MR2723325 (Review) 37A45 (05D10 11J70 11K50 28Dxx 37-01 37D40)
Einsiedler, Manfred (CH-ETHZ); **Ward, Thomas [Ward, Thomas B.]** (4-EANG)
★**Ergodic theory with a view towards number theory.**
Graduate Texts in Mathematics, 259.
Springer-Verlag London, Ltd., London, 2011. *xviii*+481 *pp.* $74.95. *ISBN* 978-0-85729-020-5

Recent decades have witnessed amazing new applications of ergodic theory and ergodic ideas to number theory and combinatorics. This introductory book, which goes beyond the standard texts and allows the reader to get a glimpse of modern developments, is a timely and welcome addition to the existing and ever-growing ergodic literature.

In the first chapter the authors provide a sample of diverse number-theoretical and combinatorial results and topics which serve as a motivation for the following chapters (and for the subsequent volume). These include Weyl's theorem on uniform distribution of polynomials, Szemerédi's theorem on arithmetic progressions and Margulis' proof of Oppenheim's conjecture.

The second chapter contains standard material on measure-preserving transformations, mean and pointwise ergodic theorems (for one-parameter actions), and some basic facts on the notions of ergodicity, weak-mixing and strong-mixing.

In Chapter 3 continued fractions are introduced, the notions of continued fraction map and Gauss measure are defined and some classical results on Diophantine approximations are derived. The reader will meet the Gauss measure again in Chapter 9, where a different proof of its ergodicity is provided via the geodesic flow.

Chapter 4 is devoted to the discussion of invariant measures for continuous maps on compact metric spaces. The topics include the classical Krylov-Bogolyubov theorem and the notion of unique ergodicity. The chapter concludes with Furstenberg's proof of Weyl's theorem on uniform distribution of polynomials.

Chapter 5 contains the background facts on conditional expectation and conditional measures. The material of this chapter is frequently used in the rest of the book and, together with some additional results presented in Chapter 6, is an essential prerequisite for Chapter 7, which is devoted to Furstenberg's proof of Szemerédi's theorem.

The measure-theoretic machinery from Chapter 5 is applied in Chapter 6 to the setting of measure-preserving systems. In particular, factors and joinings of measure-preserving systems are discussed and a proof of ergodic decomposition is given. Among the background facts that are omitted in the present volume are those relating to the theory of entropy. The authors in the preface attribute this omission to the lack of space, promising to fill this gap in a subsequent volume where this theory will be more pertinent.

Chapter 7 focuses on Furstenberg's ergodic approach to Szemerédi's theorem on arithmetic progressions. This chapter is one of the highlights of the book. The reader who patiently follows the developments of the previous chapters is rewarded with a fairly complete proof of one of the crowning achievements of ergodic theory. While the material of this chapter is at times quite sophisticated, the authors manage to preserve the friendly style which is apparent throughout the book. They also make it clear that this is just the beginning and that the ergodic methods lead to further results in combinatorics (some of which, until now, have had no conventional proofs). A small quibble I have regarding this chapter is that the authors do not sufficiently stress the role of partition results, such as van der Waerden's theorem, in the proofs of density results. While in dealing with the ergodic proof of Szemerédi's theorem van der Waerden's theorem can be avoided (as it was in the original proof of Furstenberg), the use of partition results in the proofs of the multidimensional and polynomial extensions of Szemerédi's theorem is indispensable [see H. Furstenberg and Y. Katznelson, J. Analyse Math. **34** (1978), 275–291 (1979); MR0531279 (82c:28032); J. Analyse Math. **45** (1985), 117–168; MR0833409 (87m:28007); J. Anal. Math. **57** (1991), 64–119; MR1191743 (94f:28020); V. Bergelson and A. Leibman, J. Amer. Math. Soc. **9** (1996), no. 3, 725–753; MR1325795 (96j:11013); V. Bergelson and R. McCutcheon, Mem. Amer. Math. Soc. **146** (2000), no. 695, viii+106 pp.; MR1692634 (2000m:28018); V. Bergelson, A. Leibman and R. McCutcheon, J. Anal. Math. **95** (2005), 243–296; MR2145566 (2006a:11028)]. It is also worth stressing that Ramsey-theoretical partition results are related to topological dynamics in a way which parallels the connection between density results, such as Szemerédi's theorem, and the theory of multiple recurrence for measure-preserving actions. This parallelism could be underscored by presenting one of the available topological dynamical proofs of van der Waerden's theorem rather than the purely combinatorial one which the authors give at the beginning of Chapter 7.

In Chapter 8 the authors return to some of the notions and results which were treated in previous chapters for one-parameter actions and extend the generality of exposition to actions of locally compact groups. Ergodicity, mixing and ergodic decomposition are discussed in this more general set-up. This chapter also contains the discussion on Ledrappier's "three dots" example, ergodic theorems for amenable groups and Furstenberg's notion of stationary measures.

Chapter 9 is mainly devoted to the study of geodesic flow on hyperbolic surfaces. The authors do not assume prior knowledge of Lie theory and differential geometry and introduce the required material as they go. Among other things this chapter contains Hopf's proof of the ergodicity of the geodesic flow and yet another proof of the ergodicity of the continued fraction map which is based on work of Artin.

In Chapter 10 the authors discuss an important and enlightening example: translation on the quotient of Heisenberg group.

The final chapter, Chapter 11, is another highlight of this book. Here some of the results on geodesic flow discussed in Chapter 9 are complemented with thorough discussion of the horocyclic flow. Here are some of the topics treated in this chapter: lattices in $\mathrm{SL}(2, \mathbb{R})$, the Mautner phenomenon, the Howe-Moore theorem, rigidity of invariant measures for horocyclic flows, and Ratner's proof of the equidistribution of horocyclic orbits.

The authors say in the preface that this book is "intended to provide a gentle route to a tiny sample [of] some of the dramatic interactions between ergodic theory and other parts of the subject, notably Ramsey theory, infinite combinatorics and Diophantine number theory". They definitely succeed in conveying the spirit of the ubiquity of ergodic methods. This book is highly recommended to graduate students and indeed to anyone who is interested in acquiring a better understanding of contemporary developments in mathematics.

Reviewed by *Vitaly Bergelson*