References

1. D. Ferus, K. Leschke, F. Pedit and U. Pinkall, 'Quaternionic holomorphic geometry: Plücker formula, Dirac eigenvalue estimates and energy estimates of harmonic 2-tori', *Invent. Math.* 146 (2001) 507–593.
2. F. E. Burstall, F. Pedit and U. Pinkall, 'Schwarzian derivatives and flows of surfaces', *Contemp. Math.* 308 (2002) 39–61.
3. R. Bryant, 'A duality theorem for Willmore surfaces', *J. Differential Geom.* 20 (1984) 23–53.

University of Durham TOM WILLMORE

RECURRENCE SEQUENCES
(Mathematical Surveys and Monographs 104)

By Graham Everest, Alf van der Poorten, Igor Shparlinski and Thomas Ward: 318 pp., US$79.00, isbn 0-8218-3387-1 (American Mathematical Society, Providence, RI, 2003).

Bull. London Math. Soc. 36 (2004)
DOI: 10.1112/S0024609304223268

Professor Morris Kline argues in his book *Why the professor can't teach*, that in this day and age of specialization there is an increasing need for scholars (as distinct from teachers and researchers). The task of a mathematical scholar, he says, is to study the mathematical literature and develop an overview, and then to report on this, thus making available a large body of specialised work, and promoting collaboration between experts from various backgrounds.

The present book is such a scholarly work. The four authors have much experience with recurrent sequences; moreover, their mathematical backgrounds are to a certain extent disjoint. Thus as a group they function as the scholar imagined by Kline.

The subject of recurrent sequences is an ancient one with an enormous literature, not least because some results can be obtained even with very simple mathematical tools (thus attracting many amateurs). Fortunately, the authors have been selective with their references; nevertheless, the book still contains almost 1400 of them.

Recurrent sequences occur again and again in mathematics and computer science: in the theory of power series representation of rational functions, pseudo-random number generators, cellular automata, Diophantine equations, cryptography, analysis of algorithms, time series analysis and the theory of uniform distribution of sequences. All kinds of interesting functions (certain zeta functions, Hilbert series in commutative algebra, Poincaré series) are known to be rational, and thus the coefficients of their Taylor series satisfy recurrence relations. The best-known recurrence sequence is, of course, the binary recurrence $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 1$ and $F_1 = 1$, the so-called *Fibonacci sequence* (named after Leonardo, son (filius) of Bonacci). Polynomials are other well-known examples of recurrences, but they are not dealt with *per se*, as extensive treatments already exist.

Questions involving recurrences often surface in seemingly unconnected problems; for example, the problem of determining all the imaginary quadratic number fields having class number one can be reduced to finding all the squares amongst the Fibonacci numbers! Another seemingly unconnected problem in which linear recurrences play a role is in the proof of Hilbert's tenth problem over the integers.

Several surveys on recurrences exist, but this is the most encompassing so far (indeed, the authors write that they have relied heavily on these earlier surveys). The book also deals with some emerging areas in the theory of recurrences that are currently being intensively researched: Somos sequences and elliptic divisibility sequences. The latter can be parametrized by elliptic (theta) functions, and their growth rates have non-trivial connections with the canonical height on an associated elliptic curve. It seems that results here could play a role in the resolution of Hilbert's tenth problem over the rationals (which is still an open problem).

Since the book is a survey, it does not contain many proofs. Sometimes, the basic idea of a proof is given, and sometimes new results are proved (but then in full detail). The mathematical techniques that occur most frequently in the discussion come from Diophantine approximation, p-adic analysis, and the theory of character sums. Some number-theoretic and algebraic background is also assumed of the reader. The first half of the book (roughly) is concerned with general results concerning linear recurrence sequences. In the second half, several applications are presented, together with a study of some special sequences.

Without a doubt, this book will lead to increased contacts between scientists from different backgrounds with a common interest in recurrences. (Indeed, I learned from one of the authors that this has already happened.) Paul Erdős's remark that 'everybody writes and nobody reads', seems to be particularly true for people interested in this topic—I hope that they will consult this book before submitting yet another paper on recurrences.

The mathematical community should be grateful to the authors for the painstaking work that they have done, and for the very useful book that they have produced as a result.

University of Amsterdam PIETER MOREE

AN INTRODUCTION TO INVARIANTS AND MODULI
(Cambridge Studies in Advanced Mathematics 81)

By S. MUKAI (*translated by* W. M. OXBURY): 503 pp., £65.00 (US$90.00),
ISBN 0-521-80906-1 (Cambridge University Press, 2003).

Bull. London Math. Soc. 36 (2004)

DOI: 10.1112/S0024609304233264

Moduli spaces parametrise geometric objects with given properties: for example, algebraic curves of a given genus. They play an important part, not just in algebraic geometry, but also in other, related areas of modern mathematics.

The first chapter of the book contains many classical examples of moduli spaces and invariants as motivation. It begins by showing how non-degenerate plane conics modulo Euclidean transformations can be parametrised by the surface $AB-C^3=0$ in $\mathbb{A}^3$. This chapter also includes material on invariants of finite groups, Hilbert series, Molien's theorem, polyhedral groups, and the GL(2) and SL(2) invariants of binary quartics.