t_j \check{v}(t)$$

in $L^2(\mathbb{R}^d)$ and for almost any $t \in \mathbb{R}^d$. We now multiply this once more by $p(t)$ and apply dominated convergence (by relying on the fact that $t \mapsto t_j p(t)\check{v}(t)$ lies in $L^2(\mathbb{R}^d)$) to see that

$$\lim_{s\to 0} p(t) \underbrace{\frac{1}{s}\bigl(\mathrm{e}^{2\pi \mathrm{i} s t_j} - 1\bigr)}_{|\cdot| \leqslant 2\pi |t_j|} \check{v}(t) = p(t) 2\pi \mathrm{i} t_j \check{v}(t)$$

converges in $L^2(\mathbb{R}^d)$. Multiplying by $\frac{1}{p} \in L^2(\mathbb{R}^d)$ gives convergence in $L^1(\mathbb{R}^d)$ by the Cauchy–Schwarz inequality. However, this gives, by the continuity bound in Corollary 2.5 applied to the forward Fourier transform

$$L^1(\widehat{G}) \ni F \longrightarrow \widehat{F} \in C_0(G)$$

that

$$\left\| \frac{1}{s}\bigl(\lambda_{s\mathbf{e}_j} v - v\bigr) - \lambda_\partial(\mathbf{e}_j)v \right\|_\infty \leqslant \left\| \frac{1}{s}\bigl(\mathrm{e}^{2\pi \mathrm{i} s t_j} - 1\bigr)\check{v}(t) - \widecheck{\lambda_\partial(\mathbf{e}_j)v} \right\|_1 \longrightarrow 0$$

as $s \to 0$. As this holds for all $j \in \{1, \dots, d\}$ and can be applied recursively to the partial derivatives of v, it follows that $v \in C^\infty(\mathbb{R}^d)$.

Once again the reasoning above can be reversed to see that $v \in L^2(\mathbb{R}^d)$ is smooth with respect to the regular representation if and only if $v \in C^\infty(\mathbb{R}^d)$ and its partial derivatives $\partial^\alpha v$ belong to $L^2(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ (see Exercise 6.24).

Exercise 6.24. Complete the proof of the last claim in Example 6.23.

Essential Exercise 6.25. Let π be a unitary representation of a Lie group G with Lie algebra $\mathfrak{g}$. Let $r \geqslant 1$ and $v \in \mathcal{H}_\pi$ be a C^r-smooth vector. Let $\mathcal{S}$ denote the degree r Sobolev norm. Show that $\mathcal{S}(\pi_g v) \ll_g \mathcal{S}(v)$, and that the implicit constant can be chosen to be uniformly bounded on compact subsets of G.

Exercise 6.26. Let π be a unitary representation of G, let $\mathbf{b}_1, \dots, \mathbf{b}_{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g}$, the Lie algebra of G, and let $\widetilde{\mathbf{b}}_1, \dots, \widetilde{\mathbf{b}}_{\dim \mathfrak{g}}$ be another basis of $\mathfrak{g}$. Let $r \geqslant 1$ and let $\mathcal{S}$ (respectively $\widetilde{\mathcal{S}}$) be the degree r Sobolev norm defined by $\mathbf{b}_1, \dots, \mathbf{b}_{\dim \mathfrak{g}}$ (resp. by $\widetilde{\mathbf{b}}_1, \dots, \widetilde{\mathbf{b}}_{\dim \mathfrak{g}}$). Show that we have $\mathcal{S}(v) \ll \widetilde{\mathcal{S}}(v) \ll \mathcal{S}(v)$ for any C^r-smooth $v \in \mathcal{H}_\pi$.

Show also that if $r = 1$ and both bases are orthonormal with respect to an inner product on $\mathfrak{g}$, then $\mathcal{S}(v) = \widetilde{\mathcal{S}}(v)$ for any C^1-smooth $v \in \mathcal{H}_\pi$.

Exercise 6.27. (a) Let $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the isometry group of the plane as in Section 3.3.1. Let $\pi \in \widehat{G}$ be an irreducible representation. Find and prove a description of the space of smooth vectors in $\mathcal{H}_\pi$. Also show that any $v \in \mathcal{H}_\pi$ is smooth for the restriction of π to $H = \mathbb{R}^2$.
(b) Let G be the '$ax + b$' group as in Section 3.3.2, and let $\pi_+ \in \widehat{G}$ be the irreducible representation corresponding to the set $(0, \infty) \subseteq \mathbb{R} \cong \widehat{R}$. Show that any $f \in C_c^\infty\big((0,\infty)\big)$ is a smooth vector. Can you again characterize smoothness with an appropriate moment condition?
(c) Let G be the Heisenberg group as in Section 3.3.4, and let $\pi_\xi \in \widehat{G}$ be the irreducible representation corresponding to the central character χ_ξ determined by $\xi \in \mathbb{R}^\times$. Show that any $f \in C_c^\infty(\mathbb{R})$ is a smooth vector. Can you again characterize smoothness?

6.3.2 The Total Derivative

Definition 6.28 (Total derivative). Let π be a unitary representation of G. The *total derivative* of π is defined on every C^1-smooth vector $v \in \mathcal{H}_\pi$ as the linear map $T_\pi(v)$ in $\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$ given by

$$T_\pi(v)\colon \mathbf{a} \longmapsto \pi_\partial(\mathbf{a})v$$

for $\mathbf{a} \in \mathfrak{g}$. After fixing a basis $\mathbf{b}_1, \dots, \mathbf{b}_{\dim \mathfrak{g}}$ of $\mathfrak{g}$ we can identify $T_\pi(v)$ with the tuple

$$(\pi_\partial(\mathbf{b}_1)v, \dots, \pi_\partial(\mathbf{b}_{\dim \mathfrak{g}})v) \in \mathcal{H}_\pi^{\dim \mathfrak{g}} \cong \mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi).$$

Lemma 6.29 (Closed operator). *Let π be a unitary representation of G. Then the total derivative T_π with domain*

$$D_{T_\pi} = \{v \in \mathcal{H}_\pi \mid v \text{ is } C^1\text{-smooth}\}$$

is a densely defined closed operator.

Proof. Suppose that (v_n) in D_{T_π} is a sequence with

$$(v_n, T_\pi(v_n)) \longrightarrow (v, L) \in \mathcal{H}_\pi \times \mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)$$

as $n \to \infty$ and let $\mathbf{a} \in \mathfrak{g}$. By Lemma 6.17 this implies that

$$\pi\left(\exp(t\mathbf{a})\right) v_n - v_n = \int_0^t \pi\left(\exp(s\mathbf{a})\right) \underbrace{\pi_\partial(\mathbf{a})v_n}_{=T_\pi(v_n)a} \,\mathrm{d}s$$

for any $t \in \mathbb{R}$. Since $T_\pi(v_n) \to L$ in $\mathcal{H}_\pi^{\dim \mathfrak{g}}$ as $n \to \infty$ we have

$$T_\pi(v_n)(\mathbf{a}) \longrightarrow L(\mathbf{a})$$

as $n \to \infty$. Moreover, since the integral defines a continuous operator on $\mathcal{H}_\pi$ we also obtain from this that

$$\pi\left(\exp(t\mathbf{a})\right)v - v = \int_0^t \pi\left(\exp(s\mathbf{a})\right)L(\mathbf{a})\,\mathrm{d}s.$$

Dividing this by $t \neq 0$ and using continuity of the representation π we arrive at

$$\pi_\partial(\mathbf{a})v = L(\mathbf{a})$$

for any $\mathbf{a} \in \mathfrak{g}$. However, this implies that $v \in D_{T_\pi}$ and $T_\pi(v) = L$, and hence the lemma. $\square$

We will now bring this into connection with the *adjoint* representation $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ for $g \in G$, satisfying $\exp(\mathrm{Ad}_g(v)) = g\exp(v)g^{-1}$ for $g \in G$ and $v \in \mathfrak{g}$.

Lemma 6.30 (Chain rule). *Let π be a unitary representation of G. Then*

$$T_\pi(\pi_g v) = D\pi_g(T_\pi(v))$$

for every C^1-smooth vector $v \in \mathcal{H}_\pi$, where $D\pi$ is the continuous representation defined by $D\pi_g(L) = \pi_g \circ L \circ \mathrm{Ad}_{g^{-1}}$ for any linear map $L\colon \mathfrak{g} \to \mathcal{H}_\pi$. In other words, for a C^1-smooth vector $v \in \mathcal{H}_\pi$, $\mathbf{a} \in \mathfrak{g}$, and $g \in G$, we have

$$\pi_\partial(\mathbf{a})\pi_g v = \pi_g\pi_\partial\big(\mathrm{Ad}_{g^{-1}}\mathbf{a}\big)v.$$

In particular, the vector space of C^r-smooth vectors is invariant under π_g for every $g \in G$ and $r \geqslant 1$.

Let us point out that Lemma 6.30 is indeed just a version of the chain rule adapted for unitary representations. Given a smooth vector $v \in \mathcal{H}_\pi$, Lie algebra element $\mathbf{a} \in \mathfrak{g}$, and group element $g \in G$, we wish to calculate the derivative $\pi_\partial(\mathbf{a})\pi_g v$ of $\pi_g v$ in the direction $\mathbf{a}$. Knowing that $\pi_\partial(\mathbf{b})v$ exists for all $\mathbf{b} \in \mathfrak{g}$, we will set $\mathbf{b} = \mathrm{Ad}_g^{-1}\mathbf{a}$ and use the commutative diagram

$$\begin{array}{ccc} v & \xmapsto{\ \pi\left(g^{-1}\exp(t\mathbf{a})g\right)\ } & \pi_{\exp\left(t\,\mathrm{Ad}_g^{-1}a\right)}v \\ \Big\downarrow{\scriptstyle \pi_g} & & \Big\downarrow{\scriptstyle \pi_g} \\ \pi_g v & \xmapsto{\ \pi(\exp(t\mathbf{a}))\ } & \pi_{\exp(t\mathbf{a})}\pi_g v \end{array}$$

We note that continuity of the representation is defined as in Definition 1.1(3) but that we did not claim unitarity of the representation $D\pi$ (see Section 6.3.3).

Proof of Lemma 6.30. Let $g \in G$, $\mathbf{a} \in \mathfrak{g}$, and let $v \in \mathcal{H}_\pi$ be a C^1-smooth vector. Using the fact that π_g is bounded we have

$$\pi_\partial(\mathbf{a})\pi_g v = \frac{\partial}{\partial t}\Big|_{t=0} \pi\left(\exp(t\mathbf{a})g\right) v$$
$$= \lim_{t\to 0}\frac{1}{t}\pi_g\left(\pi\left(\exp(t\operatorname{Ad}_g^{-1}\mathbf{a})\right)v - v\right) = \pi_g\pi_\partial(\operatorname{Ad}_g^{-1}\mathbf{a})v.$$

As this holds for any $\mathbf{a}\in\mathfrak{g}$ we see that $T_\pi\circ\pi_g = D\pi_g\circ T_\pi$ (where defined).

To see that $D\pi$ defines a representation on $\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)$ let $g,h\in G$ and let $L\in\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)$, and calculate

$$D\pi_g\left(D\pi_h(L)\right) = \pi_g\circ D\pi_h(L)\circ\operatorname{Ad}_{g^{-1}}$$
$$= \pi_g\circ\pi_h\circ L\circ\operatorname{Ad}_{h^{-1}}\circ\operatorname{Ad}_{g^{-1}} = D\pi_{gh}(L).$$

Moreover, notice that $D\pi_e(L)=L$.

As noted after Definition 6.28, we make the identification of $\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)$ with $\mathcal{H}_\pi^{\dim\mathfrak{g}}$ using a fixed basis of $\mathfrak{g}=\operatorname{Lie}G$. This identification gives the vector space $\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)$ the structure of a Hilbert space. With this, we also have

$$\|D\pi_g(L)\|^2 = \sum_{j=1}^{\dim\mathfrak{g}}\|\pi_g\bigl(L(\operatorname{Ad}_g\mathbf{b}_j)\bigr)\|^2 = \sum_{j=1}^{\dim\mathfrak{g}}\|L(\operatorname{Ad}_g\mathbf{b}_j)\|^2$$
$$\leqslant \dim\mathfrak{g}\max_{j=1,\dots,\dim\mathfrak{g}}\left\|\sum_{k=1}^{\dim\mathfrak{g}}\left[\operatorname{Ad}_g^{-1}\right]_{kj}\left(L(\mathbf{b}_k)\right)\right\|^2$$
$$\leqslant (\dim\mathfrak{g})^3\max_{j=1,\dots,\dim\mathfrak{g}}\left[\operatorname{Ad}_g^{-1}\right]_{kj}\|(L(\mathbf{b}_k))\|^2 \ll_g \|L\|^2.$$

In other words, $D\pi_g$ is a bounded operator on $\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)$, where $\left[\operatorname{Ad}_g^{-1}\right]_{kj}$ denotes the matrix entry of the matrix representing the linear map

$$\operatorname{Ad}_g^{-1}\colon\mathfrak{g}\to\mathfrak{g}$$

in the basis $\mathbf{b}_1,\dots,\mathbf{b}_{\dim\mathfrak{g}}$. To see the continuity of the representation $D\pi$, let $L\in\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)$, fix some $j\in\{1,\dots,\dim\mathfrak{g}\}$, and suppose (g_n) is a sequence in G with $g_n\to g$ as $n\to\infty$. Then

$$D\pi_{g_n}(L)(\mathbf{b}_j) = \pi_{g_n}\left(L\left(\operatorname{Ad}_{g_n}^{-1}\mathbf{b}_j\right)\right) = \sum_{k=1}^{\dim\mathfrak{g}}\left[\operatorname{Ad}_{g_n}^{-1}\right]_{kj}\pi_{g_n}\left(L(\mathbf{b}_k)\right)$$
$$\longrightarrow\sum_{k=1}^{\dim\mathfrak{g}}\left[\operatorname{Ad}_g^{-1}\right]_{kj}\pi_g\left(L(\mathbf{b}_k)\right) = \pi_g\left(L\operatorname{Ad}_g^{-1}\mathbf{b}_j\right) = D\pi_g(L)(\mathbf{b}_j)$$

as $n\to\infty$, This gives $D\pi_{g_n}(L)\to D\pi_g(L)$ as $n\to\infty$ as required.

The final statement follows from the argument above and induction on the degree of smoothness $r\geqslant 1$. □

We finish this subsection with an interesting exercise, which requires the following definition.

Definition 6.31 (Adjoint operator). Let T be a densely defined closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$. The adjoint operator T^* is defined on the domain

$$D_{T^*} = \left\{ w \in \mathcal{H}_2 \mid D_T \ni v \longmapsto \langle Tv, w \rangle_{\mathcal{H}_2} \text{ is bounded} \right\}$$

and satisfies

$$\langle Tv, w \rangle_{\mathcal{H}_2} = \langle v, T^* w \rangle_{\mathcal{H}_1}$$

for all $v \in D_T$ and $w \in D_{T^*}$.

We refer to [16, Lemma 13.3] for the properties of the adjoint operator.

Exercise 6.32. Show that $\pi_\partial(\mathbf{a})^* = -\pi_\partial(\mathbf{a})$ for any unitary representation π of G and element $\mathbf{a} \in \mathfrak{g}$.

6.3.3 Unitarity of the Derivative Representation

In this section we prove the following proposition which gives unitarity of the total derivative in some interesting cases.

Proposition 6.33 (Unitarity of D and Equivariance of $T_\pi^* T_\pi$). *Let G be a Lie group with Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. Suppose that $\mathfrak{g}$ is equipped with an inner product with the property that Ad_g is orthogonal for any $g \in G$, and let π be a unitary representation of G. Use an orthonormal basis of $\mathfrak{g}$ to define the isomorphism $\operatorname{End}(\mathfrak{g}, \mathcal{H}_\pi) = \mathcal{H}_\pi^{\dim \mathfrak{g}}$ and hence a Hilbert space structure on $\operatorname{End}(\mathfrak{g}, \mathcal{H}_\pi)$. Then the derivative representation $D\pi$ on $\operatorname{End}(\mathfrak{g}, \mathcal{H}_\pi)$ is unitary, and hence $T_\pi^* T_\pi$ is a densely defined closed equivariant operator from $\mathcal{H}_\pi$ to $\mathcal{H}_\pi$. If $\mathbf{b}_1, \ldots, \mathbf{b}_{\dim G}$ is an orthonormal basis of $\mathfrak{g}$ and $v \in \mathcal{H}_\pi$ is C^2-smooth, then*

$$T_\pi^* T_\pi = - \sum_{j=1}^{\dim G} \pi_\partial(\mathbf{b}_j)^2 v = \Omega v. \tag{6.28}$$

If π is irreducible, then there exists some $\lambda_\pi \geqslant 0$ with $\Omega v = \lambda_\pi v$ for all v in $\mathcal{H}_\pi$.

Notice that the Hilbert space structure of $\operatorname{End}(\mathfrak{g}, \mathcal{H}_\pi)$ and the representation D becomes clearer after noting that $\operatorname{End}(\mathfrak{g}, \mathcal{H}_\pi) \cong \mathcal{H}_\pi \otimes_{\mathbb{R}} \mathfrak{g}$ and that the latter carries a unitary representation since $\mathcal{H}_\pi$ carries a unitary representation and the real Hilbert space $\mathfrak{g}$ carries a natural representation of G that is assumed to be 'orthogonal'.

Allowing ourself to consider formal products of Lie algebra elements (giving elements of the so-called universal enveloping algebra), we may write the differential operator Ω on $\mathcal{H}_\pi$ also as $-\pi_\partial\left(\sum_{j=1}^{\dim G} \mathbf{b}_j \mathbf{b}_j\right)$.

Proof of Proposition 6.33. Let $\mathfrak{g} = \operatorname{Lie} G$ and π be as in the proposition and suppose that $\mathbf{b}_1, \dots, \mathbf{b}_{\dim G}$ is an orthonormal basis with respect to the assumed inner product on $\mathfrak{g}$. For $g \in G$ and $L \in \operatorname{End}(\mathfrak{g}, \mathcal{H}_\pi)$ we have by unitarity of π_g that

$$\begin{aligned}
\|D\pi_g L\|^2_{\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)} &= \sum_{j=1}^{\dim \mathfrak{g}} \left\|\pi_g L\big(\operatorname{Ad}_g^{-1} \mathbf{b}_j\big)\right\|^2_{\mathcal{H}_\pi} \\
&= \sum_{j=1}^{\dim \mathfrak{g}} \left\|L\big(\operatorname{Ad}_g^{-1} \mathbf{b}_j\big)\right\|^2_{\mathcal{H}_\pi} \\
&= \sum_{j=1}^{\dim \mathfrak{g}} \left\|\sum_{k=1}^{\dim \mathfrak{g}} \big[\operatorname{Ad}_g^{-1}\big]_{kj} L(\mathbf{b}_k)\right\|^2_{\mathcal{H}_\pi} \\
&= \sum_{j=1}^{\dim \mathfrak{g}} \sum_{k,\ell=1}^{\dim \mathfrak{g}} \big[\operatorname{Ad}_g^{-1}\big]_{kj} \big[\operatorname{Ad}_g^{-1}\big]_{\ell j} \langle L(\mathbf{b}_k), L(\mathbf{b}_\ell)\rangle_{\mathcal{H}_\pi},
\end{aligned}$$

where $[\operatorname{Ad}_g^{-1}]_{kj}$ again denotes the entries of the matrix representation of Ad_g^{-1} in the basis $\mathbf{b}_1, \dots, \mathbf{b}_{\dim \mathfrak{g}}$. Reordering the summation in the last expression above we obtain

$$\begin{aligned}
\|D\pi_g L\|^2_{\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)} &= \sum_{k,\ell=1}^{\dim \mathfrak{g}} \underbrace{\sum_{j=1}^{\dim \mathfrak{g}} \big[\operatorname{Ad}_g^{-1}\big]_{kj} \big[\operatorname{Ad}_g^{-1}\big]_{\ell j}}_{\delta_{k,\ell}} \langle L(\mathbf{b}_k), L(\mathbf{b}_\ell)\rangle_{\mathcal{H}_\pi} \\
&= \sum_{k=1}^{\dim \mathfrak{g}} \|L(\mathbf{b}_k)\|^2_{\mathcal{H}_\pi} = \|L\|_{\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)}
\end{aligned}$$

as required.

This implies that $T_\pi^* T_\pi$ is equivariant where it is defined. To see this, we first recall that for $g \in G$ we have

$$T_\pi \pi_g = D\pi_g \circ T_\pi$$

by Lemma 6.30. Suppose now that v is in the domain of $T_\pi^* T_\pi$ and $g \in G$. Then for all w in the domain of T_π we have

$$\begin{aligned}
\langle \pi_g T_\pi^* T_\pi v, w\rangle_{\mathcal{H}_\pi} = \langle T_\pi^* T_\pi v, \pi_g^{-1} w\rangle &= \langle T_\pi v, T_\pi \pi_{g^{-1}} w\rangle_{\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)} \\
&= \langle T_\pi v, D_{\pi_{g^{-1}}} T_\pi w\rangle_{\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)} \\
&= \langle D_{\pi_g} T_\pi v, T_\pi w\rangle_{\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)} \\
&= \langle T_\pi \pi_g v, T_\pi w\rangle_{\operatorname{End}(\mathfrak{g},\mathcal{H}_\pi)}.
\end{aligned}$$

However, this, by definition of the adjoint, implies that $T_\pi \pi_g v$ belongs to the domain of T_π^* and that

$$T_\pi^* T_\pi \pi_g v = \pi_g T_\pi^* T_\pi v.$$

This shows that the domain of $T_\pi^* T_\pi$ is invariant under π_g, and that

$$\pi_g T_\pi^* T_\pi \supseteq T_\pi^* T_\pi \pi_g$$

for all $g \in G$. Applying this for g^{-1} together with the invariance of the domain of $T_\pi^* T_\pi$, we actually obtain

$$\pi_g T_\pi^* T_\pi = T_\pi^* T_\pi \pi_g$$

for $g \in G$, as required. For the proof that $T_\pi^* T_\pi$ is densely defined and closed, we refer to [16, Th. 13.10] (see also Section 1.3.4 for similar arguments).

Now let $w_1, \ldots, w_{\dim \mathfrak{g}} \in \mathcal{H}_\pi$ be C^1-smooth vectors. We claim that the linear map L defined by

$$L\left(\sum_{j=1}^{\dim \mathfrak{g}} s_j \mathbf{b}_j\right) = \sum_{j=1}^{\dim \mathfrak{g}} s_j w_j$$

belongs to the domain $D_{T_\pi^*}$ of T_π^*, and

$$T_\pi^*(L) = -\sum_{j=1}^{\dim \mathfrak{g}} \pi_\partial(\mathbf{b}_j) w_j.$$

For this, let $v \in \mathcal{H}_\pi$ be C^1-smooth (that is, in the domain of T_π) and calculate

$$\begin{aligned}
\left\langle -\sum_{j=1}^{\dim \mathfrak{g}} \pi_\partial(\mathbf{b}_j) w_j, v \right\rangle_{\mathcal{H}_\pi} &= -\sum_{j=1}^{\dim \mathfrak{g}} \langle \pi_\partial(\mathbf{b}_j) w_j, v \rangle_{\mathcal{H}_\pi} \\
&= -\sum_{j=1}^{\dim \mathfrak{g}} \lim_{t \to 0} \frac{1}{t} \langle \pi(\exp(t\mathbf{b}_j)) w_j - w_j, v \rangle_{\mathcal{H}_\pi} \\
&= -\sum_{j=1}^{\dim \mathfrak{g}} \lim_{t \to 0} \frac{1}{t} \langle w_j, \pi(\exp(-t\mathbf{b}_j)) v - v \rangle_{\mathcal{H}_\pi} \\
&= \sum_{j=1}^{\dim \mathfrak{g}} \langle w_j, \pi_\partial(\mathbf{b}_j) v \rangle_{\mathcal{H}_\pi} = \langle L, Tv \rangle_{\mathrm{End}(\mathfrak{g}, \mathcal{H}_\pi)}.
\end{aligned}$$

As $v \in D_{T_\pi}$ was arbitrary, this gives the claim and the claim implies (6.28). The final claim in the proposition follows from Schur's lemma (Corollary 1.30). □

6.3.4 The Casimir Operator for $\mathbf{SU(2,\mathbb{R})}$

We wish to study an example of Proposition 6.33 and explicitly calculate the constants λ_π for all irreducible representations of $G = \mathrm{SU}_2(\mathbb{R})$ (which we already classified in Section 6.2). For this we will also use the basis $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathrm{SU}_2(\mathbb{R})$ in (6.4).

Corollary 6.34 (Casimir operator on $\mathrm{Sym}^n(\mathbb{C}^2)$**).** *For every $n \in \mathbb{N}_0$ the so-called* Casimir element $-(\mathbf{b}_1^2 + \mathbf{b}_2^2 + \mathbf{b}_3^2)$ *acts on* $\mathrm{Sym}^n(\mathbb{C}^2)$ *by differentiation, and equals the scalar multiplication*

$$\pi_\partial(\Omega) = -\pi_\partial(\mathbf{b}_1^2 + \mathbf{b}_2^2 + \mathbf{b}_3^2) = (n^2 + 2n)I.$$

Proof. We first note that $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathrm{SU}_2(\mathbb{R})$ as defined in (6.4) form an orthonormal basis for the inner product defined by the quadratic form det. Moreover, as discussed in Section 6.2.2, $\mathrm{SU}_2(\mathbb{R})$ acts via the adjoint representation by orthogonal matrices on $\mathrm{SU}_2(\mathbb{R})$ with respect to this inner product. Thus $G = \mathrm{SU}_2(\mathbb{R})$ and $\mathfrak{su}_2(\mathbb{R})$ equipped with this inner product satisfy the assumptions in Proposition 6.33.

Let $n \in \mathbb{N}_0$. As the representation π on $\mathrm{Sym}^n(\mathbb{C}^2)$ is an irreducible representation of $\mathrm{SU}_2(\mathbb{R})$ by Theorem 6.3, we obtain from Proposition 6.33 that

$$\pi_\partial(\Omega) = -\pi_\partial(\mathbf{b}_1^2 + \mathbf{b}_2^2 + \mathbf{b}_3^2) = \lambda_n I$$

for some $\lambda_n \geqslant 0$.

To calculate λ_n, we use the basis vectors $e_1^{\odot n} \in \mathrm{Sym}^n(\mathbb{C}^2)$. For $t \in \mathbb{R}$ we have

$$\exp(t\mathbf{b}_1) = \exp\begin{pmatrix} \mathrm{i}t & \\ & -\mathrm{i}t \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}t} & \\ & \mathrm{e}^{-\mathrm{i}t} \end{pmatrix}$$

and

$$\pi(\exp t\mathbf{b}_1)e_1^{\odot n} = \mathrm{e}^{\mathrm{i}nt} e_1^{\odot n},$$

which implies that

$$\pi_\partial(\mathbf{b}_1)e_1^{\odot n} = \mathrm{i}n e_1^{\odot n}$$

and

$$\pi_\partial(\mathbf{b}_1^2)e_1^{\odot n} = -n^2 e_1^{\odot n}. \tag{6.29}$$

For $\mathbf{b}_2, \mathbf{b}_3$ we similarly have

$$\exp(t\mathbf{b}_2) = \exp\begin{pmatrix} & \mathrm{i}t \\ \mathrm{i}t & \end{pmatrix} = \begin{pmatrix} \cos t & \mathrm{i}\sin t \\ \mathrm{i}\sin t & \cos t \end{pmatrix},$$
$$\exp(t\mathbf{b}_3) = \exp\begin{pmatrix} & -t \\ t & \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix},$$
$$\pi(\exp t\mathbf{b}_2)e_1^{\odot n} = (\cos t e_1 + \mathrm{i}\sin t e_2)^{\odot n},$$

and

$$\pi(\exp t\mathbf{b}_3)e_1^{\odot n} = (\cos te_1 + \sin te_2)^{\odot n}.$$

Expanding the latter expressions using the binomial theorem, we can take the derivative with respect to t at $t = 0$ and notice that only one term is relevant to obtain

$$\pi_\partial(\mathbf{b}_2)e_1^{\odot n} = \mathrm{i}ne_1^{\odot(n-1)} \odot e_2$$

and

$$\pi_\partial(\mathbf{b}_3)e_1^{\odot n} = ne_1^{\odot(n-1)} \odot e_2.$$

We repeat this step and obtain

$$\begin{aligned}\pi(\exp t\mathbf{b}_2)\pi_\partial(\mathbf{b}_2)e_1^{\odot n} &= \mathrm{i}n(\cos te_1 + \mathrm{i}\sin te_2)^{\odot(n-1)} \odot (\mathrm{i}\sin te_1 + \cos te_2)\\ &= \mathrm{i}n\big(\cos^{n-1} te_1^{\odot(n-1)} + \mathrm{i}(n-1)\cos^{n-2} t\sin te_1^{\odot(n-2)} \odot e_2 + \cdots\big)\\ &\qquad\qquad \odot (\mathrm{i}\sin te_1 + \cos te_2)\\ &= \mathrm{i}n\big(\mathrm{i}\sin t\cos^{n-1} te_1^{\odot n} + \mathrm{i}(n-1)\sin t\cos^{n-1} te_1^{\odot(n-2)} \odot e_2^{\odot 2} + \cdots\big)\end{aligned}$$

and

$$\begin{aligned}\pi(\exp t\mathbf{b}_3)\pi_\partial(\mathbf{b}_3)e_1^{\odot n} &= n\big(\cos te_1 + \sin te_2\big)^{\odot(n-1)} \odot \big(-\sin te_1 + \cos te_2\big)\\ &= n\big(\cos^{n-1} te_1^{\odot(n-1)} + (n-1)\cos^{n-2} t\sin te_1^{\odot(n-2)} \odot e_2 + \cdots\big)\\ &\qquad\qquad \odot \big(-\sin te_1 + \cos te_2\big)\\ &= n\big(-\sin t\cos^{n-1} te_1^{\odot n} + (n-1)\sin t\cos^{n-1} te_1^{\odot(n-2)} \odot e_2^{\odot 2} + \cdots\big),\end{aligned}$$

which implies that

$$\pi_\partial(\mathbf{b}_2^2)e_1^{\odot n} = -ne_1^{\odot n} - n(n-1)e_1^{\odot(n-2)} \odot e_2^{\odot 2}$$

and

$$\pi_\partial(\mathbf{b}_3^2)e_1^{\odot n} = -ne_1^{\odot n} + n(n-1)e_1^{\odot(n-2)} \odot e_2^{\odot 2}.$$

Together with (6.29) this gives

$$-\pi_\partial(\mathbf{b}_1^2 + \mathbf{b}_2^2 + \mathbf{b}_3^2)e_1^{\odot n} = (n^2 + 2n)e_1^{\odot n}$$

and the corollary follows. □