BOOK REVIEWS

HEIGHTS OF POLYNOMIALS AND ENTROPY IN ALGEBRAIC DYNAMICS
(Universitext)

By Graham Everest and Thomas Ward: 212 pp., £35.00, isbn 1 85233 125 9 (Springer, 1999).

'Is this a book which one could allow one's children or one's servants to read?' asked Melford-Stevenson Q.C. Although he was in no doubt about the answer in the case of *Lady Chatterley's lover*, I am certain that he would have given two outrageously reactionary thumbs up for this book. In fact, I am certain that he would have recommended it, as I do, for postgraduates and undergraduates alike to read. It ranges through a number of topics, varying from the elementary to the sophisticated, all featuring polynomials.

Let us start with something to interest an undergraduate. In 1933, D. H. Lehmer discovered a method for manufacturing large primes. Take a polynomial with integer coefficients, $F(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0$, and factorise it as $F(x) = \prod_{i=1}^{d}(x-\alpha_i)$. Define an integer, $\Delta_n(F) = \prod_{i=1}^{d}(\alpha_i^n - 1)$. If no α_i lies on the unit circle, consider the limit of $\Delta_{n+1}(F)/\Delta_n(F)$ as n tends to infinity, which measures the speed of growth of $\Delta_n(F)$. Lehmer found that the sequence of integers $\{\Delta_n(F)\}$ is more likely to produce primes if it does not grow too quickly. For example,

$$\Delta_{113}(x^3 - x - 1) = 63088004325217$$

and

$$\Delta_{127}(x^3 - x - 1) = 3233514251032733$$

are two such primes. However, $\Delta_n(F)$ has problems when F has a root lying on the unit circle. A more serviceable measure is a function introduced by Mahler:

$$m(F) = \int_0^1 \log|F(e^{2\pi i\theta})|\,d\theta = \log\left(|a_d| \cdot \prod_{i=0}^{d} \max(1, |\alpha_i|)\right).$$

This logarithmic Mahler measure is a simple yet intriguing function. It vanishes for a primitive polynomial F if and only if all the zeros of F are roots of unity. Provided that no zero is a root of unity, $\Delta_n(F)/n$ tends to $m(F)$ as n tends to infinity. In view of the connection with primes, it is of interest to find polynomials with smallest possible Mahler measure. The book under review describes the current state of knowledge on this, thereby laying the ground for an investigative project suitable for undergraduates.

Interesting as the historical background is, my interest was awakened by the connections with other areas, which the book goes on to describe. From a self-map of the d-dimensional torus, one obtains a matrix which realises its effect on the first homology group, and from this one obtains the characteristic polynomial. Ergodicity of the self-map corresponds to no zero of the characteristic polynomial being a root of unity. These observations suggest that the Mahler measure is relevant here. In fact, the authors explain why the Mahler measure of the characteristic polynomial is equal to the topological entropy of the self-map. Many similar connections may be made once one has extended $m(F)$ to polynomials of several variables. From Mahler measure on a topological torus, the authors go on to develop Mahler measure on the tori which occur in arithmetic geometry—elliptic curves. After a very useful introduction to elliptic curves, Mordell's theorem and the theory of the canonical height, the arithmetic part of the book explains how the elliptic analogue of $m(F)$ is related to the canonical height.

I imagine that this text provides an excellent basis for a beginning postgraduate course, since most of the material is not too demanding and yet it arouses the curiosity to learn more about dynamical systems, algebraic number theory or primality testing. In addition, there are 103 exercises with hints, five useful appendices sketching the prerequisites, and an extensive bibliography—all in 212 pages.

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RANDOM DYNAMICAL SYSTEMS
(Springer Monographs in Mathematics)

By LUDWIG ARNOLD: 586 pp., £57.50, ISBN 3 540 63758 3 (Springer, 1998).

STOCHASTIC DYNAMICS

Edited by HANS CRAUEL and MATTHIAS GUNDLACH: 440 pp., £49.50, ISBN 0 387 98512 3 (Springer, 1999).

Itô's first papers describing a new dynamics for the sample paths of Markov processes appeared in 1942. Itô's calculus not only gave a sound mathematical basis to the notion and study of dynamical systems 'perturbed by white noise', but also tied them in to Markov semigroups and their associated semi-elliptic operators and generalised heat equations. The need for the new calculus came from the fact that the paths of the processes involved, and in particular those of the standard model of Brownian motion $\{B_t, t \geqslant 0\}$, are not differentiable nor even locally of bounded variation. The 'white noise' driving this equation is formally the derivative of Brownian motion, and a typical Itô stochastic differential equation (SDE) on $\mathbb{R}^n$ is written as

$$dx_t = A(x_t)\,dt + \sum_{j=1}^{m} X^j(x_t)\,dB_t^j, \tag{1}$$

where A and $X^1, \ldots, X^m$ are vector fields on $\mathbb{R}^n$, and $B_t^1, \ldots, B_t^m$ refer to independent Brownian motions on $\mathbb{R}$. It could be considered as a random perturbation of the deterministic dynamical system determined by A.

In 1961, Blagovescenskii and Freidlin showed that if the coefficients of (1) have enough bounded derivatives, then it will have a solution flow $\{\xi_t, t \geqslant 0\}$ which is smooth in the initial point, although in time it is continuous but not differentiable. Behind (1) lies a probability space $\{\Omega, \mathfrak{I}, \mathbb{P}\}$, with Ω parametrising each B_t^j to give continuous paths $\{B_j^t(\omega) : t \geqslant 0\}$, for $\omega \in \Omega$, and similarly parametrising x_t so that the flow is a map $\xi_t : \mathbb{R}^m \times \Omega \longrightarrow \mathbb{R}^n$, for $t \geqslant 0$, with $\xi_t(x_0, \omega) = x_t(\omega)$. For each ω, the flow can be differentiated in x_0 to yield a process satisfying the linearisation of (1) along the trajection $x.(\omega)$. It is normal to include a shift operator $\theta_t : \Omega \longrightarrow \Omega$, for $t \geqslant 0$, preserving $\mathbb{P}$, and the flow will satisfy the *cocycle property*

$$\xi_t(\xi_s(x_0, \omega), \theta_s(\omega)) = \xi_{t+s}(x_0, \omega), \tag{2}$$

though it seems that nearly 20 years passed before the importance of this was appreciated, and the study of the ergodic and dynamical aspects of these flows got underway.

Associated to (1) is a Markov semigroup on bounded measurable functions given by $P_t f(x) = \int_\Omega f(\xi_t(x, \omega))\, d\mathbb{P}(\omega)$, with differential generator $\mathscr{A}$ given by

$$\mathscr{A}(f)(x) = Df(x)(A(x)) + \tfrac{1}{2} \sum_j D^2 f(x)(X^j(x), X^j(x)). \tag{3}$$

If this posesses an invariant probability measure ρ on $\mathbb{R}^n$, then the transformations

$$\begin{aligned} &\Theta_t : \Omega \times \mathbb{R}^n \longrightarrow \Omega \times \mathbb{R}^n, \\ &\Theta_t(\omega, x) = (\theta_t(\omega), \xi_t(x, \omega)), \quad t \geqslant 0, \end{aligned} \tag{4}$$

will be a semigroup preserving the measure μ, where $\mu = \Omega \otimes \rho$. This enabled Carverhill to apply Oseledets' multiplicative ergodic theorem (MET) to the derivative of ξ_t to obtain Lyapunov exponents for non-linear SDEs and carry over some of the smooth ergodic theory of Pesin as developed by Ruelle. That work was stimulated by questions of Ludwig Arnold, who, together with his ex-students and co-workers, is responsible for much of the progress in this, and related areas, which forms the basis of his monograph.

Professor Arnold's object of study is random dynamical systems (RDS). Essentially, these are 'skew products' ξ_t as above, satisfying a cocycle property (2) with time $t \in T$ where T is $\mathbb{R}(\geqslant 0)$, $\mathbb{R}$, $\mathbb{Z}$ or $\mathbb{N}$. This therefore includes random diffeomorphisms and a class of ordinary differential equations with random coefficients, as well as SDEs (the introduction above was biased); a chapter describes in detail how they are generated. There is emphasis on 2-sided time. This is somewhat strange to those used to Markov processes, but is a key point in enabling a more detailed analysis to be given of the dynamics. In particular, invariant measures μ for (4) which have $\mathbb{P}$ as marginal, but are not product measures (that is, random invariant measures for ξ_t), are crucial as a tool and as an object of study. (For example, the 'dynamical' or 'D'-bifurcations discussed in Arnold's book correspond to changes in these measures; changes in ρ are called 'phenomenological' or 'P'-bifurcations.)

The monograph is a most substantial and careful piece of work. The foundations are laid with great care, and the measure-theoretic niceties of cocycles and their perfection are nailed down. Oseledets' MET is proved in detail, with versions for bundles to allow, in particular, for the important cases which arise of RDS on projective and other homogeneous spaces. Generalised rotation numbers are described. Following this is a long chapter on invariant manifolds, for example, stable and unstable manifolds in the sense of Pesin, in which is included a version for RDS of Hartman–Grobman linearisation. Normal form theory comes next, with a careful exposition first describing the non-random theory. For random systems this is very complex computationally (106 cohomological equations are mentioned for a noisy Duffing–van der Pol oscillator). For SDEs there is the extra theoretical problem, which arises often in this analysis, that the objects involved, for example, Oseledets spaces, are usually not only random but are anticipative; they depend on all time. Non-anticipation was essential for Itô's calculus as originally described, and the anticipative versions are harder to use.

The final chapter on bifurcation theory is well worth reaching, however. Arnold's account of the concepts of stochastic bifurcation theory, the notions of random attractors, and Baxendale's conditions for bifurcations, is a delight to read. Many examples, especially bifurcations associated to noisy versions of Duffing–van der Pol equations, are examined at length. The distinction between the P-bifurcations and D-bifurcations mentioned above is emphasised. The first type depend only on the generator (3) of the SDE (in the white noise case), that is, on the one point motion, whereas the second type depend on the SDE or flow. Many different equations (1) can give rise to the same generator (3), and the qualitative behaviour of their flows can be quite different. In practical applications, it is therefore especially important to be clear about what is being modelled and how any noise should enter into the equation.

Despite its nearly 600 pages, the monograph had to omit several topics, as the author points out: in particular, the infinite-dimensional theory needed for stochastic or random PDEs and functional equations, the highly-developed theory of products of random matrices, and geometrical aspects of stochastic flow theory. For some sections, the reader will want the thesis of H. Crauel [**1**], on which key sections are based, and the book of Kunita [**2**] to obtain further details or to complete proofs. However, the foundations of the subject are firmly laid here. It is a major achievement, which will be of immense value. Anyone who works near this area will be deprived if they do not have a volume ready to refer to, and the lengthy section on Oseledets' MET should be of use to a wider audience.

The volume *Stochastic dynamics* arose from a conference in honour of Professor Arnold's 60th birthday. It is a very useful complement to his monograph. In particular, the bifurcation theory there is extended in several articles. In general, 'stochastic dynamics' is given a wider interpretation here than in the monograph; infinite-dimensional dynamics, SPDE theory, and geometric behaviour all appear. Articles on computational aspects reflect the increasing interest in them. The quality of the articles makes this book a fitting response to Professor Arnold's achievement.

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THE REAL FATOU CONJECTURE
(Annals of Mathematics Studies 144)

By Jacek Graczyk and Grzegorz Świątek: 148 pp., £18.95, isbn 0 691 00258 4 (Princeton University Press, 1998).

One of the main open problems in the theory of dynamical systems is the problem of describing the structurally stable maps. (A map is called structurally stable in some topology if there exists some neighbourhood of this map such that all maps in this neighbourhood are topologically conjugate.) This problem goes back to Poincaré, Fatou and Andronov. In his famous paper, Fatou expressed the belief that 'general' rational maps are expanding on the Julia set. In today's language, we would formulate his conjecture in the following way: 'Hyperbolic rational (polynomial) maps are dense in the space of rational (polynomial) maps'. Here, we say that a map is hyperbolic if it is expanding on its Julia set or, equivalently, if iterates of all its critical points converge to attracting cycles. This conjecture is very closely related to the local connectivity of the Mandelbrot set: if the Mandelbrot set is locally connected, then hyperbolic quadratic polynomials are dense in the space of all quadratic polynomials.

The authors of the book under review consider the simplest non-trivial case of the Fatou conjecture; namely, they consider a family of real quadratic maps $x \mapsto ax(1-x)$. For this family, they prove that the conjecture indeed holds. Thus they prove the following important theorem.

THEOREM 1. *In the real quadratic family*

$$f_a : x \mapsto ax(1-x),$$

where $0 < a < 4$, *the map* f_a *has an attracting cycle, and thus is hyperbolic, for an open and dense set of parameters* a.

The whole book is devoted to the proof of this theorem. The proof is well structured and contains many results which are very useful for other applications in one-dimensional dynamics. Unfortunately, these results are too technical to state here; however, one of them ought to be mentioned. It appears that if f_a is a non-renormalizable map, then one can construct a sequence of induced maps with exponential decay of moduli. Moreover, the speed of the exponential decay depends only on the modulus of the given map f_a. The proof of the main theorem is based on this fact and, undoubtedly, it will play a key role in proofs of other theorems.

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MODEL CATEGORIES
(Mathematical Surveys and Monographs 63)

By MARK HOVEY: 207 pp., US$54.00, ISBN 0 8218 1359 5
(American Mathematical Society, 1998).

The theory of model categories was introduced in 1967 by Quillen [**5**] to unify a number of contexts in which one can do homotopy theory, notably with chain complexes, spaces, and simplicial sets, groups or rings. Shortly afterwards, he used it to give an algebraic description of spaces whose homotopy groups are rational vector spaces [**6**], which has led to a rich and beautiful theory with many concrete examples. Later, Bousfield showed how to use exotic model structures on the category of simplicial sets to solve some foundational problems in the theory of homological localisation [**1**, Appendix]. Model categories of simplicial rings were also used as the foundation for homology theories for associative algebras (Hochschild homology) and commutative algebras (André–Quillen homology). Although these applications flourished, for some time there was relatively little work on the general theory of model categories. In the last decade, several developments have conspired to change this. The first begins with the work of Hopkins and Miller, in which they outlined an obstruction theory for showing that certain ring spectra can be made strictly (rather than just homotopically) commutative. As the outline has been filled in, the rôle of Quillen's theory has become more and more prominent: it involves model categories of ring spectra, model categories of simplicial operads, and so on. Partially inspired by this, Elmendorf *et al*. gave a new and much more satisfactory topological foundation for stable homotopy theory [**2**]. In this context, it is easy to construct new topological categories of ring spectra, module spectra, and so on. Another way of revamping the foundations of the stable category was discovered by Smith [**4**]. In each case there is a corresponding homotopy theory, which is best understood using model categories.

Another completely independent development was the spectacular work of Voevodsky in algebraic K-theory, including his proof of the Milnor Conjecture. This involves setting up a homotopy category of simplicial schemes, and comparing a corresponding stable homotopy category with the ordinary topological stable homotopy category. This point of view is most visible in the joint work of Voevodsky and Morel, which again relies on model categories to control homotopical behaviour.

The book under review is thus very timely. It starts with an account of the definitions, and a development of the homotopy theory of model categories. This is probably the first time in which the important notion of cofibrant generation has appeared in a book, and the consideration of the 2-category of model categories and Quillen adjunctions is another interesting feature. Hovey then treats the basic examples of spaces, simplicial sets and chain complexes of modules over a ring or comodules over a Hopf algebra (the last of which is new). Many subtle details (usually left to the reader) are treated in full. Next, there is a new discussion of monoidal products on model categories (such as the smash product of pointed spaces, or of various kinds of spectra), with conditions under which they are homotopically well-behaved; these considerations are essential in the new applications discussed earlier. Hovey also extends Quillen's discussion of pointed model categories by developing a theory of fibre and cofibre sequences and the comparison between them. This involves a suspension functor from the pointed model category to itself; the category is said to be *stable* if this functor is an equivalence. The book develops a general theory of stable model categories, and gives mild criteria under which the associated homotopy categories are stable homotopy categories in the sense of [**3**]. Again, this covers many of the new applications. There is also a chapter showing that the homotopy category of any model category is enriched over the homotopy category of spaces, a remarkable result with origins in work of Dwyer and Kan.

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QUANTUM GROUPS AND THEIR REPRESENTATIONS
(Texts and Monographs in Physics)

By Anatoli Klimyk and Konrad Schmüdgen: 552 pp., £49.00, isbn 3 540 63452 5 (Springer, 1997).

ALGEBRAS OF FUNCTIONS ON QUANTUM GROUPS: PART I
(Mathematical Surveys and Monographs 56)

By Leonid I. Korogodski and Yan S. Soibelman: 150 pp., US$49.00, isbn 0 8218 0336 0 (American Mathematical Society, 1998).

There is, as Klimyk and Schmüdgen point out in their Preface, no recognised mathematical definition of a 'quantum group'. And as a comparison of their book with that by Korogodski and Soibelman will show, different discussions of these non-existent objects can be almost disjoint in their content. Is there actually anything there to discuss?

Nothing there, perhaps, but plenty to discuss. Start with the pointless attitude of algebraic geometry, shifting attention from a space of points to the algebra of functions on that space as an equivalent notion. This algebra is necessarily commutative, but there may be non-commutative algebras very close to it which one would like to think of in the same way. A 'quantum space' is the space that one would then like to think of, a virtual space. If the space we start from is a group, then its algebra of functions A has not only a (pointwise) multiplication, but also a second operation, reflecting the group operation in the underlying space, namely a *comultiplication* taking a function f of one variable to a function Δf of two variables given by $\Delta f(x, y) = f(xy)$. Reflecting the inverse operation in the underlying space is the *antipode*, the operation S on the algebra of functions given by $Sf(x) = f(x^{-1})$; reflecting the identity e is the *counit*, the functional $\varepsilon : f \mapsto f(e)$. This is an example of the general notion of a *Hopf algebra*, an algebra with a comultiplication $\Delta : A \to A \otimes A$, an antipode $S : A \to A$, and a counit $\varepsilon : A \to \mathbb{C}$ satisfying appropriate axioms; even if non-commutative, such an algebra can possibly be regarded as an algebra of (virtual) functions on a 'quantum group'.

This fanciful way of thinking is most mathematically fruitful when the non-commutative 'algebra of functions' retains some similarity to a genuine (commutative) Hopf algebra of functions $\mathcal{O}(G)$. It should be a deformation of $\mathcal{O}(G)$ in the sense that it depends on a continuous parameter q and becomes equal to $\mathcal{O}(G)$ for one value of this parameter; and it should stay close to commutativity in the sense that there should be some manageable general relation between xy and yx. In quantum group theory this replacement for commutativity is provided by a further element of structure, the *universal R-matrix*, which makes our 'algebra of functions' into a *coquasitriangular* Hopf algebra.

There is enough structure here to provide scope for a wide range of approaches and emphases. The concept of a Hopf algebra A has an in-built duality, it being possible to regard the comultiplication as a multiplication on an appropriate dual of A, which then becomes a *quasitriangular* Hopf algebra. If we are deforming the algebra of functions on a Lie group G, then this dual algebra is a deformation of the universal enveloping algebra of the Lie algebra of G. This is the initial object of attention in the book by Klimyk and Schmüdgen. They prepare the way with a chapter on Hopf algebras and a chapter on q-calculus, which is an essential ingredient in the construction of quantum groups. They then devote the remaining hundred pages of their Part I to a detailed study of the simplest quantum group, that based on the classical Lie algebra $\mathfrak{sl}(2)$. This Part I is called 'An introduction to quantum groups', but anyone who reads all of it will have passed well beyond an introduction to intimacy with this particular quantum group and some closely related ones. Its representations and their tensor products are completely determined, with detailed formulae for the Clebsch–Gordan coefficients and related quantities such as $6j$-coefficients. Both the deformed universal enveloping algebra and the algebra of functions are discussed, a chapter being devoted to each, and quantum spaces appear in the form of quantum spheres. Part I finishes with a chapter on q-oscillator algebras, which are defined by relations inspired by those of quantum groups and which have been popular in the physics literature.

The general theory of quantised enveloping algebras for simple Lie algebras is presented in Part II of the book, with treatments of the (non-trivial) quantised versions of the Weyl group, the Poincaré–Birkhoff–Witt theorem and the Harish-Chandra homomorphism, and an account of the representation theory of quantised enveloping algebras, including general work on Clebsch–Gordan coefficients (to which Klimyk has himself contributed) and the result of Schmüdgen and Schüler that the differential calculus on a quantised version of a classical group is not a deformation of the classical calculus; the tangent space inevitably has a different dimension from the classical one.

Klimyk and Schmüdgen are kind to their readers. Proofs are given in full, and there are helpful explanations of the basic concepts (like the elaboration of the definition of h-adic topology on page 25). It is unfortunate that there is a lapse on the very first page (where the last equation contains $\Delta \otimes \mathrm{id}$, although Δ has not been defined in terms of tensor products), but in the rest of the book errors are rare. Occasionally explicitness displaces elegance (for example, in the Wigner–Eckart theorem, where the authors could have made things easier by remembering that they were in a Hopf algebra), and if index notation is to be used, then it is a pity not to take advantage of its full power by placing indices correctly; but the book has the virtue of comprehensiveness in its chosen range of topics. It is easy to dip into and use as a reference book, and there is a good (though not totally complete) index of symbols which aids this. The bibliography is irritatingly split into two lists, one for books and one for papers, though the labels do not tell us which list to look in, and the items are not sorted alphabetically by label.

Comprehensive though this selection of topics may seem, Korogodski and Soibelman show us a bookful of material, just as close to the foundations of the subject, for which Klimyk and Schmüdgen have no room. They adopt Drinfeld's approach to quantum groups, which shows that there is more justification for the use of the word 'quantum' than is apparent in the other book. In quantum mechanics, the dynamical variables of classical mechanics (functions on phase space) are replaced by non-commuting quantities depending on Planck's constant h, not in any old way, but so as to maintain a relation between the Poisson bracket of the functions on phase space and the first-order (in h) terms in the quantum commutator. There is a similar relation in quantum group theory: if the parameter q in a quantised enveloping algebra $U_q(\mathfrak{g})$ is written as $q = e^h$, so that the classical enveloping algebra is described by $h = 0$, and if the cocommutator (dual to the commutator in the quantised algebra of functions) is expanded in powers of h, then the first-order term (called the 'classical r-matrix') gives an extra structure on the Lie algebra of G (a *cobracket* $\delta: \mathfrak{g} \to \mathfrak{g}$) making it into a *Lie bialgebra*. The existence of an R-matrix in the quantised enveloping algebra is detected by a property of the Lie bialgebra which is also called quasitriangularity. A Lie bialgebra structure yields a symplectic structure on the classical Lie group G which is compatible with the group operation, making G into a *Poisson Lie group*.

Korogodski and Soibelman devote the first of their four chapters to the study of Poisson Lie groups and Lie bialgebras, including a proof of the celebrated Belavin–Drinfeld theorem which classifies (most) quasitriangular Lie bialgebra structures on finite-dimensional complex simple Lie algebras. They also give the complete description of the symplectic leaves (the maximal symplectic submanifolds) of the Poisson Lie groups corresponding to the Drinfeld–Jimbo quantised enveloping algebras.

Quantum groups (in the form of quantised enveloping algebras) make their appearance only in the second chapter, where they are regarded as being obtained from a Lie bialgebra by a process of *quantisation*. It is only recently that Etingof and Kazhdan have settled the question of whether a given Lie bialgebra can be quantised and to what extent the quantisation is unique; the result is given in this chapter, though it would be out of the question to present the proof. The chapter continues with succinct treatments of quasitriangular Hopf algebras, the centre and the representation theory of quantised universal algebras. Following the same order as Klimyk and Schmüdgen, Korogodski and Soibelman proceed to discuss quantised algebras of functions; but the two discussions are quite different. The former authors dealt with representations of such algebras only in the form of corepresentations which are roughly the same as the representations of their duals, the quantised enveloping algebras. This is to ignore the fact that the quantised algebras of functions are themselves non-commutative algebras with an interesting representation theory, which is fully described by Korogodski and Soibelman (who is responsible for the ideas and much of their implementation). The main result is that the irreducible *-representations of the quantised algebra of functions on a compact simple Lie group are in one-to-one correspondence with the symplectic leaves of the appropriate Poisson Lie group.

The final chapter, now intersecting once again with the book of Klimyk and Schmüdgen, contains the construction of the quantum Weyl group which makes it possible to define all the root vectors in a quantised enveloping algebra (initially only the simple roots are defined), and hence to formulate the Poincaré–Birkhoff–Witt theorem; they are also needed for the formula for the universal R-matrix which was discovered by Soibelman.

This book is not an easy read. Korogodski and Soibelman leave the reader to do a lot of the work, and their dismissive remarks that this work is easy are not always to be trusted. (They characterise as 'straightforward' a number of computations which either have quite sharp corners or would be better done in a less than straightforward way: for example, the proof of compatibility of the quantised Serre relations with the Drinfeld–Jimbo coproduct, which takes a full page in Klimyk and

Schmüdgen even after they have prepared the ground of q-analysis.) Some of their proofs appear to me not to prove anything, which may mean that I do not meet their expectations of their readers but in at least one case is because they are attempting to prove an incorrect statement. There is no index.

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p-AUTOMORPHISMS OF FINITE p-GROUPS
(London Mathematical Society Lecture Note Series 246)

By E. I. KHUKHRO: 204 pp., £24.95 (US$39.95, LMS Members' price £18.70), ISBN 0 521 59717 X (Cambridge University Press, 1998).

This is a beautiful book. It may attract a smaller audience than it deserves on account of its rather specialised title. Those who jib at having mathematical symbols in the title of a book will indeed read it as 'Automorphisms of prime-power order of groups of prime-power order, the primes being the same'. So the first question to be answered is 'why should we care?'. There are two answers. The first is that this book is a delightful introduction to very general techniques in group theory, such as the use of Lie algebras, and to central ideas, such as powerful groups, in the theory of p-groups. But more important than its use as an introduction to techniques are the results that are proved. We should be concerned with p-automorphisms of p-groups because they give an insight into the structure of arbitrary p-groups.

The general theme of this book is Khukhro's deep result that a p-group P that possesses an automorphism that is of order a small power of p, and whose centraliser in P is small, has a normal subgroup of small index and small derived length.

Most sensible classification theorems of p-groups make precise, in some way, the fact that all p-groups look like the dihedral groups of order 2^n. Note that these dihedral groups have an automorphism of order 2, fixing only two elements, and a normal subgroup of small index (namely 2) of small derived length (namely 1).

Of course, every p-group of order greater than p does have an automorphism of prime-power order, so one really does have to worry about the meaning here of 'small'. The precise result is as follows.

THEOREM 12.15. *If a finite p-group P admits an automorphism of order dividing p^n that fixes at most p^m elements of P, then P contains a characteristic subgroup of (p,m,n)-bounded index which is soluble of (p,n)-bounded derived length; in fact of derived length at most $2k(p^n)$, where k is Kreknin's function.*

Kreknin's function derives from a similar result for Lie algebras; it is known that $k(a) \leqslant 2^{a-1}$, so the bounds given are rather extravagant.

This result should be compared with S. McKay's theorem, proved independently by I. Kiming, which is as follows.

THEOREM 13.1. *If a finite p-group P admits an automorphism of order dividing p^n that fixes exactly p elements of P, then P has a subgroup P_1 of (p,n)-bounded index which is nilpotent of class 2 (abelian if $p = 2$).*

Not only is this result (for which a new proof is given) far stronger than suggested by Theorem 12.15, but very realistic bounds for the index of P_1 in P can be given, and more importantly a very good description of the groups in question can also be given. So we may regard the groups coming under McKay's theorem as being well understood. However, McKay's theorem does not apply to all p-groups, whereas Khukhro's theorem, for some m and n, does always apply.

Now McKay's theorem was conceived as part of the successful coclass project, which does claim to give a description of all p-groups; that is, it describes all p-groups of coclass r modulo a normal subgroup of order bounded by an explicit function of p and r. A most beautiful proof of the coclass conjectures, with quite realistic bounds, is due to Shalev [**1**]. For a proof that exhibits the structure of the p-groups in question (modulo the small normal subgroup) see [**2**].

The unavoidable problem with the coclass theory is that the small normal subgroup may in fact be very large, even swallowing up the whole of P, unless the coclass of P is small enough. So the question arises of whether there is any hope that a better structure theorem can be produced based on Khukhro's theorem, or perhaps on the following theorem of Medvedev.

THEOREM 14.1. *If a finite p-group P admits an automorphism of order p that fixes at most p^m elements of P, then P has a subgroup P_1 of (p, m)-bounded index which is nilpotent of m-bounded class.*

Medvedev's theorem bounds the class of P_1 in terms of m, whereas Khukhro's bound to the derived length of the subgroup of small index is in terms of p and n. This is an unexpected feature of Khukhro's theorem; having a p-automorphism of small order says nothing about a p-group; it is having an automorphism with a small centraliser that should give strong information.

Of course, Medvedev's theorem applies to all p-groups of order greater than p for some value of m; but, unlike McKay's theorem, it does not come with an understanding of the groups in question modulo small edge effects of (p, m)-bounded size.

The question then arises of whether the results in this book may lead to a much closer understanding of p-groups. For this discussion it is convenient to consider not a p-group P, and an automorphism of P of order dividing p^n and fixing at most p^m points, but rather the class $\mathcal{G}(p, m, n)$ of p-groups G which are split extensions of a specified normal subgroup P by a cyclic group of order dividing p^n that centralises at most p^m elements of P. These concepts are almost equivalent. Now an easy result (Lemma 2.12) shows that $\mathcal{G}(p, m, n)$ is closed under taking normal subgroups, in the sense that if G with specified normal subgroup P lies in $\mathcal{G}(p, m, n)$, then so does G/N with specified normal subgroup P/N, whenever N is a normal subgroup of G contained in P.

The next step is to construct a graph $\mathcal{T}(p, m, n)$ whose vertices are the groups in $\mathcal{G}(p, m, n)$ (up to isomorphism preserving the specified subgroup), where a group G with (non-trivial) specified subgroup P is joined by an edge to the group G/N, with specified subgroup P/N, where N is the intersection of P with the centre of G. To understand $\mathcal{G}(p, m, n)$, one could try to understand $\mathcal{T}(p, m, n)$; and the first step would be to understand the infinite chains in this graph. Such an infinite chain corresponds to a pro-p-group, namely the inverse limit of the groups in the chain.

Now Khukhro's theorem states that these pro-p-groups are soluble; and this is a powerful condition on a pro-p-group. It follows at once that every just-infinite pro-p-group arising in this way is abelian-by-finite, and hence is an extension of a lattice over the p-adic integers by a finite p-group acting irreducibly.

Whether this approach will lead to any further understanding of the groups in question is not clear to me. If so, there is much work to be done before we get to this point. It is a feature of this book that it is almost free of examples. The only specific groups that play a major role are free groups and relatively free groups. The reason is that while any p-group affords an example, no one seems to know how to produce a sequence of examples (or a pro-p-group as above) that would give an interesting lower bound to the functions appearing in Khukhro's theorem or Medvedev's theorem.

The book has been written in a most readable style. It is entirely suited to anyone wishing to learn group theory at a graduate level. Thus the author not only begins at the beginning, but he does not make excessive assumptions of mathematical sophistication. For example, he points out that it is surprising that a non-abelian free group is residually soluble although it maps onto every finite simple group. He has also gone to the trouble of drawing beautiful pictures in TEX. So if, unfortunately, you are not interested in group theory, you should still buy a copy of this book and try to write a TEX program that will produce an animated version of the two steam trains colliding on page 96.

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COMPLETELY REGULAR SEMIGROUPS
(Canadian Mathematical Society Series of Monographs and Advanced Texts 23)

By MARIO PETRICH and NORMAN R. REILLY: 481 pp., £74.50, ISBN 0 471 19571 5
(John Wiley & Sons, 1999).

INVERSE SEMIGROUPS: THE THEORY OF PARTIAL SYMMETRIES

By MARK V. LAWSON: 411 pp., £40.00, ISBN 981 02 3316 7
(World Scientific, 1998).

GROUPOIDS, INVERSE SEMIGROUPS, AND THEIR OPERATOR ALGEBRAS
(Progress in Mathematics 170)

By ALAN L. T. PATERSON: 274 pp., CHF98.00, ISBN 0 8176 4051 7
(Birkhäuser, 1999).

Since non-abelian group theorists made progress by studying 'abelian-like' conditions, such as nilpotency and solubility, it is not surprising that a substantial part of semigroup theory has been concerned with 'group-like' conditions. In particular, there is a substantial theory concerning semigroups S with an extra unary operation $x \mapsto x^{-1}$ satisfying the laws $xx^{-1}x = x$, $(x^{-1})^{-1} = x$. Extra laws can be adjoined to define varieties, the most important being given in the following table.

Groups	Completely regular semigroups	Inverse semigroups
$xx^{-1} = yy^{-1}$	$x^{-1}x = xx^{-1}$	$xx^{-1}yy^{-1} = yy^{-1}xx^{-1}$

It is clear from the definitions that the class CR of completely regular semigroups and the class Inv of inverse semigroups both contain the class of groups. While the defining law of CR seems more natural than that of Inv, it is Inv that has hitherto received more attention, since inverse semigroups have a natural representation as partial symmetries.

The book by Petrich and Reilly is a natural sequel to Petrich's encyclopædic account [**3**] of inverse semigroups, published fifteen years ago. Although the study of CR goes back to the 'dawn' of semigroup theory, in Clifford's seminal article [**1**], an intensive and thorough study is much more recent, and aficionados of this area will be delighted that such a thorough and readable account is now available. Whether CR will ever be as important as Inv remains to be seen, but this book will make the subject accessible to potential users in a way that it has not been before.

Lawson's book is quite different in spirit. While it does develop certain areas of 'pure' inverse semigroup theory, the whole emphasis of the book is on the applicability of the notion, on the 'naturalness' of the inverse semigroup concept as a means of studying local symmetries. He traces the origin of the inverse semigroup concept back to the 'pseudogroups' of Veblen and Whitehead [**4**], who realised that the *Erlanger-Programm*, with its vision of groups as the key to all geometry, could never be the whole answer, and this historical observation colours the whole approach of the book. Semigroupers have too often followed the internal logic of their subject, with little attention to actual and potential connections with other areas of mathematics. Lawson's persuasive account may persuade them to take a broader view, and may also convince other mathematicians that they might need to know some semigroup theory.

Paterson, whose main interest is in C^*-algebras and operator theory, is already convinced, and at an early stage we are introduced to a theorem from [**2**] to the effect that a semigroup is an inverse semigroup if and only if it is isomorphic to a *-semigroup of partial isometries on a Hilbert space. The emphasis of the book is, however, on groupoids rather than inverse semigroups, where a groupoid is defined as a set G with a *partial* binary operation $(a, b) \mapsto ab$ and an everywhere defined unary operation $a \mapsto a^{-1}$ satisfying the following.

(1) $(a^{-1})^{-1} = a$ for every a in G.

(2) If ab, bc are defined, then so are $(ab)c$ and $a(bc)$, and $(ab)c = a(bc)$.

(3) The product bb^{-1} is defined for all b, and if ab is defined, then $a^{-1}(ab) = b$ and $(ab)b^{-1} = a$.

This notion is not unrelated to the notion of an inverse semigroup: members of a special class of groupoids, called *inductive*, can be made into inverse semigroups by the simple expedient of adjoining a zero and defining all products to be 0 if not already defined.

Paterson gives a readable account of the contribution of these ideas in operator algebras. The contents of the book are perhaps best described by quoting from the back cover.

The representation theories of locally compact and r-discrete groupoids are developed in the third chapter, and it is shown that the C^*-algebras of r-discrete groupoids are the covariance C^*-algebras for inverse semigroup actions on locally compact Hausdorff spaces. A final chapter associates a *universal* r-discrete groupoid with any inverse semigroup.

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MORITA EQUIVALENCE AND CONTINUOUS-TRACE C*-ALGEBRAS
(Mathematical Surveys and Monographs 60)

By Iain Raeburn and Dana P. Williams: 327 pp., US$65.00, isbn 0 8218 0860 5 (American Mathematical Society, 1998).

Cohomological methods have revolutionised the theory of C*-algebras in recent decades. K-theory has played a particularly prominent role, but one of the earliest significant appearances of cohomology in a C*-algebraic context is to be found in work of Dixmier and Douady [**2**] from 1963 on the classification of continuous-trace C*-algebras using an invariant based on Čech cohomology.

The continuous-trace C*-algebras possess a structure sufficiently tractable to make them amenable to classification, yet rich enough to display a range of interesting properties. A trivial example of a continuous-trace C*-algebra is the algebra $C_0(X, K(H))$ of continuous functions from a locally compact Hausdorff space X to the algebra of compact operators $K(H)$ on a separable Hilbert space. More general examples arise as section algebras of certain continuous bundles of C*-algebras over such X, where the fibre at each point x is the algebra $K(H_x)$ for some Hilbert space H_x whose dimension depends on x. (The precise definition, though rather technical, implies that a continuous-trace C*-algebra over X has a natural $C_0(X)$-module structure.)

In essence, Dixmier and Douady showed that for a general separable continuous-trace C*-algebra, the obstruction to being trivial is purely topological. They discovered a map δ from the continuous-trace algebras over a given locally compact Hausdorff space X to the third Čech cohomology group $H^3(X; \mathbb{Z})$ which is surjective if X is paracompact. Moreover, if A and B satisfy certain separability assumptions, then $\delta(A) = \delta(B)$ if and only if A and B are $C_0(X)$-stably isomorphic, that is, the C*-tensor products $A \otimes K(\ell^2(\mathbb{N}))$ and $B \otimes K(\ell^2(\mathbb{N}))$ are $C_0(X)$-isomorphic. An account of this work was given by Dixmier in his classical treatise [**1**] on C*-algebras.

The authors' primary aim in the book under review is to give a self-contained account of the Dixmier–Douady theory in a formulation which takes account of subsequent developments in some other areas. They give a sharper form of the main classification theorem, and present related later results, such as a description, due to

Philips and Raeburn, of the automorphisms of a continuous-trace C*-algebra over X in terms of elements of the group $H^2(X; \mathbb{Z})$. Central to their statement and proof of the classification results is the theory of Morita equivalence of C*-algebras, which was developed by Rieffel in the 1970s as a generalisation, in terms of C*-algebras, of Mackey's theory of induced representations of locally compact groups.

C*-algebras A and B are *Morita equivalent* if there exists a C*-algebra C containing full projections p and q such that $p+q = 1$, $A \cong pCp$ and $B \cong qCq$. ('Full' means that the closed two-sided ideal of C generated by the projection is C itself.) The significance of this for the Dixmier–Douady theory is that by a result of Brown, Green and Rieffel, stable equivalence and Morita equivalence coincide for separable C*-algebras. Morita equivalence, however, possesses an important functoriality property which stable equivalence does not.

Although the above definition of Morita equivalence is succinct, it is not always easy to work with in practice. It is difficult, for example, to show directly that the relation so defined is transitive. To prove this requires a more fundamental, though technically more complex, definition involving certain two-sided Hilbert modules, known as *imprimitivity* or *equivalence* bimodules. One of the particular strengths of this book is the inclusion of a detailed account of the theory of imprimitivity bimodules, which covers Rieffel's imprimitivity theorem and its connection with Mackey's imprimitivity theorem for groups. Although other books on Hilbert modules have appeared recently, their emphasis has been different, and they have not covered this difficult, but important topic. The account here collects together and clarifies many results on Morita equivalence which cannot otherwise be found in one place.

The first two-thirds of the book, some seven chapters, forms the body of the work. There are successive chapters on: the compact operators; Hilbert C*-modules; Morita equivalence; and sheaves, cohomology and bundles. Chapter 5 contains the main classification results. Further developments and discussion are given in Chapters 6 and 7, the main theme of which is group actions on continuous-trace C*-algebras. The final third of the book consists of a number of appendices on a wide range of topics which, though not all breaking new ground, form a useful reference compendium. Topics covered include the spectrum of a C*-algebra, tensor products of C*-algebras, and Rieffel's imprimitivity theorem and its application to group representations. The inclusion of this material helps to make the main part of the text accessible to anyone acquainted with the rudiments of C*-algebra theory, as given in, say, the first few chapters of [**1**].

I hope that I have conveyed that what at first sight seems a somewhat specialised monograph, is in fact of much more general scope. The exposition is stimulating and well written, and should be regarded as essential reading for any research student in C*-algebras. Indeed, the book has a strong claim to be on the shelves of anybody, student or veteran, working in the subject.

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University of Glasgow SIMON WASSERMANN

GEOMETRY OF NONPOSITIVELY CURVED MANIFOLDS
(Chicago Lectures in Mathematics)

By PATRICK B. EBERLEIN: 449 pp., US$45.00 (£35.95), ISBN 0 226 18198 7
(University of Chicago Press, 1996).

NONPOSITIVE CURVATURE: GEOMETRIC AND ANALYTIC ASPECTS
(Lecture in Mathematics ETH Zürich)

By JÜRGEN JOST: 108 pp., DM.38.–, ISBN 3 7643 5736 3 (Birkhäuser, 1997).

The study of Riemannian manifolds in which all sectional curvatures are nonpositive has flourished in recent decades. Nonpositive sectional curvature proves to be an excellent condition under which to pursue the interplay between curvature and global properties of the manifold. There are many interesting examples of manifolds with nonpositive curvature, the best known and most important of which are locally symmetric spaces, that is, quotients of symmetric spaces. The symmetric spaces that have nonpositive curvature are the Euclidean spaces and the spaces of noncompact type. There is also a variety of constructions giving metrics of nonpositive curvature on manifolds that do not admit locally symmetric metrics. On the other hand, the restriction on the sign of the sectional curvature imposes strong restrictions on the manifold. For example, all homotopy groups of the manifold except for the fundamental group must vanish. In contrast, the analogous condition on Ricci curvature imposes no restrictions except on surfaces; it was proved by Lohkamp [**6**] that any differentiable manifold with dimension at least 3 admits a metric with all Ricci curvatures negative.

The book by Eberlein provides a comprehensive introduction to manifolds with nonpositive curvature, accessible to a reader whose background includes a first graduate level course in differential geometry and the rudiments of Lie groups. The book begins with a thorough survey of the foundations. The reader is referred to references for some proofs, but the treatment is largely self-contained. The consequences of nonpositive curvature are usually easiest to express in the case of a simply connected manifold. One studies a general manifold M of nonpositive curvature by lifting the metric to the universal cover $\tilde{M}$ and viewing M as the quotient of $\tilde{M}$ by the fundamental group, which acts on $\tilde{M}$ by isometries. Nonpositive curvature implies that geometrically interesting functions on $\tilde{M}$, such as the distance from any given point, are convex. It follows easily that $\tilde{M}$ must be diffeomorphic to a ball. The picture of a manifold with a metric of constant negative curvature as the quotient of the Poincaré disc is typical. For any manifold M with nonpositive curvature, it is possible to define a boundary sphere $\tilde{M}(\infty)$ that is analogous to the boundary sphere of the Poincaré disc. The action of the fundamental group on $\tilde{M}$ extends to an action by homeomorphisms on $\tilde{M} \cup \tilde{M}(\infty)$. Much can be learned by studying the action of the fundamental group on $\tilde{M}(\infty)$.

The sphere at infinity of a symmetric space (of noncompact type) with rank greater than 1 has an especially rich structure. The rank of any manifold M with nonpositive curvature is the least integer k such that every geodesic of $\tilde{M}$ lies in a submanifold, called a k-flat, that is a totally geodesically isometrically embedded copy of the Euclidean space $\mathbb{R}^k$. For a symmetric space, this geometric definition agrees with the usual algebraic definition as the dimension of a maximal abelian subalgebra of the Lie algebra of the isometry group. When the flats have dimension at least 2, $\tilde{M}(\infty)$ has a simplicial complex structure in which the boundary of each flat is a finite union of simplices. This structure is known as a Tits building; the division of the boundary of the flat corresponds to the partition of the maximal abelian subalgebra into Weyl chambers. The structure can also be described using the Tits metric on the sphere at infinity, which was introduced by Gromov. Eberlein provides a wealth of detail about the symmetric spaces of noncompact type and the associated Tits structures; close to half of the book is devoted to these matters. There is no comparable treatment in the literature. Helgason's book [**5**], although it is an invaluable resource for the theory of symmetric spaces in general, is able to devote only about 30 pages to the special features of the spaces of noncompact type. Eberlein's exposition meets a long-felt need.

The later parts of Eberlein's book cover many of the results from the last 30 years. Two themes predominate. The first is the close relationship between the geometry and the fundamental group of the manifold. A typical result in this direction is the Flat Torus Theorem, which says that the existence of an abelian subgroup of rank at least 2 in the fundamental group of a compact manifold with nonpositive curvature implies that the manifold contains a totally geodesic and isometrically immersed flat torus.

The second theme is rigidity. The general idea here is that apparently mild hypotheses are actually strong enough to force the manifold to be locally isometric to one of a restricted class of model spaces. The prototypical result is the celebrated Mostow rigidity theorem [**7**].

Suppose that M is a compact locally symmetric space that is irreducible (*in the sense that no finite covering splits as a product*) *and is not a quotient of the hyperbolic plane. Let M' be a locally symmetric space whose fundamental group is isomorphic to that of M. Then M' is isometric to M up to a scaling factor.*

Mostow's original theorem has been extended in a number of directions. Prasad and Margulis showed that it is enough to assume that M has finite volume, and Margulis generalized the result to his super-rigidity theorem. The higher-rank rigidity theorem extends Mostow's result in a different direction. It states that if M is a finite-volume manifold with nonpositive curvature whose rank, in the sense defined above, is at least 2, then $\tilde{M}$ either splits as the Riemannian product or is a symmetric space of noncompact type. This result also holds if M is homogeneous. Eberlein gives a simplified proof of Mostow's theorem along the lines of Mostow's original proof, and sketches a proof of higher-rank rigidity. The proofs are based on the Tits structure discussed above; in the case of higher-rank rigidity, one must first prove the existence of flats and the Tits structure. A striking and quite different proof of Mostow's theorem can be found in [**3**].

Still another approach to Mostow's theorem is the subject of Jost's book: harmonic mapping methods can be used to prove the Mostow rigidity and Margulis super-rigidity theorems. This approach to Mostow rigidity was developed in a series of papers by Jost and Yau. As Jost explains, the rigidity theorems follow rather easily from results about harmonic mappings, the simplest form of which is as follows.

A harmonic map of a compact locally symmetric space with nonpositive curvature into a manifold with nonpositive curvature operator must be totally geodesic.

This is a beautiful result in its own right, which is implicit in Jost's account, although (surprisingly to me) it is not explicitly formulated in the book. Jost briefly and efficiently develops the necessary geometric and analytic background; many proofs are sketched rather than presented in detail. His focus never strays far from his central theme. The result is an energetic and pleasantly quick introduction to a wonderful part of modern geometry.

The two books cover several of the highlights of recent work on manifolds of nonpositive curvature, but is not possible for them to contain the whole subject. The survey papers [**4**] and [**3**] are suggested to the reader interested in other perspectives.

Both books are welcome additions to the literature. They belong in any serious library, and on the shelf of a geometer interested in manifolds with nonpositive curvature.

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Northwestern University, Evanston, IL KEITH BURNS

A MATHEMATICAL INTRODUCTION TO STRING THEORY: VARIATIONAL PROBLEMS, GEOMETRIC AND PROBABILISTIC METHODS
(London Mathematical Society Lecture Note Series 225)

By Sergio Albeverio, Jürgen Jost, Sylvie Paycha and Sergio Scarlatti: 135 pp., £22.95 (LMS Members' price £17.20), ISBN 0 521 55610 4 (Cambridge University Press, 1997).

In string theory, perturbative computations require the use of important tools from the theory of Riemann surfaces, and algebraic and differential topology. Although various review articles exist, scattered in the literature, on these topics, this is the first book with a comprehensive and pedagogical exposition of the many definitions and main theorems required in understanding and evaluating string perturbative amplitudes.

Among the topics covered, one can find the theory of harmonic maps, the ζ-function, evaluation of functional determinants, the study of Gaussian functional integrals, the theory of determinant bundles, Teichmüller spaces, index theory for families of elliptic operators, and so on.

The presentation is careful and pedagogical, and the authors have been very careful in making sure that the reader needs to be acquainted with only the basic rudiments of string theory. With this necessary prerequisite, the reader does not need to go and look in any other sources to understand and follow the definitions and theorems explained in the book.

The book is clear and elegant, and it represents a very valuable addition to the literature. It is ideal for graduate students in both physics and mathematics, or for those professionals who are interested in understanding some of the mathematical intricacies of string theory.

CERN LUIS ALVAREZ-GAUME

THE BOOK OF INVOLUTIONS
(American Mathematical Society Colloquium Publications 44)

By MAX-ALBERT KNUS, ALEXANDER MERKURJEV, MARKUS ROST and JEAN-PIERRE TIGNOL: 593 pp., US\$69.00, ISBN 0 8218 0904 0 (American Mathematical Society, 1998).

An involution is an anti-automorphism of order two of an algebra. For example, let K be a field, and let $b: V \times V \rightarrow K$ be a non-singular symmetric or skew-symmetric K-bilinear form. Then such an involution, $\sigma_b: \mathrm{End}_K(V) \rightarrow \mathrm{End}_K(V)$, is given by sending f to its adjoint with respect to b.

This volume is a compendious study of algebras with involution, a subject with many facets which becomes particularly interesting for central simple algebras. In the 1930s, A. A. Albert began the systematic study of involutions on central simple algebras in order to classify Riemann algebras, which are subalgebras of $M_{2g}(\mathbf{Q})$ associated to the study of correspondences on a Riemann surface of genus g. *Inter alia*, the authors give a new proof of Albert's classification.

Throughout the book, the relation between algebras and groups recurs continually in various manifestations. For example, as observed by A. Weil, the connected components of automorphism groups of central simple algebras with involution are generally classical algebraic simple adjoint groups. In fact, in their final chapter, the authors complete Weil's classification programme for outer forms of D_4 by means of their notion of a trialitarian algebra. Recall that triality is a topic of super-gravitas for physicists working in the super-space of things like super-gravity, and the final chapter ought to prove very interesting to the super-readers who make it that far!

This is not meant to imply that the volume is hard-going; far from it. Despite the authors' protestations to the contrary, the book is excellently written, and the chapters on algebraic groups and Galois cohomology alone would make the book an ideal read for aspiring postgraduate students of an algebraic persuasion. In addition, there is plenty of material to enlighten even those of us who already know something about the subject. For example, the authors study trace forms on algebras without the use of Galois descent in a manner which was new to me. Also, they do not discount the case when the characteristic is equal to two until the last possible moment, which makes this volume almost unique among books on hermitian forms.

All in all, this book recommends itself to anyone who wants a thorough reference source, complete with an ample selection of enlightening exercises and historical notes, which deals with exceptional Jordan algebras, Clifford algebras and modules, Tits' algebras and algebraic groups in a modern manner. The authors derive considerable inspiration from the classical papers of Jacques Tits and, in recompense, Tits has donated a very complimentary preface.

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INDEX OF BOOK REVIEWS