Preface

Arithmetic geometry and algebraic dynamical systems are flourishing areas of mathematics. Both subjects have highly technical aspects, yet both offer a rich supply of down-to-earth examples. Both have much to gain from each other in techniques and, more importantly, as a means for posing (and sometimes solving) outstanding problems. Both require such technical competence that it is unlikely new graduate students will have the time or energy to master both. This book is intended as a starting point for either topic, but is in content no more than an invitation. We hope to show that a rich common vein of ideas permeates both areas, and hope that further exploration of this commonality will result.

Central to both topics is a notion of complexity. In arithmetic geometry `height' measures arithmetical complexity of points on varieties, while in dynamical systems `entropy' measures the orbit complexity of (algebraic) maps. The connections between these two notions in explicit examples lie at the heart of the book. The fundamental objects which appear in both settings are polynomials, so we are concerned principally with heights of polynomials. By working with polynomials rather than algebraic numbers we avoid local heights and p-adic valuations.

Our approach is computational and examples are included to illustrate the basic notions. There are 102 exercises in the text (some well-known and some original). Since the background required for some of them goes a little beyond the text, hints at solutions for all of them are included. There are also 14 `questions': these are either open-ended suggestions or known open problems. Some of the main results are proved under simplifying assumptions (for example, certain results about many-variable polynomials are proved for two variables), and some stated theorems are not proved at all. This is part of a deliberate attempt to advertise the subjects without losing the reader in a mass of detail.

What the book contains

Starting with Lehmer's beautiful paper from 1933, we sketch some of the history of what has become known as Mahler's measure of a polynomial. In his paper, Lehmer was concerned mainly with constructing large primes in a way that generalises the Mersenne primes (which is also a topic of current interest). Here we already see a coincidence of ideas appearing: all of his basic ingredients arise as dynamical data. His famous problem (to bound below his measure of growth in a uniform way) can be interpreted in dynamical terms. In the first chapter, we describe some partial results on Lehmer's problem. These have been selected to satisfy two criteria: each is definitive for its own special case, and each proof offers great aesthetic appeal with little technical anguish. Thus Schinzel's lower bound for the measure of integral polynomials with real zeros is included because it does both. The same applies to Smyth's resolution of Lehmer's problem in the non-

reciprocal case (though we present a simpler weaker version), and to Zagier's proof of Zhang's theorem. Dobrowolski's result is not included: this is elegant and important but was excluded because it is unclear whether it is the last word on the general case. Similarly, we omit further harder results of Dobrowolski, Schinzel and Lawton bounding the non-zero measures in terms of the number of non-zero coefficients. Clearly the selection has involved painful choices and we hope an interested reader would go on into the wider literature.

The second chapter introduces dynamical systems associated to integral polynomials at two levels. The first deals with toral endomorphisms (coming from monic polynomials), and some proofs are merely illustrated by examples. The second deals with automorphisms of ``solenoids'' and complete proofs are given at the expense of greater technical demands. In both settings the objects from Chapter 1 appear again in dynamical terms. A central result here is that the integer sequences studied by Lehmer arise as periodic points, and the Mahler measure appears as an entropy. The notion of expansiveness for dynamical systems is introduced, and it is shown that Lehmer's growth rate measure only exists for expansive systems. This is vital preparatory material for Chapter 4, which is an account of certain dynamical systems arising from polynomials in several variables.

In Chapter 3, we present some truly beautiful mathematics concerned with generalising notions in Chapter 1 to the many variable case. It is here that Mahler's insight has proved to be so valuable: he noticed (in 1960) that Lehmer's measure has an integral representation, and we have dubbed this `Mahler's Lemma'. Despite being simple (a direct application of Jensen's formula from complex analysis) it is of fundamental importance as it allows a natural definition of measure for polynomials in several variables. The chapter is devoted to three main topics. The first is to present some of the fascinating specific calculations over the years, including some recent calculations. In particular we state some of Boyd's recent examples (motivated by work of Deninger) where Mahler measures for some two-variable polynomials arise as values of L-functions of elliptic curves. The original intention to include some material on this connection was abandoned because of the massive background required to even state the basic conjectures. We go on to give Smyth's geometric proof of the classification of polynomials with vanishing measure. Finally, we give a thorough account of Lawton's ingenious proof of a limit formula which shows that many-variable measures are limits of single-variable measures.

Chapter 4 describes the dynamical system associated to a polynomial in several variables. The periodic points give an analogue of Lehmer's original growth rate for polynomials in several variables. The entropy formula is described, with some examples discussed in detail.

The first of the two chapters on elliptic curves, Chapter 5, contains mainly classical material. More than twenty years before Lehmer's paper, Mordell had used a notion of height to prove his famous and influential theorem on the finite generation of the group of rational points on an elliptic curve. We give a thorough account of how the notion of height is used to prove this, assuming for brevity the weak Mordell theorem. We go on to develop the more functorial notion of the canonical height, and go far enough to make it clear that the canonical height is some kind of analogue of Mahler's measure.

By this point, one theme should become clear. There is a linkage between Mahler's measure and arithmetic dynamical systems. There is another linkage between Mahler's measure and the canonical height. It is reasonable to ask if this triangle of ideas can be filled in by constructing a family of ``elliptic'' dynamical systems whose topological entropy is related to the canonical height and whose ergodicity is related to the (non-)vanishing of the canonical height. Chapter 6 therefore serves two purposes. One is to give a brief and self-contained introduction to the theory of elliptic curves over the complex numbers, sufficient to fix the idea that a complex elliptic curve is an analogue of the circle. The other is to make a deliberate attempt to reconcile the notions of Mahler's measure and the canonical height. In practice, this means re-interpreting the theory of the archimedean local height as it appears in Lang and Silverman (and as worked out originally by Tate). This local height is only one component of the global height, but all the ideas we want to illustrate arise here. The full picture for the possible elliptic dynamical systems is probably adelic in character, but the difficulties all lie at the archimedean places (and those coming from primes of bad reduction). Also, at the archimedean place there all the classical functions at our disposal. We develop elliptic analogues of Lehmer's original quantities and show they have the expected properties. In particular, we note that (for real points on real curves) it is possible to realise the values of the elliptic Mahler measure as the entropy of a transcendental map (in the rational case) whose periodic points are counted asymptotically by the division polynomial evaluated at the point.

This we hope completes the circle of ideas in a reasonable way for a book at this level.

Readership and background

The book can be read by post-graduates (or advanced under-graduates). It arose out of lectures given to post-graduate and MMath students at the University of East Anglia. With a few exceptions (notably the version of the Nullstellensatz in Chapter 5, Haar measure, elementary abelian harmonic analysis and the Gelfand transform in Chapters 2 and 4) the only background required is the typical mix of algebra and complex analysis taught in most British universities. Thus basic ring and group theory are assumed, as is complex analysis as far as Cauchy's theorem, residue calculations and some understanding of uniform and absolute convergence

of power series. Three appendices contain less standard material, a fourth contains a simple proof of an important estimate due to Mahler, a fifth contains hints for some of the exercises, and the last appendix contains a glossary of symbols.

Relationship between chapters

Chapters 1 and 3 are self-contained and comprise an introduction to the Mahler measure of polynomials. Chapters 5 and 6 are self-contained and comprise an introduction to elliptic curves, canonical heights and the elliptic Mahler measure. Chapters 2 and 4 use some material from 1 and 3 respectively, but are otherwise a fairly self-contained introduction to the simplest possible family of algebraic dynamical systems.

Sources

Much in this book is in the literature, and there are excellent texts on some of the topics considered. For elliptic curves and canonical heights, there are the texts of Silverman, although we interpret the height in a new way. For algebraic dynamical systems there is Schmidt's monograph, and for a general introduction to the field of dynamical systems and measures of orbit complexity there is the comprehensive introduction by Katok and Hasselblatt. The material on Mahler's measure (both classical and elliptic) is taken from papers not having appeared in book form before. For background on ergodic theory and dynamical systems, there are the books of Walters or Petersen.

Notation and numbering

Notation is explained in Appendix F. Theorems, lemmas, definitions and remarks are numbered X.Y for the Yth statement in Chapter X. Exercises are numbered separately by chapter, and Questions are numbered separately independent of chapter. Bold numbers in the index indicate definitions or main entries.

Acknowledgments

Besides the patient support of Susan and Tania, we would like to acknowledge the help received from David Boyd, Manfred Einseidler, Wayne Lawton, Andrezj Schinzel, Joe Silverman and Christopher Smyth. Also, for the careful reading of sections of the manuscript at various stages, we thank Paola D'Ambros, Bríd ní Flathúin, Richard Miles, Christian Röttger and the class of MTH4A22 (Spring 1998) at UEA.