Chapter 4
Weak Containment and the Fell Topology

Recall from Definition 1.19 that a unitary representation π is contained in a unitary representation ρ, written as $\pi < \rho$, if there exists a ρ-invariant closed subspace $\mathcal{V} \subseteq \mathcal{H}_\rho$ such that the restriction $\rho|_{\mathcal{V}}$ of ρ to this subspace is isomorphic to π. In this chapter we will introduce the weaker, and often more useful, notion of weak containment. We also recall that in Corollary 1.73 we have shown that irreducible representations can be used to approximate any other cyclic representation. In fact, the approximation could be done uniformly on compact sets in G, which opens the door to many other discussions. In fact our discussions of weak containment and the Fell topology will both use the compact-open topology on $\mathcal{P}(G)$. We will also discuss these notions in detail for abelian groups as in Chapter 2, and for the examples of metabelian groups in Chapter 3.

4.1 Weak Containment

We start with the following important notion.

Definition 4.1 (Weak containment). Let π and ρ be unitary representations of G. We say that π is *weakly contained in* ρ and write $\pi \prec \rho$ if every diagonal π-matrix coefficient can be approximated, uniformly within compact subsets of G, by finite sums of diagonal ρ-matrix coefficients. That is, if for every $v \in \mathcal{H}_\pi$, every compact $Q \subseteq G$, and every $\varepsilon > 0$ there exists some $n \geqslant 1$ and $w_1, \ldots, w_n \in \mathcal{H}_\rho$ such that

$$\Big\|\varphi_v^\pi - \sum_{j=1}^n \varphi_{w_j}^\rho\Big\|_{Q,\infty} = \max_{g\in Q}\Big|\langle \pi_g v, v\rangle - \sum_{j=1}^n \langle \rho_g w_j, w_j\rangle\Big| \leqslant \varepsilon. \tag{4.1}$$

We note that by allowing sums of ρ-matrix coefficients we are able to ignore any multiplicity questions for the representations π or ρ (see also Exercise 4.2).

Exercise 4.2 (Elementary observations). Let π and ρ be unitary representations of G.
(a) Show that $\pi < \rho$ implies $\pi \prec \rho$ (where in fact we could set $Q = G$ and $\varepsilon = 0$ in the definition).
(b) Now let ρ be any unitary representation, and define $\pi = \rho \oplus \rho$ to be the direct sum of ρ with itself. Show that $\pi \prec \rho$ (where again we could set $Q = G$ and $\varepsilon = 0$ in the definition). However, if $\dim \mathcal{H}_\rho < \infty$ or ρ is irreducible, then we cannot have $\pi < \rho$.
(c) Let π and ρ be unitary representations of G. Show that $\pi \prec \rho$, $\pi^\infty \prec \rho$, and $\pi \prec \rho^\infty$ are mutually equivalent.

For an abelian group G we again write $\widehat{G}$ for its Pontryagin dual (using additive notation as in Chapter 2), $t \in \widehat{G}$ for its elements, and χ_t for the associated character and unitary representation on $\mathbb{C}$.

Exercise 4.3 (Abelian regular representation). Let λ be the regular representation of an abelian group G and let $t \in \widehat{G}$. Show that the character χ_t is weakly contained in λ. Show moreover that χ_t is not contained in λ if G is non-compact.

We hope that the abelian case above will help to motivate the notion of weak containment, and now return to the general case.

Essential Exercise 4.4 (Transitivity). Let π, ρ, γ be unitary representations of G with $\pi \prec \rho$ and $\rho \prec \gamma$. Show that $\pi \prec \gamma$.

Exercise 4.5 (Twisting weak containment). Let π and ρ be unitary representations of G and let χ be a unitary character of G. Show that $\pi \prec \rho$ implies that $\chi \otimes \pi \prec \chi \otimes \rho$ (see Lemma 1.26).

Exercise 4.6 (Restricting weak containment). Let $H < G$ be a closed subgroup and $\pi \prec \rho$ be two unitary representations of G. Show that $\pi|_H \prec \rho|_H$.

We start with the following simple reduction.

Lemma 4.7 (Normalization of vector and sum). *Let π and ρ be unitary representations of G. In the definition of weak containment $\pi \prec \rho$ it suffices to consider unit vectors $v \in \mathcal{H}_\pi$. Moreover, if $\pi \prec \rho$ and $v \in \mathcal{H}_\pi$ has $\|v\| = 1$, then for every compact subset $Q \subseteq G$ and $\varepsilon > 0$, there exist vectors $w_j \in \mathcal{H}_\rho$ for $j = 1, \dots, n$ with $\sum_{j=1}^n \|w_j\|^2 = 1$ satisfying (4.1).*

Proof. The first claim in the lemma follows simply from normalizing a given $v \in \mathcal{H}_\pi \smallsetminus \{0\}$. So suppose now that $\pi \prec \rho$ and $v \in \mathcal{H}_\pi$ has $\|v\| = 1$. Indeed, if $e \in Q$ and $\varepsilon \in (0,1)$ (as we may assume), then (4.1) implies that $|1 - s| = \left|\varphi_v^\pi(e) - \sum_{j=1}^n \varphi_{w_j}^\rho(e)\right| < \varepsilon$ for $s = \sum_{j=1}^n \|w_j\|^2$, and so $s > 0$. This gives, with $\widetilde{w}_j = s^{-\frac{1}{2}} w_j$ for $j = 1, \dots, n$, that

$$\Big\|\varphi_v^\pi - \sum_{j=1}^n \varphi_{\widetilde{w}_j}^\rho\Big\|_{Q,\infty} \leqslant \varepsilon + \Big\|\sum_{j=1}^n \varphi_{w_j}^\rho - \sum_{j=1}^n \varphi_{\widetilde{w}_j}^\rho\Big\|_{Q,\infty}$$

$$\leqslant \varepsilon + |1-s^{-1}| \underbrace{\Big\|\sum_{j=1}^n \varphi_{w_j}^\rho\Big\|_{Q,\infty}}_{=s} \leqslant \varepsilon + |s-1| < 2\varepsilon.$$

Since $\sum_{j=1}^n \|\widetilde{w}_j\|^2 = 1$, this implies the lemma. □

We now show that for irreducible representations of G we do not have to consider sums of matrix coefficients in the definition of weak containment. For this, recall from Proposition 1.74 that the compact-open and the weak* topologies on $\mathcal{P}^1(G)$ agree. For this reason, we will not specify the topology on $\mathcal{P}^1(G)$ whenever we use either of the two.

Proposition 4.8 (Irreducible representations). *Let π be an irreducible unitary representation of G. Suppose that, for a unit vector $v \in \mathcal{H}_\pi$, there exists a sequence (ϕ_n) in $\mathcal{P}^1(G)$ of the form*

$$\phi_n = \sum_j \phi_{n,j}$$

with $\phi_{n,j} \in \mathcal{P}^{\leqslant 1}(G)\smallsetminus\{0\}$ for all j, so that (ϕ_n) converges to φ_v^π as $n \to \infty$. Then there exists a sequence $(j(n))$ of indices in the sum with the property that the sequence $(\widetilde{\phi}_{n,j(n)})$ in $\mathcal{P}^1(G)$ converges to φ_v^π as $n \to \infty$, where

$$\widetilde{\phi}_{n,j(n)} = \phi_{n,j(n)}(e)^{-1}\phi_{n,j(n)} \in \mathcal{P}^1(G)$$

for all $n \geqslant 1$.

In particular, if $\pi \prec \rho$ for a unitary representation ρ of G, then there exists for every unit vector $v \in \mathcal{H}_\pi$, every compact subset $Q \subseteq G$, and every $\varepsilon > 0$, a unit vector $w \in \mathcal{H}_\rho$ such that $\|\varphi_v^\pi - \varphi_w^\rho\|_{Q,\infty} < \varepsilon$.

Proof. Suppose first that $\pi \prec \rho$ for a unitary representation ρ of G, and that $v \in \mathcal{H}_\pi$ is a unit vector. Then, by Lemma 4.7, for every compact $Q \subseteq G$ and $\varepsilon > 0$ we can find vectors $w_1, \dots, w_J \in \mathcal{H}_\rho$ with corresponding matrix coefficients $\phi_j = \varphi_{w_j}^\rho$ for $j = 1, \dots, J$ so that

$$\Big\|\varphi_v^\pi - \sum_{j=1}^J \phi_j\Big\|_{Q,\infty} < \varepsilon,$$

satisfying

$$\sum_{j=1}^J \phi_j(e) = \sum_{j=1}^J \|w_j\|^2 = 1$$

and hence

$$\phi = \sum_{j=1}^{J} \phi_j \in \mathcal{P}^1(G)$$

approximates φ_v^π, as in the first part of the proposition. Using $\varepsilon = \frac{1}{n}$ and σ-compactness of G, it follows that it suffices to prove the first part of the proposition.

Let $v \in \mathcal{H}_\pi$ have $\|v\| = 1$ and recall from Proposition 1.68 that $\varphi_v^\pi \in \mathcal{P}^1(G)$ is extremal. Since by definition

$$\mathcal{P}^{\leqslant 1}(G) = [0,1]\mathcal{P}^1(G),$$

it follows that φ_v^π is also extremal when viewed as an element of $\mathcal{P}^{\leqslant 1}(G)$. This is the reason why the proposition should be true: We approximate an extremal point by convex combinations, which should imply that most of the vectors used in the convex combination approximate the extremal vector. However, to give a complete argument we have to work a bit harder.

By the assumption, there exists a sequence (ϕ_n) with

$$\phi_n = \sum_{j=1}^{J(n)} \phi_{n,j} \in \mathcal{P}^1(G)$$

with $\phi_{n,j} \in \mathcal{P}^{\leqslant 1}(G) \smallsetminus \{0\}$ for all n, so that $\phi_n \to \varphi_v^\pi$ as $n \to \infty$ (in the weak* or compact-open topology). We define $s_{n,j} = \phi_{n,j}(e)$ and

$$\widetilde{\phi}_{n,j} = s_{n,j}^{-1} \phi_{n,j} \in \mathcal{P}^1(G)$$

for $j = 1, \dots, J(n)$. Using this notation, we have $\sum_{j=1}^{J(n)} s_{n,j} = \phi(e) = 1$ and may define the probability measure

$$\mu_n = \sum_{j=1}^{J(n)} s_{n,j} \delta_{\widetilde{\phi}_{n,j}} \tag{4.2}$$

on $\mathcal{P}^{\leqslant 1}(G)$.

We now show that μ_n has barycentre[†] ϕ_n in the following sense. Recalling that $L^\infty(G)$ is the dual Banach space to $L^1(G)$, let us write

$$\mathrm{ev}_f \colon \mathcal{P}^{\leqslant 1}(G) \ni \phi \longmapsto \int_G f \phi \,\mathrm{d}m$$

for the evaluation map for any $f \in L^1(G)$. We note that in the weak* topology the continuous linear functionals on $L^\infty(G) \supseteq \mathcal{P}^{\leqslant 1}(G)$ are precisely of this form. Because of this, the equation

[†] We refer to [21, Sec. 8.6.2] for more details on this notion.

$$\begin{aligned}\int_{\mathcal{P}^{\leqslant 1}(G)} \mathrm{ev}_f \, \mathrm{d}\mu_n &= \sum_{j=1}^{J(n)} s_{n,j} \int_{\mathcal{P}^{\leqslant 1}(G)} \mathrm{ev}_f \, \mathrm{d}\delta_{\widetilde{\phi}_{n,j}} \\ &= \sum_{j=1}^{J(n)} s_{n,j} \int_G f \widetilde{\phi}_{n,j} \, \mathrm{d}m \\ &= \int_G f \underbrace{\sum_{j=1}^{J(n)} s_{n,j} \widetilde{\phi}_{n,j}}_{=\sum_{j=1}^{J(n)} \phi_{n,j} = \phi_n} \, \mathrm{d}m = \int f \phi_n \, \mathrm{d}m\end{aligned}$$

for all $f \in L^1(G)$ is interpreted as ϕ_n being the barycentre of μ_n.

Now recall that the barycentre ϕ_n converges to φ_v^π in the weak* topology. Therefore we obtain a sequence of probability measures (μ_n) whose barycentres ϕ_n converge to φ_v^π as $n \to \infty$. As $\mathcal{P}^{\leqslant 1}(G)$ (equipped with the weak* topology) is a compact metric space by Corollary 1.73, the same holds for the space of Borel probability measures on $\mathcal{P}^{\leqslant 1}(G)$ (see [21, Prop. 8.27]). Hence we can choose a subsequence (μ_{n_ℓ}) with $\lim_{\ell\to\infty} \mu_{n_\ell} = \nu$ for some probability measure ν on $\mathcal{P}^{\leqslant 1}(G)$. Let $f \in L^1(G)$. By continuity of ev_f on $\mathcal{P}^{\leqslant 1}(G)$, we have

$$\begin{aligned}\int_{\mathcal{P}^{\leqslant 1}(G)} \mathrm{ev}_f \, \mathrm{d}\nu &= \lim_{\ell\to\infty} \int_{\mathcal{P}^{\leqslant 1}(G)} \mathrm{ev}_f \, \mathrm{d}\mu_{n_\ell} \\ &= \lim_{\ell\to\infty} \int_G f \phi_{n_\ell} \, \mathrm{d}m = \int_G f \varphi_v^\pi \, \mathrm{d}m.\end{aligned}$$

It follows that the barycentre of ν is given by φ_v^π.

We claim that, by extremality of φ_v^π, we have $\nu = \delta_{\varphi_v^\pi}$. Assuming the claim, it follows that we even have

$$\lim_{n\to\infty} \mu_n = \delta_{\varphi_v^\pi}. \tag{4.3}$$

Indeed, in a compact metric space, if every convergent subsequence has the same limit, then this limit is the limit of the whole sequence. Moreover, we conclude now from the definition of μ_n in (4.2), from (4.3), and Urysohn's lemma, that there exists a sequence $(j(n))$ with $1 \leqslant j(n) \leqslant J(n)$ for all $n \in \mathbb{N}$ such that $\widetilde{\phi}_{j(n),n}$ converges to φ_v^π in the weak* topology as $n \to \infty$. Since $\widetilde{\phi}_{j(n),n}$ and φ_v^π lie in $\mathcal{P}^1(G)$ for all $n \in \mathbb{N}$, Proposition 1.74 shows that this is equivalent to φ_v^π being the limit in the compact-open topology. This shows that the claim implies the proposition.

To prove the claim, we suppose (for the purposes of a contradiction) that $\nu \neq \delta_{\varphi_v^\pi}$. Then there exists some $\phi_0 \in \operatorname{supp} \nu \smallsetminus \{\varphi_v^\pi\}$ and some $f_0 \in L^1(G)$ with

$$\int_G f_0\phi_0 \,\mathrm{d}m \neq \int_G f_0\varphi_v^\pi \,\mathrm{d}m.$$

Multiplying f_0 by -1 or $\pm\mathrm{i}$ if necessary, we assume that

$$\Re\int_G f_0\phi_0 \,\mathrm{d}m < s < \Re\int_G f_0\varphi_v^\pi \,\mathrm{d}m$$

for some $s \in \mathbb{R}$, define the weak* neighbourhood

$$O = \{\phi \in \mathcal{P}^{\leqslant 1}(G) \mid \Re\int_G f_0\phi_0 \,\mathrm{d}m < s\}$$

with $\nu(O) \in (0,1)$, and split ν into the convex combination

$$\nu = \nu(O)\big(\tfrac{1}{\nu(O)}\nu|_O\big) + \nu(O^c)\big(\tfrac{1}{\nu(O^c)}\nu|_{O^c}\big).$$

However, this implies that φ_v^π is, as the barycentre of ν, the non-trivial convex combination of the barycentre of $\frac{1}{\nu(O)}\nu|_O$ that belongs to O and another element of $\mathcal{P}^{\leqslant 1}(G)$. This contradicts extremality of φ_v^π, which proves the claim and finishes the proof of the proposition. □

Exercise 4.9. Show that any probability measure μ on $\mathcal{P}^{\leqslant 1}(G)$ has a barycentre also in $\mathcal{P}^{\leqslant 1}(G)$.

By Proposition 4.8, the statement $\pi \prec \rho$ for an irreducible unitary representation π is equivalent to the statement that for any $v \in \mathcal{H}_\pi$ there is a $w \in \mathcal{H}_\rho$ with the property that v and w behave very similarly on a large compact subset (described by φ_v^π). Moreover, in the case that $\pi = \chi$ is a character, we may call the vector $w \in \mathcal{H}_\rho$ with this behaviour an *approximate eigenvector* for the character χ. If π is not irreducible, then the same can be said, but only using a vector in the Hilbert space $\mathcal{H}_\rho^\infty$.

Exercise 4.10 (Approximate eigenvectors). Let π be a unitary representation of G, and let χ be a unitary character of G. Show that $\chi \prec \pi$ if and only if for any compact set $Q \subseteq G$ and $\varepsilon > 0$ there exists a vector w such that $\|\pi_g w - \chi(g)w\| < \varepsilon$ for all $g \in Q$.

The following example shows that weak containment has surprising properties for non-abelian groups.

Example 4.11. Let $G = \mathbb{R}_{>0} \ltimes \mathbb{R}$ be the '$ax+b$' group as in Section 3.3.2. Recall that G has the old representations defined by characters on $A = G/H$ and the two new representations π^+ and π^-, where π^+ is defined by the measure μ_+ on $\widehat{H}$ with support $[0,\infty)$, and π^- is defined by the measure μ_- with support $(-\infty,0]$. We claim that $\mathbb{1} \prec \pi^+$. To see this define, for $\varepsilon \in (0,1)$,

$$v_\varepsilon = \frac{1}{\sqrt{|\log\varepsilon|}}\mathbb{1}_{[\varepsilon^2,\varepsilon]} \in \mathcal{H}_+ = L^2_{\mu_+}(\mathbb{R})$$

with

$$\|v_\varepsilon\|^2_{\mathcal{H}_+} = \frac{1}{|\log\varepsilon|}\int_{\varepsilon^2}^{\varepsilon}\frac{\mathrm{d}t}{t} = 1.$$

For $h_b \in H$ we have

$$\big|\langle\pi^+(h_b)v_\varepsilon, v_\varepsilon\rangle - 1\big| \leqslant \frac{1}{|\log\varepsilon|}\int_{\varepsilon^2}^{\varepsilon}\underbrace{\big|\mathrm{e}^{2\pi\mathrm{i}bt}-1\big|}_{\leqslant 2\pi|b|\varepsilon}\frac{\mathrm{d}t}{t} \leqslant 2\pi|b|\varepsilon$$

by the mean value theorem, which shows that $\varphi^{\pi^+}_{v_\varepsilon} = \langle\pi^+(h_b)v_\varepsilon, v_\varepsilon\rangle$ converges to 1 uniformly on compact subsets of H. For $g_a \in A$ with $a \in (\varepsilon, \frac{1}{\varepsilon})$, we also have

$$\pi^+(g_a)v_\varepsilon = \mathbb{1}_{[\varepsilon^2,\varepsilon]}\circ\widehat{\theta_{g_a}} = \mathbb{1}_{[\varepsilon^2/a,\varepsilon/a]}$$

and so

$$\big|\langle\pi^+(g_a)v_\varepsilon, v_\varepsilon\rangle - 1\big| = \Big|\frac{1}{|\log\varepsilon|}\int_{\max(\varepsilon^2,\varepsilon^2/a)}^{\min(\varepsilon,\varepsilon/a)}\frac{\mathrm{d}t}{t} - 1\Big| = \frac{|\log a|}{|\log\varepsilon|} \longrightarrow 0$$

uniformly on compact subsets of A. Therefore both $\|\pi^+(h_b)v_\varepsilon - v_\varepsilon\| \to 0$ and $\|\pi^+(g_a)v_\varepsilon - v_\varepsilon\| \to 0$ uniformly on compact subsets of H, resp. A. This implies that $\mathbb{1} \prec \pi^+$ (see Exercise 4.12).

Exercise 4.12. (a) Complete the argument above to show that $\mathbb{1} \prec \pi^+$.
(b) Extend the argument above by showing that $\chi \prec \pi^+$ and $\chi \prec \pi^-$ for any character χ of A.

4.2 Amenability, Property (T), and Spectral Gap

Using the notion of weak containment, we can formulate two important properties that the group G might have. For this, the following weakening of the notion of invariant vectors from Section 1.1.3 will be useful.

Definition 4.13 (Almost invariant vectors). A unitary representation π of G has *almost invariant vectors* if for every compact set $Q \subseteq G$ and $\varepsilon > 0$ there exists a unit vector $u \in \mathcal{H}_\pi$ with $\|\pi_g u - u\| < \varepsilon$ for all $g \in Q$. We also say, for a compact subset $Q \subseteq G$ and $\varepsilon > 0$, that a unit vector $u \in \mathcal{H}_\pi$ is (Q,ε)-*invariant* if $\|\pi_g u - u\| < \varepsilon$ for all $g \in Q$.

The following corollary to Proposition 4.8 links this notion to weak containment.

Corollary 4.14 (Almost invariant vectors). *Let π be a unitary representation of G. Then $\mathbb{1}_G \prec \pi$ if and only if π has almost invariant vectors.*

PROOF. Suppose that π has almost invariant vectors. Let $Q \subseteq G$ be compact and $\varepsilon > 0$. Then there exists a unit vector $v \in \mathcal{H}_\pi$ with $\|\pi_g v - v\| < \varepsilon$ for all $g \in Q$. However, this implies that

$$|1 - \varphi_v(g)| = |\|v\|^2 - \langle \pi_g v, v\rangle| = |\langle v - \pi_g v, v\rangle| < \varepsilon$$

for all $g \in Q$. As the compact subset Q and $\varepsilon > 0$ were arbitrary, we obtain $\mathbb{1}_G \prec \pi$.

For the converse, suppose that $\mathbb{1}_G \prec \pi$. By Proposition 4.8 there exists, for every compact subset $Q \subseteq G$ and $\varepsilon > 0$, a unit vector $v \in \mathcal{H}_\pi$ with

$$\|1 - \varphi_v\|_{Q,\infty} < \varepsilon^2.$$

For $g \in Q$ we now have

$$\|\pi_g v - v\|^2 = \langle \pi_g v, \pi_g v\rangle - 2\Re\langle \pi_g v, v\rangle + \|v\|^2 = 2 - 2\Re\varphi_v(g) < 2\varepsilon^2.$$

Taking the square root, we obtain $\|\pi_g v - v\| < \sqrt{2}\varepsilon$ for $g \in Q$. It follows that π has almost invariant vectors. □

Definition 4.15 (Amenability). We say that G is *amenable* if the trivial representation $\mathbb{1}_G$ is weakly contained in the regular representation λ_G, or equivalently if the regular representation λ_G has almost invariant vectors.

In particular, we see from Exercise 4.3 that every abelian group is amenable.

Exercise 4.16. Show that the solvable groups considered in Chapter 3 are amenable.

Definition 4.17 (Kazhdan's property (T)). We say that G has *property* (T) if, for any unitary representation ρ of G, we have that $\mathbb{1}_G \prec \rho$ holds only if $\mathbb{1}_G < \rho$, or equivalently if any representation that has almost invariant vectors in fact has invariant vectors.

For non-compact locally compact groups, amenability and property (T) are in some sense opposite properties. The first is a 'soft' property permitting many different behaviours for the actions of the group, the second a 'rigidity' property that sharply constrains the possible behaviours of actions.† Some groups have neither property, including finitely-generated free groups and (as we will show later) $\mathrm{SL}_2(\mathbb{R})$. Exercise 4.18 identifies the very limited class of groups with both properties.

Exercise 4.18. Show that G is compact if and only if G is amenable and has property (T).

We will briefly encounter property (T) again in Sections 7.2 and 7.3, and refer to [21, Ch. 10] for a further discussion of both amenability and property (T), and to the monograph of Bekka, de la Harpe, and Valette [2] for an extensive treatment of property (T).

Exercise 4.19. Let $\mathbb{F}_2 = \langle a, b\rangle$ be the free group generated by a and b. Show that $\mathbb{F}_2$ is not amenable, and that $\mathbb{F}_2$ does not have property (T).

† We will explain the notation in the notion 'property (T)' in Chapter **??**.

4.2.1 (Uniform) Spectral Gap

Definition 4.20 (Spectral gap). Let π be a unitary representation of G, and write

$$\mathcal{H}_\pi^G = \{v \in \mathcal{H}_\pi \mid \pi_g v = v \text{ for all } g \in G\}$$

for the subspace of invariant vectors. We say that π has *spectral gap* if π restricted to $(\mathcal{H}_\pi^G)^\perp$ does not have almost invariant vectors. Equivalently, π has spectral gap if there exists a compact subset $Q \subseteq G$ and some $\varepsilon > 0$ with the property that there are no (Q, ε)-invariant unit vectors in $(\mathcal{H}_\pi^G)^\perp$.

Comparing Definitions 4.13, 4.17, and 4.20, we see that a unitary representation of a group with property (T) always has a spectral gap. The next lemma shows that more is true.

Definition 4.21 (Uniform spectral gap). A collection S of unitary representations of G has *uniform spectral gap* if there exist $\varepsilon > 0$ and a compact subset $Q \subseteq G$ such that for every $\pi \in S$ and any $v \in \left(\mathcal{H}_\pi^G\right)^\perp$ there is some $g \in Q$ with $\|\pi_g v - v\| \geqslant \varepsilon \|v\|$.

Lemma 4.22 (Uniform spectral gap). *Suppose G has property* (T)*. Then all its unitary actions have uniform spectral gap.*

Proof. By our standing assumption, G can be written as $G = \bigcup_{n=1}^\infty Q_n$ for compact sets Q_n with $Q_n \subseteq Q_{n+1}^o$ for all $n \geqslant 1$. Suppose now that the lemma does not hold. Then for every $n \geqslant 1$ there is a unitary representation π_n of G on $\mathcal{H}_{\pi_n}$ so that π_n fails Definition 4.21 for Q_n and $\varepsilon = \frac{1}{n}$. Restricting to $\left(\mathcal{H}_{\pi_n}^G\right)^\perp$, we may assume that π_n has no non-trivial invariant vectors but that there exists a vector v_n with $\|v_n\| = 1$ and with

$$\sup_{g \in Q_n} \|\pi_n(g) v_n - v_n\| < \tfrac{1}{n}.$$

Now define $\mathcal{H}_\pi = \bigoplus_{n \geqslant 1} \mathcal{H}_{\pi_n}$ with the unitary representation $\pi = \bigoplus_n \pi_n$ of G on $\mathcal{H}_\pi$. It follows that π has no non-zero G-invariant vectors, but does have almost invariant vectors. This contradicts the assumption that G has property (T). □

The phrase 'spectral gap' and, in particular, the word 'gap' is made more clear in the next result.

Proposition 4.23 (Spectral gap in terms of convolution operators). *Let π be a unitary representation of G. Then the unitary representation has spectral gap if and only if there exists some non-negative function $f \in L^1(G)$ with $\int f \,\mathrm{d}m_G = 1$ for which there is some $\delta > 0$ with*

$$\left\| \pi_*(f)|_{(\mathcal{H}_\pi^G)^\perp} \right\|_{\mathrm{op}} \leqslant 1 - \delta < 1. \tag{4.4}$$

That is, the norm of the bounded operator $\pi_*(f)$, when restricted to the orthogonal complement of the space of invariant functions, is strictly less than one.

Moreover, f can be chosen in $C_c(G)$ with $f = f^* \geqslant 0$, or alternatively equal to $\frac{1}{m(P)}\mathbb{1}_P$ for some compact $P \subseteq G$ with $P^o \supseteq Q \cup \{e\}$, where $Q \subseteq G$ is as in the definition of spectral gap. Finally, if a collection S of unitary representations of G has uniform spectral gap, then f and the bound in (4.4) *can be chosen uniformly across S.*

Notice that the operator $\pi_*(f)$ in (4.4) satisfies $\pi_*(f)v = v$ for all $v \in \mathcal{H}_\pi^G$. If in addition $f = f^*$, then $\pi_*(f)$ is self-adjoint and (4.4) does indeed give a gap in the spectrum of $\pi_*(f)$.

Proof of Proposition 4.23. One direction of the equivalence is relatively straightforward. Indeed, suppose that π is a unitary representation of G and $f \in L^1(G)$ satisfies $f \geqslant 0$, $\int f \,\mathrm{d}m_G = 1$, and

$$\left\| \pi_*(f)|_{(\mathcal{H}_\pi^G)^\perp} \right\|_{\mathrm{op}} \leqslant \lambda = 1 - \delta < 1. \tag{4.5}$$

Then there exists a compact set $Q \subseteq G$ with

$$\int_{G \smallsetminus Q} f \,\mathrm{d}m_G < \tfrac{\delta}{3}.$$

This implies that, for any $v \in (\mathcal{H}_\pi^G)^\perp$, there exists some $g \in Q$ with

$$\|\pi_g v - v\| \geqslant \tfrac{\delta}{3}\|v\|.$$

Indeed, if this were not true, then we would have $\|\pi_g v - v\| < \frac{\delta}{3}\|v\|$ for all g in Q, and hence, by Exercise 1.54) applied to $\mathrm{d}\nu = \bigl(\int_Q f \,\mathrm{d}m_G\bigr)^{-1} f|_Q \,\mathrm{d}m_G$, we have

$$\begin{aligned}\|\pi_*(f)v - v\| &\leqslant \underbrace{\Bigl\|\pi_*(f|_Q)v - \int_Q f \,\mathrm{d}m_G v\Bigr\|}_{< \frac{\delta}{3}\int_Q f\,\mathrm{d}m_G \|v\|} \\ &\qquad + \underbrace{\|\pi_*(f|_{G\smallsetminus Q})v\|}_{<\frac{\delta}{3}\|v\|} + \underbrace{\int_{G\smallsetminus Q} f \,\mathrm{d}m_G}_{<\frac{\delta}{3}} \|v\| < \delta\|v\|,\end{aligned}$$

which contradicts (4.5).

Suppose now, for the converse, that the unitary representation π of G has spectral gap in the sense of Definition 4.20, and let (Q, ε) be chosen with the property that there are no (Q, ε)-invariant unit vectors in $(\mathcal{H}_\pi^G)^\perp$. We will show, starting with $Q \subseteq G$, $\varepsilon > 0$, and a compact set

$$P \supseteq P^o \supseteq Q \cup \{e\}$$

as in the proposition, that $f = \frac{1}{m(P)} \mathbb{1}_P$ satisfies (4.4). Note that with this f we have $f * f^* \in C_c(G)$ by Exercise 1.44, $(f * f^*)^* = f * f^*$, and that $f * f^*$ also satisfies (4.4). Hence the second half of the equivalence, and all of the additional claims in the second part of the proposition, will follow if we show that $f = \frac{1}{m(P)} \mathbb{1}_P$ gives rise to an operator $\pi_*(f)|_{(\mathcal{H}_\pi^G)^\perp}$ of norm no greater than $1 - \delta$ and with gap δ depending only on P, Q, and ε. At the heart of the argument, we will use strict convexity of the norm in $\mathcal{H}_\pi$.

For this, a uniform continuity property for applying π_g will be useful. To achieve this, we define $w = \pi_*(f^*)v$ for $v \in \left(\mathcal{H}_\pi^G\right)^\perp$. Thus

$$\begin{aligned}\|\pi_g w - \pi_h w\| &= \|\pi_g \pi_*(f^*)v - \pi_h \pi_*(f^*)v\| \\ &= \|\pi_*(\lambda_g f^* - \lambda_h f^*)v\| \\ &\leqslant \|\lambda_g f^* - \lambda_h f^*\|_1 \|v\| = \|f^* - \lambda_{g^{-1}h} f^*\|_1 \|v\|.\end{aligned}$$

This shows that there exists a uniform $\eta > 0$ (independent of π, $\mathcal{H}$, and v) such that

$$\|\pi_g w - \pi_h w\| < \tfrac{\varepsilon}{4}\|v\| \tag{4.6}$$

whenever $g^{-1}h \in B_\eta(e)$ holds. Using compactness of Q, the assumption that $Q \cup \{e\} \subseteq P^o$, and decreasing η if necessary, we can also assume that

$$g B_\eta^G(e) \subseteq P$$

for all $g \in Q \cup \{e\}$.

If

$$\|w\| = \|\pi_*(f^*)v\| < \tfrac{3}{4}\|v\|, \tag{4.7}$$

then we have basically achieved our goal. Hence we will assume for now that

$$\|w\| = \|\pi_*(f^*)v\| \geqslant \tfrac{3}{4}\|v\|.$$

Since $w \in \left(\mathcal{H}_\pi^G\right)^\perp$ is non-zero, there exists some $g_1 \in Q$ satisfying

$$\|\pi_{g_1} w - w\| \geqslant \varepsilon \|w\|.$$

Then, by the above assumption on w, we also have

$$\|\pi_{g_1} w - w\| \geqslant \tfrac{3\varepsilon}{4}\|v\|. \tag{4.8}$$

Moreover, the uniform continuity estimate in (4.6) gives

$$\|\pi_{h_0} w - w\| < \tfrac{\varepsilon}{4}\|v\| \tag{4.9}$$

for all $h_0 \in B_0 = B_\eta^G(e) \subseteq P$, and

$$\|\pi_{h_1} w - \pi_{g_1} w\| < \tfrac{\varepsilon}{4}\|v\| \tag{4.10}$$

for all $h_1 \in B_1 = g_1 B_\eta^G(e) \subseteq P$. In particular,

$$\|\pi_{h_0} w - \pi_{h_1} w\| > \tfrac{\varepsilon}{4}\|v\|$$

for $h_0 \in B_0$ and $h_1 \in B_1$, and so $B_0 \cap B_1 = \emptyset$. With these preparations, and setting $s = \frac{m(B_0)}{m(P)}$, we now obtain that

$$\begin{aligned}\pi_*\left(\tfrac{1}{m(P)}\mathbb{1}_P\right)w = \underbrace{\tfrac{m(B_0)}{m(P)}}_{=s}\underbrace{\pi_*\left(\tfrac{1}{m(B_0)}\mathbb{1}_{B_0}\right)w}_{=w_0} \\ + \underbrace{\tfrac{m(B_1)}{m(P)}}_{=s}\underbrace{\pi_*\left(\tfrac{1}{m(B_1)}\mathbb{1}_{B_1}\right)w}_{=w_1} \\ + \underbrace{\tfrac{m(\widetilde{P})}{m(P)}}_{=1-2s}\underbrace{\pi_*\left(\tfrac{1}{m(\widetilde{P})}\mathbb{1}_{\widetilde{P}}\right)w}_{=\widetilde{w}}\end{aligned}$$

where $\widetilde{P} = P\smallsetminus(B_0 \cup B_1)$. Using (4.9) and (4.10) gives $\|w_0 - w\| \leqslant \frac{\varepsilon}{4}\|v\|$ and $\|w_1 - \pi_{g_1} w\| \leqslant \frac{\varepsilon}{4}\|v\|$ by Exercise 1.54. Together with (4.8), we have

$$\|w_0 - w_1\| \geqslant \tfrac{\varepsilon}{4}\|v\|. \tag{4.11}$$

We now show the desired estimate by a detailed calculation. First note that

$$\begin{aligned}\left\|\pi_*(\tfrac{1}{m(P)}\mathbb{1}_P)w\right\|^2 &= \left\langle sw_0+sw_1+(1-2s)\widetilde{w}, sw_0+sw_1+(1-2s)\widetilde{w}\right\rangle \\ &= s^2\|w_0\|^2+s^2\|w_1\|^2+(1-2s)^2\|\widetilde{w}\|^2+2s^2\Re\langle w_0,w_1\rangle \\ &\quad + 2s(1-2s)\Re\langle w_0,\widetilde{w}\rangle+2s(1-2s)\Re\langle w_1,\widetilde{w}\rangle \\ &\leqslant \Big(2s^2+(1-2s)^2+2s^2+2s(1-2s)+2s(1-2s)\Big)\|w\|^2 \\ &= \|w\|^2 \leqslant \|v\|^2\end{aligned}$$

by the Cauchy–Schwarz inequality and the bounds $\|w_0\|, \|w_1\|, \|\widetilde{w}\| \leqslant \|w\|$. Hence it is sufficient to improve the inequality for one term only, and we will then obtain (4.4).

For this, notice that (4.11) implies

$$\begin{aligned}\left(\tfrac{\varepsilon}{4}\right)^2\|v\|^2 \leqslant \|w_0-w_1\|^2 &= \|w_0\|^2+\|w_1\|^2-2\Re\langle w_0,w_1\rangle \\ &\leqslant 2\|w\|^2-2\Re\langle w_0,w_1\rangle \\ &\leqslant 2\|v\|^2-2\Re\langle w_0,w_1\rangle,\end{aligned}$$

which gives

$$\Re\langle w_0, w_1\rangle \leqslant \left(1 - \tfrac{\varepsilon^2}{32}\right)\|v\|^2.$$

Combining this with the above gives, for an arbitrary $v \in (\mathcal{H}_\pi^G)^\perp$, either (4.7) or for $w = \pi_*(f^*)v$ with $f = \frac{1}{m(P)}\mathbb{1}_P$, that

$$\left\|\pi_*\left(\tfrac{1}{m(P)}\mathbb{1}_P\right)w\right\|^2 \leqslant \left(1 - s^2\tfrac{\varepsilon^2}{32}\right)\|v\|^2.$$

In either case, we have shown that $\pi_*(f)\pi_*(f^*)$ is a contraction on $(\mathcal{H}_\pi^G)^\perp$. Since $\pi_*(f^*) = \pi_*(f)^*$ by Section 1.4.3, $(\mathcal{H}_\pi^G)^\perp$ is invariant, and

$$\|AA^*\| = \|A\|^2$$

for any operator A on a Hilbert space, we have

$$\|\pi_*(f)\pi_*(f^*)|_{(\mathcal{H}_\pi^G)^\perp}\| = \|\pi_*(f)|_{(\mathcal{H}_\pi^G)^\perp}\|^2.$$

This implies that $\pi_*(f)$ is also a contraction on $(\mathcal{H}_\pi^G)^\perp$. Moreover, the contraction constant depends only on P, Q, and ε. □

4.2.2 (Uniform) Spectral Gap and Ergodic Theory

In ergodic theory, the existence of a spectral gap has many interesting consequences. The first of these is an unexpected and striking quantitative ergodic theorem.

Suppose that G acts on a locally compact space X and μ is an invariant and ergodic probability measure on X such that the induced unitary representation has spectral gap. We note that, using the notation of Definition 4.20, the G-action is ergodic if and only if $L^2_\mu(X)^G = \mathbb{C}\mathbb{1}$. Now let $f \in C_c(G)$ be as in Proposition 4.23. Then

$$v - \int v\,\mathrm{d}\mu \in \left(L^2_\mu(X)^G\right)^\perp$$

and hence

$$\left\|\pi_*(f)^n v - \int v\,\mathrm{d}\mu\right\|_{L^2_\mu} \leqslant (1-\delta)^n\|v\|_{L^2_\mu} \tag{4.12}$$

for $v \in L^2_\mu(X)$. That is, the average

$$\begin{aligned}\pi_*(f)^n v(x) &= \pi_*(f^{*n})v(x)\\ &= \int_G\cdots\int_G f(g_n)\cdots f(g_1)v(g_n\cdots g_1\centerdot x)\,\mathrm{d}m(g_1)\cdots\mathrm{d}m(g_n)\end{aligned}$$

converges exponentially fast in the square mean norm to the integral $\int v \, d\mu$. This is an odd kind of ergodic theorem, corresponding to a 'random walk'

$$g_1, g_2 g_1, \dots, g_n \cdots g_1$$

on G, and, via

$$g_1 \cdot x, (g_2 g_1) \cdot x, \dots (g_n \cdots g_1) \cdot x,$$

also on X. In fact, we have to use a particular density f^{*n} on the group to form the averages, but it is striking to obtain an exponential rate for all L^2-functions.† We will refine this phenomenon in Section 8.8 for particular representations of $\mathrm{SL}_2(\mathbb{R})$, and for more general groups we refer to the monograph of Gorodnik and Nevo [30].

Another consequence of the existence of a uniform spectral gap concerns the topology of the space of ergodic measures for the action. For actions of $\mathbb{Z}$, one can find many examples of continuous actions for which the set of ergodic invariant measures is dense in the much larger convex set of invariant probability measures (examples include the full shift on $\mathbb{F}_2^{\mathbb{Z}}$ discussed in Section 3.2.1, hyperbolic toral automorphisms, or the time-one map in an Anosov flow). As was discovered by Glasner and Weiss [28] and by Burger and Sarnak [5] in quite different contexts, uniform spectral gap implies that the structure of the set of ergodic measures is more well-behaved.

Corollary 4.24 (Ergodic measures with uniform spectral gap). *Let G act continuously on a locally compact σ-compact metric space X. Suppose that (μ_n) is a sequence of invariant ergodic probability measures on X such that the induced unitary representations have uniform spectral gap. If $\mu_n \to \mu$ as $n \to \infty$ in the weak* topology for some probability measure μ, then μ is also G-invariant and ergodic, with the same spectral gap. In particular, if G has property* (T)*, then the set of ergodic measures is a closed subset of the convex set of invariant measures.*

PROOF. By the hypotheses and Proposition 4.23 there exists $f = f^* \in C_c(X)$ with $f \geqslant 0$, $\int f \, dm_G = 1$ such that

$$\left\| \pi_*(f)v - \int v \, d\mu_n \right\|_{L^2_{\mu_n}} \leqslant (1-\delta) \|v\|_{L^2_{\mu_n}} \tag{4.13}$$

for all $v \in L^2_{\mu_n}(X)$, and so in particular for all $v \in C_c(X)$. Now notice that

$$(\pi_*(f)v)(x) = \int_{\operatorname{supp} f} f(g) v(g^{-1} \cdot x) \, dm(g)$$

† This is in complete contrast to the case of a single transformation, where Rokhlin towers (see [Sec. 2.9][20]) can be used to show that no uniform rate occurs in the convergence of ergodic averages; see Krengel [42].

for $x \in X$ defines another continuous function of compact support. Squaring both sides of (4.13) and expanding the definition shows that it is an inequality between quantities derived from integrals of various continuous functions (for example, $|v|^2$ on the right-hand side) against μ_n. Thus the inequality (4.13) depends continuously on μ_n, and so also holds for the limit measure μ. Moreover, (4.13) now extends by density of $C_c(X) \subseteq L^2_\mu(X)$ and continuity (with respect to $\|\cdot\|_{L^2_\mu}$) to all functions $v \in L^2_\mu(X)$.

If now $v \in L^2_\mu(X)$ is G-invariant then, by applying (4.13) to the function

$$v - \int v \,\mathrm{d}\mu,$$

we get

$$\left\| v - \int v \,\mathrm{d}\mu \right\|_{L^2_\mu} \leqslant (1-\delta) \left\| v - \int v \,\mathrm{d}\mu \right\|_{L^2_\mu}$$

which shows that

$$v = \int v \,\mathrm{d}\mu,$$

and so the limit measure μ is ergodic.

The final claim follows from the above, together with Lemma 4.22. □

Exercise 4.25. Square and expand (4.13) to verify that both sides depend continuously on μ_n for a fixed continuous function $v \in C_c(X)$.

Exercise 4.26. Let G act continuously on X, preserving a probability measure μ. Suppose the G-action is ergodic, and the induced representation has spectral gap. Deduce a 'random walk' pointwise ergodic theorem for $v \in L^2_\mu(X)$ with exponential rate of convergence at almost every $x \in X$.

4.3 Characterizing Weak Containment

To avoid cumbersome notation we also introduce a shorthand for positive-definite functions appearing in Definition 4.1 and the results of Section 4.1.

Definition 4.27 (Positive-definite functions associated to π). For a unitary representation π of G we define the set

$$\mathcal{P}^1_\pi = \left\{ \phi^\pi = \sum_{j=1}^n \varphi^\pi_{v_j} \mid n \in \mathbb{N}, v_1, \dots, v_n \in \mathcal{H}_\pi \text{ and } \sum_{j=1}^n \|v_j\|^2 = 1 \right\},$$

and refer to its elements $\phi^\pi \in \mathcal{P}^1_\pi$ as *positive-definite functions associated* to π. Moreover, we define $\overline{\mathcal{P}^1_\pi}$ to be the closure of $\mathcal{P}^1_\pi$ in the compact-open topology, and will refer to its elements ϕ^π as positive-definite functions *weakly associated* to π.

By comparing this with the definition of weak containment (see Definition 4.1 and Lemma 4.7), we see that a representation π of G is weakly contained in another representation ρ of G if and only if for any unit vector $v \in \mathcal{H}_\pi$ we have $\varphi_v^\pi \in \overline{\mathcal{P}_\rho^1}$.

We defined weak containment as a weakening of actual containment (Definition 1.19), and we have seen in the abelian setting examples where weak containment cannot be replaced by containment (see Exercise 4.3). We will see that weak containment will become more useful and important after we have obtained equivalent formulations. This motivates the following result, which goes back to the work of Fell [23] and Eymard [22].

Theorem 4.28 (Weak containment). *Let π and ρ be unitary representations of G. Then the following are equivalent:*

($\pi \prec_{\mathrm{diag}} \rho$) *For every unit vector $v \in \mathcal{H}_\pi$ we have $\varphi_v^\pi \in \overline{\mathcal{P}_\rho^1}$.*
($\pi \prec_{\mathrm{op}} \rho$) *We have $\|\pi_*(f)\|_{\mathrm{op}} \leqslant \|\rho_*(f)\|_{\mathrm{op}}$ for all $f \in L^1(G)$ (or $C_c(G)$).*
($\pi \prec_{\mathrm{mc}} \rho$) *For every $v, w \in \mathcal{H}_\pi$, compact $Q \subseteq G$ and $\varepsilon > 0$ there exist $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{H}_\rho$ such that we again have the uniform approximation of the matrix coefficients*

$$\left| \langle \pi(g)v, w \rangle - \sum_{j=1}^n \left\langle \rho(g)a_j, b_j \right\rangle \right| < \varepsilon$$

for all $g \in Q$ with the additional constraint† *that*

$$\sum_{j=1}^n \|a_j\| \|b_j\| \leqslant \|v\| \|w\|.$$

If any (and hence all) of these conditions are satisfied then we again say that π is weakly contained *in ρ, and write $\pi \prec \rho$.*

We will prove the theorem in three steps.

Proof that ($\pi \prec_{\mathrm{diag}} \rho$) implies ($\pi \prec_{\mathrm{op}} \rho$). Let $f \in C_c(G)$ and define

$$f_0 = f^* * f \in C_c(G).$$

As this implies (by the work in Section 1.4.3) that both $\pi_*(f_0) = \pi_*(f)^* \pi_*(f)$ and $\rho_*(f_0) = \rho_*(f)^* \rho_*(f)$ are self-adjoint positive operators, we have

$$\|\pi_*(f_0)\|_{\mathrm{op}} = \sup_{\substack{v \in \mathcal{H}_\pi \\ \|v\|=1}} \langle \pi_*(f_0)v, v \rangle$$

and

† The reader should recall from the proof of Lemma 4.7 that if $e \in Q$ then this additional constraint is — up to ε — automatic in Definition 4.1. This is unclear in the first estimate of ($\pi \prec_{\mathrm{mc}} \rho$).

$$\|\rho_*(f_0)\|_{\text{op}} = \sup_{\substack{w\in\mathcal{H}_\rho\\ \|w\|=1}} \langle \rho_*(f_0)w, w\rangle.$$

Fix some unit vector $v \in \mathcal{H}_\pi$, let $Q = \operatorname{supp} f_0$ and $\varepsilon > 0$. By assumption, we can find $\phi^\rho = \sum_{j=1}^n \varphi^\rho_{w_j} \in \mathcal{P}^1_\rho$ with $w_1,\dots,w_n \in \mathcal{H}_\rho$ and $\sum_{j=1}^n \|w_j\|^2 = 1$ such that

$$\|\varphi^\pi_v - \phi^\rho\|_{Q,\infty} < \varepsilon.$$

We multiply $\varphi^\pi_v(g) - \phi^\rho(g)$ by $f_0(g)$ and integrate over $Q = \operatorname{supp} f_0$ to obtain

$$\left|\langle \pi_*(f_0)v, v\rangle - \sum_{j=1}^n \langle \rho_*(f_0)w_j, w_j\rangle\right| < \varepsilon\|f_0\|_1.$$

Therefore

$$\begin{aligned}\langle \pi_*(f_0)v, v\rangle &< \sum_{j=1}^n \langle \rho_*(f_0)w_j, w_j\rangle + \varepsilon\|f_0\|_1\\ &< \sum_{j=1}^n \|\rho_*(f_0)\|_{\text{op}}\|w_j\|^2 + \varepsilon\|f_0\|_1 = \|\rho_*(f_0)\|_{\text{op}} + \varepsilon\|f_0\|_1.\end{aligned}$$

As $\varepsilon > 0$ and the unit vector $v \in \mathcal{H}_\pi$ were arbitrary, we deduce that

$$\|\pi_*(f_0)\|_{\text{op}} \leqslant \|\rho_*(f_0)\|_{\text{op}}.$$

Now recall that

$$\pi_*(f_0) = \pi_*(f^* * f) = \pi_*(f)^*\pi_*(f),$$

and similarly

$$\rho_*(f_0) = \rho_*(f)^*\rho_*(f),$$

which together with the identity $\|A^*A\|_{\text{op}} = \|A\|^2_{\text{op}}$ for any operator A on a Hilbert space gives $\|\pi_*(f)\|_{\text{op}} \leqslant \|\rho_*(f)\|_{\text{op}}$. By approximating $f \in L^1(G)$ by elements in $C_c(G)$ this inequality extends to $f \in L^1(G)$, and hence $(\pi \prec_{\text{op}} \rho)$**.** follows. □

Proof that $(\pi \prec_{\text{op}} \rho)$ implies $(\pi \prec_{\text{mc}} \rho)$. For this step we are going to use the weak* topology on the closed unit ball $B_1 = \overline{B_1(0)}$ of

$$L^\infty(G) \cong L^1(G)',$$

the subset $\mathcal{M}^{\leqslant 1}_\rho \subseteq B_1$ defined as the set consisting of all finite sums

$$\sum_{j=1}^n \varphi^\rho_{a_j,b_j} = \sum_{j=1}^n \left\langle \rho(g)a_j, b_j\right\rangle$$

of ρ-matrix coefficients $\varphi^\rho_{a_i,b_j}$ with the side constraint that

$$\sum_{j=1}^{n} \|a_j\| \|b_j\| \leqslant 1,$$

and the weak* closure $\overline{\mathcal{M}_\rho^{\leqslant 1}}$ of $\mathcal{M}_\rho^{\leqslant 1}$. Notice that the nature of the definition of $\mathcal{M}_\rho^{\leqslant 1}$ implies that $\mathcal{M}_\rho^{\leqslant 1}$ and $\overline{\mathcal{M}_\rho^{\leqslant 1}}$ are also convex subsets.

We claim that for a given $v, w \in \mathcal{H}_\pi$ with $\|v\| = \|w\| = 1$ the π-matrix coefficient

$$\varphi_{v,w}^\pi = \langle \pi(g)v, w\rangle$$

belongs to $\overline{\mathcal{M}_\rho^{\leqslant 1}}$. Suppose this is not the case. Then, by a corollary of the Hahn–Banach lemma (see [21, Th. 8.73]), and since $\overline{\mathcal{M}_\rho^{\leqslant 1}}$ is closed and convex, this implies that there is an $\mathbb{R}$-valued $\mathbb{R}$-linear functional L and some $s \in \mathbb{R}$ such that

$$L(\phi) \leqslant s < L(\varphi_{v,w}^\pi)$$

for all $\phi \in \overline{\mathcal{M}_\rho^{\leqslant 1}}$. It is easy† to verify that

$$L_{\mathbb{C}}(\phi) = L(\phi) - \mathrm{i}L(\mathrm{i}\phi)$$

for $\phi \in L^\infty(G)$ defines a $\mathbb{C}$-valued $\mathbb{C}$-linear functional on $L^\infty(G)$ with the property that $L = \Re L_{\mathbb{C}}$. However, since $\overline{\mathcal{M}_\rho^{\leqslant 1}}$ is closed in the weak* topology we may ensure that L and $L_{\mathbb{C}}$ are continuous in the weak* topology, which implies that $L_{\mathbb{C}}$ is given by an evaluation map (see [21, Lem. 8.13]). Together, it follows that there exists some function $f \in L^1(G)$ such that

$$L(\phi) = \Re\int_G f\phi \,\mathrm{d}m$$

for all $\phi \in L^\infty(G)$. Therefore,

$$\Re\int_G f\varphi_{a,b}^\rho \,\mathrm{d}m \leqslant s < \Re\int_G f\varphi_{v,w}^\pi \,\mathrm{d}m$$

for all $a, b \in \mathcal{H}_\rho$ with $\|a\|, \|b\| \leqslant 1$. Using the definition of $\rho_*(f)$ and $\pi_*(f)$ this gives

$$\Re \langle \rho_*(f)a, b\rangle \leqslant s < |\langle \pi_*(f)v, w\rangle| \leqslant \|\pi_*(f)\|.$$

As $a, b \in \mathcal{H}_\rho$ are arbitrary with $\|a\|, \|b\| \leqslant 1$ this gives

$$\|\rho_*(f)\| \leqslant s < \|\pi_*(f)\|,$$

† Indeed, any $\mathbb{R}$-valued $\mathbb{R}$-linear functional $L_{\mathbb{R}}$ on a complex vector space V has the form $L_{\mathbb{R}} = \Re L_{\mathbb{C}}$ for a $\mathbb{C}$-valued $\mathbb{C}$-linear functional $L_{\mathbb{C}}$ defined by $L_{\mathbb{C}}(\phi) = L_{\mathbb{R}}(\phi) - \mathrm{i}L_{\mathbb{R}}(\mathrm{i}\phi)$ for $v \in V$.

which contradicts the assumption ($\pi \prec_{\mathrm{op}} \rho$). It follows that $\varphi^{\pi}_{v,w}$ can be approximated in the weak* topology by elements of $\mathcal{M}^{\leqslant 1}_{\rho}$. To see that the matrix coefficient $\varphi^{\pi}_{v,w}$ can be approximated in the compact-open topology by finite sums, as claimed in ($\pi \prec_{\mathrm{mc}} \rho$), we will need Lemma 4.29 below. $\square$

Lemma 4.29 (Approximation in the compact-open topology). *Suppose π and ρ are unitary representations of G and that elements $v, w \in \mathcal{H}_\pi$ have the property that the matrix coefficient $\varphi^{\pi}_{v,w} \in C_b(G)$ can be approximated arbitrarily well in the weak* topology by finite sums*

$$\phi = \sum_{j=1}^{n} \varphi^{\rho}_{a_j, b_j}$$

of matrix coefficients defined by vectors $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{H}_\rho$ that satisfy the constraint

$$\sum_{j=1}^{n} \|a_j\| \|b_j\| \leqslant \|v\| \|w\|.$$

Then $\varphi^{\pi}_{v,w}$ can also be approximated arbitrarily well by such sums in the compact-open topology.

As we will see, the proof is an adaptation of the method of proof of Proposition 1.74

Proof of Lemma 4.29. Let π be a unitary representation, let $v, w \in \mathcal{H}_\pi$, and let us use the shorthand $\varphi_{v,w} = \varphi^{\pi}_{v,w}$. We fix some $\varepsilon > 0$ and some compact $K \subseteq G$. By continuity of the representation there exists a compact neighbourhood U of $e \in G$ such that

$$\big|\varphi_{v,w}(gh) - \varphi_{v,w}(g)\big| = \big|\langle \pi_g(\pi_h v - v), w\rangle\big| < \varepsilon \tag{4.14}$$

for all $g \in G$ and $h \in U$. We define $f_0 = \frac{1}{m(U)} \mathbb{1}_U$ and choose some open neighbourhood $V = V^{-1} \subseteq U$ of $e \in G$ such that

$$\big\|\lambda_h f_0 - f_0\big\|_1 < \varepsilon$$

for all $h \in V$. Moreover, we use translates of V to cover K, so that

$$\bigcup_{\ell=1}^{n} g_\ell V \supseteq K, \tag{4.15}$$

and use the elements $g_1, \dots, g_m$ to define $f_\ell = \frac{1}{m(U)} \mathbb{1}_{g_\ell U}$ for $\ell = 1, \dots, m$ and the weak* neighbourhood

$$\mathcal{N}_{f_0,\dots,f_m;\varepsilon} = \bigcap_{\ell=0,\dots,m} \Big\{\phi \in L^\infty(G) \mid \Big|\int (\phi - \varphi_{v,w}) f_\ell \,\mathrm{d}m\Big| < \varepsilon\Big\}$$

of $\varphi_{v,w}$. Suppose now that

$$\phi = \sum_{j=1}^{n} \varphi^{\rho}_{a_j,b_j} \in \mathcal{N}_{f_0,\dots,f_m;\varepsilon} \tag{4.16}$$

for some unitary representation ρ and vectors $a_1,\dots,a_n,b_1,\dots,b_n \in \mathcal{H}_\rho$ with $\sum_{j=1}^{n} \|a_j\|\|b_j\| \leqslant \|v\|\|w\|$. The pairing of ϕ and f_ℓ can be expressed differently as

$$\int_G \phi f_\ell \,\mathrm{d}m = \frac{1}{m(U)} \sum_{j=1}^{n} \int_U \big\langle \rho_{g_\ell h} a_j, b_j \big\rangle \,\mathrm{d}m(h) = \sum_{j=1}^{n} \Big\langle \rho_{g_\ell} \underbrace{\rho_*(f_0) a_j}_{=\widetilde{a_j}}, b_j \Big\rangle.$$

Using the continuity properties in (4.14) twice we also have, for any $g \in g_\ell U$,

$$\int_G \varphi_{v,w} f_\ell \,\mathrm{d}m = \frac{1}{m(U)} \int_{g_\ell U} \varphi_{v,w} \,\mathrm{d}m = \varphi_{v,w}(g) + \mathrm{O}(\varepsilon).$$

Combining the last two formulas with the assumption (4.14), we obtain

$$\sum_{j=1}^{n} \big\langle \rho_{g_\ell} \widetilde{a_j}, b_j \big\rangle = \varphi_{v,w}(g) + \mathrm{O}(\varepsilon) \tag{4.17}$$

for $\ell = 0,\dots,m$.

Finally, for $\widetilde{a_j} = \rho_*(f_0) a_j$ we will need the following uniform continuity property, namely that $h \in V$ implies

$$\|\rho_h \widetilde{a_j} - \widetilde{a_j}\| = \|\rho_h \rho_*(f_0) a_j - \rho_*(f_0) a_j\| \leqslant \|\lambda_h f_0 - f_0\|_1 \|a_j\| \leqslant \varepsilon \|a_j\| \tag{4.18}$$

for $j = 1,\dots,n$. Therefore, combining the above for $g = g_\ell h \in g_\ell V$ we obtain

$$\begin{aligned} \sum_{j=1}^{n} \big\langle \rho_g \widetilde{a_j}, b_j \big\rangle &= \sum_{j=1}^{n} \big\langle \rho_{g_\ell h} \widetilde{a_j}, b_j \big\rangle \\ &= \sum_{j=1}^{n} \big(\big\langle \rho_{g_\ell} \widetilde{a_j}, b_j \big\rangle + \mathrm{O}(\varepsilon \|a_j\|\|b_j\|) \big) && \text{(by (4.18))} \\ &= \varphi_{v,w}(g) + \mathrm{O}_{v,w}(\varepsilon) && \text{(by (4.17))} \end{aligned}$$

and

$$\sum_{j=1}^{n} \|\widetilde{a_j}\|\|b_j\| \leqslant \sum_{j=1}^{n} \|a_j\|\|b_j\| \leqslant \|v\|\|w\|.$$

We now can replace the sum of matrix coefficients in (4.16) by the sum

$$\widetilde{\phi} = \sum_{j=1}^{n} \varphi^{\rho}_{\widetilde{a_j}, b_j}$$

and, using the cover in (4.15), we obtain the desired statement. □

PROOF THAT $(\pi \prec_{\mathrm{mc}} \rho)$ IMPLIES $(\pi \prec_{\mathrm{diag}} \rho)$. Let $v \in \mathcal{H}_\pi$ have $\|v\| = 1$, and let $Q \subseteq G$ be a compact subset containing e. Fix some $\varepsilon \in (0, 1]$. Applying $(\pi \prec_{\mathrm{mc}} \rho)$ we find $a_j, b_j \in \mathcal{H}_\rho$ with

$$\left| \langle \pi(g)v, v \rangle - \sum_{j=1}^{n} \left\langle \rho(g) a_j, b_j \right\rangle \right| < \varepsilon^4$$

for all $g \in Q$ and

$$\sum_{j=1}^{n} \|a_j\| \|b_j\| \leqslant \|v\|^2 = 1.$$

Since $e \in Q$ we also have

$$\left| \sum_{j=1}^{n} \left\langle a_j, b_j \right\rangle - \|v\|^2 \right| < \varepsilon^4. \tag{4.19}$$

Together these inequalities will show that a_j and b_j are close to each other for 'most' j, which will allow us to switch to diagonal matrix coefficients.

Indeed, let

$$\mathrm{Bad} = \{ j \mid \Re \left\langle a_j, b_j \right\rangle \leqslant (1 - \varepsilon^2) \|a_j\| \|b_j\| \}.$$

Then using (4.19) we obtain

$$\begin{aligned} 1 - \varepsilon^4 &< \Re \sum_{j=1}^{n} \left\langle a_j, b_j \right\rangle \\ &\leqslant \sum_{j \in \mathrm{Bad}} (1 - \varepsilon^2) \|a_j\| \|b_j\| + \sum_{j \notin \mathrm{Bad}} \|a_j\| \|b_j\| \\ &\leqslant 1 - \varepsilon^2 \sum_{j \in \mathrm{Bad}} \|a_j\| \|b_j\|, \end{aligned}$$

which gives

$$\sum_{j \in \mathrm{Bad}} \|a_j\| \|b_j\| < \varepsilon^2.$$

Dropping those bad indices altogether, we may worsen our estimate by ε^2 to

$$\left| \langle \pi(g)v, v \rangle - \sum_{j=1}^{n} \left\langle \rho(g) a_j, b_j \right\rangle \right| < 2\varepsilon^2,$$

but may assume without loss of generality that

$$\Re \left\langle a_j, b_j \right\rangle > (1-\varepsilon^2)\|a_j\|\|b_j\| \tag{4.20}$$

for all j. Multiplying a_j by a scalar and b_j by its inverse, we may also assume that $\|a_j\| = \|b_j\|$. In that case (4.20) gives

$$\|a_j - b_j\|^2 = 2\|a_j\|^2 - 2\Re \left\langle a_j, b_j \right\rangle < 2\varepsilon^2 \|a_j\|^2,$$

and so

$$\left|\left\langle \rho(g)a_j, a_j \right\rangle - \left\langle \rho(g)a_j, b_j \right\rangle\right| = \left|\left\langle \rho(g)a_j, a_j - b_j \right\rangle\right| \leqslant \sqrt{2}\varepsilon \|a_j\|^2.$$

Thus we may replace b_j by a_j and worsen the approximation only by another 2ε. However, this means that we have obtained a uniform approximation of $\langle \pi(g)v, v\rangle$ on Q by a sum of diagonal matrix coefficients for ρ. Applying Lemma 4.7 gives **(**$\pi \prec_{\mathrm{diag}} \rho$**)**. □

Exercise 4.30. Show that the following condition is also equivalent to weak containment:

($\pi \prec_{\mathrm{measure}} \rho$**)** We have $\|\pi_*(\nu)\| \leqslant \|\rho_*(\nu)\|$ for all $\nu \in \mathcal{M}(G)$.

4.3.1 Weak Containment for Abelian Groups

Using the characterization **(**$\pi \prec_{\mathrm{op}} \rho$**)** of weak containment in Theorem 4.28 it is quite convenient to characterize weak containment for abelian groups using the support of unitary representations as in Definition 2.63.

Corollary 4.31 (Weak containment for abelian groups). *Let G be an abelian group, and let π and ρ be unitary representations of G. Then $\pi \prec \rho$ if and only if* $\operatorname{supp} \pi \subseteq \operatorname{supp} \rho$. *In particular, for any $t \in \widehat{G}$, we have $\chi_t \prec \rho$ if and only if $t \in \operatorname{supp} \rho$.*

Proof. By the spectral theorem (Corollaries 2.12 and 2.62) we have

$$\|\pi_*(f)\|_{\mathrm{op}} = \|\check{f}\|_{\operatorname{supp}\pi,\infty}$$

for $f \in L^1(G)$, and similarly for ρ. If now $\operatorname{supp}\pi \subseteq \operatorname{supp}\rho$, then we obtain

$$\|\pi_*(f)\|_{\mathrm{op}} = \|\check{f}\|_{\operatorname{supp}\pi,\infty} \leqslant \|\check{f}\|_{\operatorname{supp}\rho,\infty} = \|\rho_*(f)\|_{\mathrm{op}}$$

for all $f \in L^1(G)$, which implies $\pi \prec \rho$ by Theorem 4.28.

Suppose now that $\operatorname{supp}\pi$ is not contained in $\operatorname{supp}\rho$, and choose an element $t_0 \in \operatorname{supp}\pi \smallsetminus \operatorname{supp}\rho$. By Urysohn's lemma, there exists some $F \in C_c(\widehat{G})$ with $F(t_0) = 1$ and $F|_{\operatorname{supp}\rho} = 0$. By Corollary 2.5, there exists some function $f \in L^1(G)$ with $\|\check{f} - F\|_\infty < \frac{1}{2}$. Therefore

$$\|\check{f}\|_{\operatorname{supp}\rho,\infty} = \|\check{f} - F\|_{\operatorname{supp}\rho,\infty} < \tfrac{1}{2}$$

and

$$\|\check{f}\|_{\operatorname{supp}\pi,\infty} \geqslant |\check{f}(t_0)| > \tfrac{1}{2},$$

which gives $\|\pi_*(f)\|_{\mathrm{op}} > \|\rho_*(f)\|_{\mathrm{op}}$ and, by Theorem 4.28, that π is not weakly contained in ρ. □

4.3.2 Cyclic Representations

Using the characterization of weak containment in Theorem 4.28, we can characterize weak containment in terms of the generator, and explain the meaning of $\overline{\mathcal{P}_\rho^1}$ more clearly.

Corollary 4.32 (Cyclic vectors suffice). *Let ρ be a unitary representation of G. Then, for any cyclic representation π with generator $v_0 \in \mathcal{H}_\pi$ of norm $\|v_0\| = 1$, we have $\pi \prec \rho$ if and only if $\varphi_{v_0}^\pi \in \overline{\mathcal{P}_\rho^\infty}$. Moreover, we have*

$$\begin{aligned}[0,1]\overline{\mathcal{P}_\rho^1} &= \textit{closure of } [0,1]\mathcal{P}_\rho^1 \textit{ in the compact-open topology}\\ &= \textit{closure of } [0,1]\mathcal{P}_\rho^1 \textit{ in the weak* topology}\\ &= \big\{\varphi_{v_0}^\pi \mid \pi \prec \rho \textit{ is cyclic with generator } v_0 \textit{ of norm } \|v_0\| \leqslant 1\big\}.\end{aligned}$$

The equivalent condition $(\pi \prec_{\mathrm{op}} \rho)$ in Theorem 4.28 makes the following lemma useful for the proof of Corollary 4.32 and later discussions.

Lemma 4.33 (Convolution formula for operator norm). *Let π be a unitary representation of G, and let $f \in C_c(G)$. Then*

$$\begin{aligned}\|\pi_*(f)\|_{\mathrm{op}} &= \sup_{v \in \mathcal{H}_\pi} \lim_{n\to\infty} \big(\langle \pi_*(f^* * f)^{*n} v, v\rangle\big)^{\frac{1}{2n}}\\ &= \sup_{v \in \mathcal{H}_\pi} \lim_{n\to\infty} \left(\int_G (f^* * f)^{*n} \varphi_v^\pi \,\mathrm{d}m\right)^{\frac{1}{2n}},\end{aligned}$$

where the supremum could also be taken over a dense subset of $\mathcal{H}_\pi$.

Proof. Let $f \in C_c(G)$ and define $f_0 = f^* * f \in C_c(G)$ so that

$$T = \pi_*(f_0) = \pi_*(f)^* \pi_*(f)$$

is a positive self-adjoint operator. Fix some $v \in \mathcal{H}_\pi$ and let μ_v^T denote the spectral measure on $[0,\infty) \subseteq \mathbb{R}$ for the self-adjoint operator T (see [21, Sec. 12.4]) so that, in particular,

$$\langle \pi_*(f_0)v, v\rangle = \int_0^\infty x \, \mathrm{d}\mu_v^T(x).$$

We apply Hölder's inequality for the conjugate pair of exponents $(n, \frac{n}{n-1})$ to the last expression and obtain

$$\int_0^\infty x \cdot 1 \, \mathrm{d}\mu_v^T(x) \leqslant \underbrace{\left(\int_0^\infty x^n \, \mathrm{d}\mu_v^T(x)\right)^{\frac{1}{n}}}_{\longrightarrow S_v} \underbrace{\left(\int_{\mathbb{R}} \mathrm{d}\mu_v^T\right)^{1-\frac{1}{n}}}_{\longrightarrow \|v\|^2}$$

as $n \to \infty$, where[†] the limit S_v can be expressed using the continuous functional calculus for T (see [21, Sec. 12.4]) as

$$S_v = \sup \operatorname{supp} \mu_v^T = \lim_{n\to\infty} \left(\int_0^\infty x^n \, \mathrm{d}\mu_v^T(x)\right)^{\frac{1}{n}} = \lim_{n\to\infty} \langle \pi_*(f_0)^n v, v\rangle^{\frac{1}{n}}.$$

Combining these, we obtain that

$$\langle \pi_*(f_0)v, v\rangle \leqslant S_v \|v\|^2.$$

Taking the supremum over all $v \in \mathcal{H}_\pi$ (or v from a dense subset of $\mathcal{H}_\pi$) with $\|v\| \leqslant 1$ gives

$$\|\pi_*(f_0)\|_{\mathrm{op}} = \sup_{\|v\|\leqslant 1} \langle \pi_*(f_0)v, v\rangle \leqslant \sup_{v \in \mathcal{H}_\pi} S_v,$$

since we assumed that $\pi_*(f_0)$ is positive and self-adjoint.

Also note that

$$S_v = \lim_{n\to\infty} \langle \pi_*(f_0)^n v, v\rangle^{\frac{1}{n}} \leqslant \lim_{n\to\infty} \left(\|\pi_*(f_0)^n\|_{\mathrm{op}} \|v\|^2\right)^{\frac{1}{n}} = \|\pi_*(f_0)\|_{\mathrm{op}}$$

and so

$$\sup_{v\in\mathcal{H}_\pi} S_v \leqslant \|\pi_*(f_0)\|_{\mathrm{op}},$$

which implies that

$$\|\pi_*(f_0)\|_{\mathrm{op}} = \sup_{v\in\mathcal{H}_\pi} \lim_{n\to\infty} \langle \pi_*(f_0)^n v, v\rangle^{\frac{1}{n}} = \sup_{v\in\mathcal{H}_\pi} \lim_{n\to\infty} \left(\int f_0^{*n} \varphi_v^\pi \, \mathrm{d}m\right)^{\frac{1}{n}}$$

Finally, since $f_0 = f^* * f$ for $f \in C_c(G)$, the lemma follows from the identity $\|\pi_*(f^* * f)\|_{\mathrm{op}} = \|\pi_*(f)\|_{\mathrm{op}}^2$. □

[†] We note that the limits exist for the following simple reason: if $F \in L^\infty(X, \mu)$ is a non-negative bounded function on a finite measure space (X, μ), then $\left(\int_X F^n \, \mathrm{d}\mu\right)^{\frac{1}{n}}$ converges to $\|F\|_\infty$ as $n \to \infty$.

PROOF OF COROLLARY 4.32. We start by proving the equality of the four sets. Continuity of scalar multiplication implies for $\phi^\rho \in \overline{\mathcal{P}_\rho^1}$ and $s \in [0,1]$ that $s\phi^\rho \in s\overline{\mathcal{P}_\rho^1}$ belongs to the closure of $[0,1]\mathcal{P}_\rho^1$ in the compact-open topology. By Proposition 1.74, the compact-open topology is stronger than the weak* topology, so the compact-open closure of $[0,1]\mathcal{P}_\rho^1$ is contained in its weak* closure.

Suppose now that ϕ^ρ belongs to the weak* closure of $[0,1]\mathcal{P}_\rho^1$. By the GNS construction in Theorem 1.71 and Corollary 1.73, there exists a cyclic unitary representation π with generator v_0, so that $\phi^\rho = \varphi_{v_0}^\pi$. Since $\|\phi^\rho\|_\infty \leqslant 1$, we also have $\|v_0\| \leqslant 1$. We need to show that $\pi \prec \rho$ by using the assumption that the matrix coefficient $\varphi_{v_0}^\pi$ of the generator $v_0 \in \mathcal{H}_\pi$ belongs to the weak* closure of $[0,1]\mathcal{P}_\rho^1$. Equivalently, we assume that the matrix coefficient

$$\varphi_{v_0}^\pi = \lim_{k\to\infty} \varphi_{w_k}^{\rho^\infty}$$

is equal to the limit (in the weak* topology) of matrix coefficients of vectors $w_k \in \mathcal{H}_\rho^\infty$ for the representation ρ^∞ with $\|w_k\| \leqslant 1$ for all $k \geqslant 1$. We note that this implication will prove, in particular, also the non-trivial part of the first statement in the corollary.

For this we will apply the convolution formula in Lemma 4.33 and the notion $(\prec_{\mathrm{op}})$ from Theorem 4.28. First note that if $f_0 \in C_c(G)$, then

$$\begin{aligned}\left|\int_G f_0\varphi_{v_0}^\pi \,\mathrm{d}m\right| &= \lim_{k\to\infty}\left|\int_G f_0\varphi_{w_k}^{\rho^\infty}\,\mathrm{d}m\right| \\ &= \lim_{k\to\infty}\left|\left\langle \rho_*^\infty(f_0)w_k, w_k\right\rangle\right| \leqslant \|\rho_*^\infty(f_0)\|_{\mathrm{op}} = \|\rho_*(f_0)\|_{\mathrm{op}}.\end{aligned}$$

To obtain other vectors from our generator v_0, we let $v = \pi_*(\psi)v_0$ for some $\psi \in C_c(G)$. Now apply the above for

$$f_n = \psi^* * (f^* * f)^{*n} * \psi \in C_c(G)$$

for $f \in C_c(G)$ and $n \in \mathbb{N}$. This gives

$$\begin{aligned}\left|\int_G (f^* * f)^{*n}\varphi_v^{*n}\,\mathrm{d}m\right| &= \left|\left\langle \pi_*(f^* * f)^{*n}\pi_*(\psi)v_0, \pi_*(\psi)v_0\right\rangle\right| \\ &= \left|\left\langle \pi_*(f_n)v_0, v_0\right\rangle\right| \\ &= \left|\int f_n\varphi_{v_0}^\pi\,\mathrm{d}m\right| \leqslant \|\rho_*(f_n)\| \leqslant \|\rho_*(\psi)\|^2\|\rho_*(f)\|^{2n}.\end{aligned}$$

Taking the $2n$th root and the limit as $n\to\infty$, we arrive at

$$\lim_{n\to\infty}\left(\int_G (f^* * f)^{*n}\varphi_v^\pi\,\mathrm{d}m\right)^{\frac{1}{2n}} \leqslant \|\rho_*(f)\|_{\mathrm{op}}.$$

As v_0 is a generator of π by assumption, the subspace $\pi_*\left(C_c(G)\right)v_0$ is dense by Corollary 1.48, and hence the convolution formula in Lemma 4.33 implies $\|\pi_*(f)\|_{\mathrm{op}} \leqslant \|\rho_*(f)\|_{\mathrm{op}}$. The characterization ($\prec_{\mathrm{op}}$) of weak containment in Theorem 4.28 now gives $\pi \prec \rho$, as $f \in C_c(G)$ was arbitrary. Hence the set in the second line in the statement of the corollary is contained in the set in the third line.

Suppose now that $\pi \prec \rho$ is cyclic with generator v_0 of norm $\|v_0\| \leqslant 1$. Then $\widetilde{v_0} = \|v_0\|^{-1}v_0$ satisfies $\varphi^{\pi}_{\widetilde{v_0}} \in \overline{\mathcal{P}^1_\rho}$ by definition of weak containment, and hence $\varphi^{\pi}_{v_0}$ belongs to $[0,1]\overline{\mathcal{P}^1_\rho}$.

As already mentioned, if π is cyclic with generator v_0 satisfying $\|v_0\| = 1$ and $\varphi^{\pi}_{v_0} \in \overline{\mathcal{P}^1_\rho}$, then the argument above shows that $\pi \prec \rho$. As the converse of the first claim in the corollary is clear by definition, this gives the corollary. □

4.3.3 Approximation by Irreducible Representations

The following strengthens Corollary 1.73 by restricting the collection of irreducible representations needed for approximating the matrix coefficient of a given unitary representation.

Proposition 4.34 (Weakly contained irreducible representations). *Let ρ be a unitary representation of the group G and let $v \in \mathcal{H}_\rho$ be a vector with $\|v\| = 1$. Then φ^{ρ}_v can be approximated in the compact-open topology by sums $\sum_{j=1}^n \varphi^{\pi_j}_{v_j}$ with $v_j \in \mathcal{H}_{\pi_j}$ for $j = 1,\dots,n$ and $\sum_{j=1}^n \|v_j\|^2 = 1$ for irreducible unitary representations $\pi_1,\dots,\pi_n$ weakly contained in ρ.*

Proof. We claim that the extremal points of the convex set $\overline{\mathcal{P}^1_\rho}$ are also extremal in $\mathcal{P}^1(G)$, and so correspond to irreducible representations by Proposition 1.68. For this, let ϕ be an extremal point of $\overline{\mathcal{P}^1_\rho} \subseteq \mathcal{P}^1(G)$, and suppose that π is a cyclic unitary representation of G with generator v such that $\phi = \varphi^{\pi}_v$ (see Proposition 1.67) and $\|v_0\|^2 = \|\phi\| = 1$. By the characterization of weak containment for cyclic representations in Corollary 4.32 we also have $\pi \prec \rho$. If now ϕ is not an extremal point of $\mathcal{P}^1(G)$, then Proposition 1.68 shows that π is not irreducible and we can find non-trivial subspaces $\mathcal{V}_1$, $\mathcal{V}_2$ such that $\mathcal{H}_\pi = \mathcal{V}_1 \oplus \mathcal{V}_2$, non-zero vectors $v_1 \in \mathcal{V}_1$, $v_2 \in \mathcal{V}_2$ with $v = v_1 + v_2$ and hence $\phi = \varphi^{\pi}_v = \varphi^{\pi}_{v_1} + \varphi^{\pi}_{v_2}$. Moreover, we have $\|v_1\|^{-2}\varphi^{\pi}_{v_1} \neq \|v_2\|^{-2}\varphi^{\pi}_{v_2}$ since otherwise $v = v_1 + v_2 \in \mathcal{V}_1 \oplus \mathcal{V}_2$ would not generate $\mathcal{H}_\pi = \mathcal{V}_1 \oplus \mathcal{V}_2$, but instead the graph of an isomorphism between the cyclic representation $\pi|_{\mathcal{V}_1}$ and $\pi|_{\mathcal{V}_2}$ (see Proposition 1.60). It follows that $\pi|_{\mathcal{V}_j} < \pi \prec \rho$ for $j = 1, 2$ and hence $\|v_1\|^{-2}\varphi^{\pi}_{v_1}, \|v_2\|^{-2}\varphi^{\pi}_{v_2} \in \overline{\mathcal{P}^1_\rho}$ which, together with the identity $\phi = \varphi^{\pi}_v = \varphi^{\pi}_{v_1} + \varphi^{\pi}_{v_2}$ contradicts extremality of $\phi \in \overline{\mathcal{P}^1_\rho}$. This proves the claim that the extremal points of $\overline{\mathcal{P}^1_\rho}$ are also extremal in $\mathcal{P}^1(G)$, and so correspond to irreducible unitary representations of G (see Proposition 1.68).

By Corollary 4.32, the weak* closure of the convex set $[0,1]\mathcal{P}^1_\rho$ is given by $\mathcal{C} = [0,1]\overline{\mathcal{P}^1_\rho}$. Hence $\mathcal{C} \subseteq \mathcal{P}^{\leqslant 1}(G)$ is weak* compact by the Banach–Alaoglu theorem. Moreover, the extreme points of $\mathcal{C}$ are 0 and the extreme points of $\overline{\mathcal{P}^1(G)}$ belonging to $\mathcal{C}$. Applying now the proof of Corollary 1.73 with this in mind and replacing $\mathcal{P}^{\leqslant 1}(G)$ there with $\mathcal{C}$ gives the proposition. □

Exercise 4.35 (Characterizing weak containment using $\widehat{G}$). Let ρ and γ be unitary representations of G. Show that $\rho \prec \gamma$ if and only if for any $\pi \in \widehat{G}$ with $\pi \prec \rho$ we also have $\pi \prec \gamma$.

4.3.4 Weak Containment for some Metabelian Groups

In the context of metabelian groups as in Chapter 3, the following corollary to Proposition 4.34 is of interest as it helps to analyze which irreducible representations are weakly contained in a given unitary representation.

Corollary 4.36 (Weak containment and abelian subgroups). *Let H be a closed abelian subgroup of the group G. Then*

$$\overline{\bigcup_{\substack{\pi \in \widehat{G}\\ \pi \prec \rho}} \operatorname{supp}\bigl(\pi|_H\bigr)} = \operatorname{supp}\bigl(\rho|_H\bigr) \tag{4.21}$$

for any unitary representation ρ of G.

Proof. If $\pi \in \widehat{G}$ is weakly contained in ρ, then we may restrict the approximation claim in Definition 4.1 to H, and obtain $\pi|_H \prec \rho|_H$. This shows that $\operatorname{supp} \pi|_H \subseteq \operatorname{supp} \rho|_H$ by Corollary 4.31 and the first inclusion in (4.21) follows.

For the converse, we apply Propositions 4.8 and 4.34 (surprisingly often). Suppose now that $t \in \operatorname{supp} \rho|_H$, so that $\chi_t \prec \rho|_H$ by Corollary 4.31. By Proposition 4.8 applied to the irreducible representation $\chi_t \prec \rho|_H$, there exists a sequence of vectors (w_n) in $\mathcal{H}_\rho$ so that $\varphi^{\rho|_H}_{w_n}$ approximates χ_t in the compact-open topology. Applying Proposition 4.34 (for G) to each $\varphi^\rho_{w_n}$, there exist $\pi_j \in \widehat{G}$ weakly contained in ρ and $v_j \in \mathcal{H}_{\pi_j}$ for $j = 1, \dots, J(n)$ so that $\sum_{j=1}^{J(n)} \|v_j\|^2 = 1$ and $\sum_{j=1}^{J(n)} \varphi^{\pi_j}_{v_j}$ approximates $\varphi^\rho_{w_n}$ arbitrarily well. Restricting to H, we may also assume that $\sum_{j=1}^{J(n)} \varphi^{\pi_j|_H}_{v_j}$ approximates χ_t. Applying the first part of Proposition 4.8 (again for H), it follows that χ_t can also be approximated by matrix coefficients $\varphi^{\pi_n|_H}_{v_n}$ with $\pi_n \in \widehat{G}$ weakly contained in ρ and $v_n \in \mathcal{H}_{\pi_n}$ a unit vector.

Once more we apply Proposition 4.34 (for H) and see that $\varphi_{v_n}^{\pi_n|_H}$ can be approximated arbitrarily well by a sum

$$\sum_{j=1}^{J(n)} c_j \chi_{t_{n,j}}$$

with $\chi_{t_{n,j}} \prec \pi_n|_H$ and $c_j \geqslant 0$ for $j = 1, \ldots, J(n)$ and $\sum_{j=1}^{J(n)} c_j = 1$. Combining this with the above, we see that the same is true for χ_t. Finally, we apply Proposition 4.8 again to see that χ_t can be approximated in the compact-open topology by characters χ_{t_n} with $\chi_{t_n} \prec \pi_n|_H$. However, by Corollary 4.31 this shows that $t_n \in \operatorname{supp} \pi_n|_H$ and, by Corollary 2.5, that $t = \lim_{n\to\infty} t_n$ also belongs to the closure appearing on the left-hand side of (4.21). □

Let us study again the question of whether an irreducible representation can be weakly contained in other irreducible representations.

- For abelian groups, Corollary 4.31 shows that $\chi_{t_0} \prec \chi_{t_1}$ if and only if $t_0 = t_1$.
- Let $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ as in Section 3.3.1. Restricting the irreducible representations χ_m, χ_n, π^r, and π_s for $m, n \in \mathbb{Z}$ and $r, s \in (0, \infty)$ to the normal abelian subgroup $H \cong \mathbb{R}^2$, we see from Corollary 4.36 that $\pi^r \prec \pi_s$ implies $r = s$, that $\chi_m \not\prec \pi^r$, and that $\pi^r \not\prec \chi_m$. For χ_m and χ_n we apply Corollary 4.31 to $\mathrm{SO}_2(\mathbb{R}) \cong G/H$ and see that $\chi_m \prec \chi_n$ implies $m = n$.
- For the '$ax + b$' group $G = \mathbb{R}_{>0} \ltimes \mathbb{R}$ from Section 3.3.2, we already saw in Example 4.11 and Exercise 4.12 that $\chi \prec \pi^{\pm}$ for any character χ on $A = \mathbb{R}_{>0}$. Corollary 4.36 for the maximal subgroup H isomorphic to $\mathbb{R}$ shows, however, that $\pi^- \not\prec \pi^+$ and $\pi^+ \not\prec \pi^-$. Moreover, Corollary 4.31 (applied to $A \cong G/H$) shows that no character is weakly contained in another.

Exercise 4.37. (a) Analyze the question of weak containment for all pairs of irreducible unitary representations of the Heisenberg group in Section 3.3.4.
(d) Repeat this for the solvable group Sol in Exercise 3.23.

Exercise 4.38 (Weak containment for the isometry group of the plane). Let ρ be a unitary representation of the isometry group $\mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ of the plane, and let X be the set $\operatorname{supp} \rho|_{\mathbb{R}^2}$. Prove the following characterizations of weak containment for irreducible unitary representations.
(a) Show that $\pi^r \prec \rho$ if and only if $r\mathbb{S}^1 \subseteq X$.
(b) Show that if 0 is an accumulation point of X, then $\chi_n \prec \rho$ for all $n \in \mathbb{Z}$.
(c) Show that if 0 is not an accumulation point of X, then for every $n \in \mathbb{Z}$ we have $\chi_n \prec \rho$ if and only if $\chi_n < \rho$.

4.4 Fell Topology*

The Fell topology on the unitary dual $\widehat{G}$ of G is useful to give $\widehat{G}$ additional structure, which can be helpful to better understand and visualize $\widehat{G}$. We already saw this in Chapter 2, since the topology on the Pontryagin dual for a locally compact abelian group will turn out to be a special case of the Fell topology (see Sections 2.4 and 4.5.1). However, unlike our discussion above of weak containment, we will only need the Fell topology for non-abelian groups when we want to discuss its properties in a construction or an example.

A word of warning before we get started: The Fell topology can have, much like the Zariski topology in algebraic geometry, unusual properties — and in particular, for non-abelian or non-compact groups it is often not Hausdorff. Despite this, we will speak of convergence and limits even if the limit of a sequence may not be uniquely determined by the sequence.

At times it is also useful to have the Fell topology defined on a larger class of unitary representations. For this we recall our standing assumptions that we only wish to consider separable Hilbert spaces, so that it is sufficient to consider unitary representations on $\mathcal{V}_n = \mathbb{C}^n$ for $n \in \mathbb{N}$ or $\mathcal{V}_\infty = \ell^2(\mathbb{N})$.

Definition 4.39 (The set of unitary representations). For the group G, we define

$$\mathscr{U}(G) = \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \bigl\{\pi \mid \pi \text{ is a unitary representation on } \mathcal{V}_n\bigr\}.$$

For $\pi \in \mathscr{U}(G)$, we will again write $\mathcal{H}_\pi = \mathcal{V}_n$ if π is a unitary representation on $\mathcal{V}_n$ for $n \in \mathbb{N} \cup \{\infty\}$. We note that $\mathscr{U}(G)$ is a set.

As we will see, the Fell topology will have a very strong relationship with the notion of weak containment in Definition 4.1 and Theorem 4.28. Due to the equivalent notions of weak containment in Theorem 4.28, we also have the choice of defining the Fell topology using compact-open approximation of diagonal matrix coefficients as in $(\pi \prec_{\mathrm{diag}} \rho)$, or operator norms for convolution operators as in $(\pi \prec_{\mathrm{op}} \rho)$ from Theorem 4.28. The former may seem more natural, but is notationally slightly heavier and at times requires longer arguments, so we will use operator norms more prominently.

Definition 4.40 (Fell open sets). For any $f \in L^1(G)$ and $\alpha \in \mathbb{R}$, we define the principal Fell open subset

$$\mathscr{FO}(f, \alpha) = \bigl\{\pi \in \mathscr{U}(G) \mid \|\pi_*(f)\|_{\mathrm{op}} > \alpha\bigr\}.$$

We declare the principal Fell open sets to be a sub-basis of the Fell topology on $\mathscr{U}(G)$. That is, a Fell open set is, by definition, a union of finite intersections of principal Fell open sets as above.

Lemma 4.41 (Fell topology). *The Fell topology is second countable.*

Proof. Let $\{f_1, f_2, \dots\} \subseteq L^1(G)$ be L^1-dense, $\mathbb{Q} = \{r_1, r_2, \dots\}$, $f \in L^1(G)$, and $\alpha > 0$. We claim that this implies

$$\mathscr{FO}(f, \alpha) = \bigcup_{\substack{j,k \in \mathbb{N} \\ \|f - f_j\|_1 < r_k - \alpha}} \mathscr{FO}(f_j, r_k). \tag{4.22}$$

Suppose first that $\pi \in \mathscr{FO}(f_j, r_k)$ for some $(j,k) \in \mathbb{N}^2$ with

$$\|f - f_j\|_1 < r_k - \alpha.$$

In particular, $\pi \in \mathscr{FO}(f_j, r_k)$ means that $\|\pi_*(f_j)\|_{\mathrm{op}} > r_k$. Then we have

$$\|\pi_*(f)\|_{\mathrm{op}} \geqslant \|\pi_*(f_j)\|_{\mathrm{op}} - \|\pi_*(f - f_j)\|_{\mathrm{op}} > r_k - \|f - f_j\|_1 > \alpha$$

by the triangle inequality, or equivalently $\pi \in \mathscr{FO}(f, \alpha)$.

So suppose now that $\pi \in \mathscr{FO}(f, \alpha)$, so that $\|\pi_*(f)\|_{\mathrm{op}} > \alpha$. Then there exists some $r_k \in \mathbb{Q}$ with $\|\pi_*(f)\|_{\mathrm{op}} > r_k > \alpha$ and some $j \in \mathbb{N}$ with

$$\|f - f_j\|_1 < \|\pi_*(f)\|_{\mathrm{op}} - r_k.$$

This implies that $\|\pi_*(f - f_j)\|_{\mathrm{op}} \leqslant \|f - f_j\|_1 < \|\pi_*(f)\|_{\mathrm{op}} - r_k$, and so

$$\|\pi_*(f_j)\|_{\mathrm{op}} \geqslant \|\pi_*(f)\|_{\mathrm{op}} - \|\pi_*(f - f_j)\|_{\mathrm{op}} > r_k,$$

or equivalently that $\pi \in \mathscr{FO}(f_j, r_k)$ also belongs to the right-hand side of (4.22).

Taking all possible finite intersections of principal Fell open sets of the form $\mathscr{FO}(f_j, r_k)$ with $(j,k) \in \mathbb{N}$ we obtain, because of (4.22) and the definition of the Fell topology, a countable basis of the Fell topology. □

The following is a corollary to Theorem 4.28 and the above definition of the Fell topology.

Corollary 4.42 (Fell topology and weak containment). *For unitary representations $\pi, \rho \in \mathscr{U}(G)$ we have $\pi \prec \rho$ if and only if $\pi \in \overline{\{\rho\}}$, where the closure is taken with respect to the Fell topology. Moreover, for a sequence (ρ_n) in $\mathscr{U}(G)$ and $\pi \in \mathscr{U}(G)$ the following conditions are equivalent.*

(i) *ρ_n converges to π as $n \to \infty$ in the Fell topology.*
(ii) *For any strictly monotonely increasing sequence (n_k) in $\mathbb{N}$ we have*

$$\pi \prec \bigoplus_{k=1}^{\infty} \rho_{n_k}.$$

(iii) *For any $f \in L^1(G)$ (or $C_c(G)$), we have*

$$\liminf_{n\to\infty} \|\rho_{n,*}(f)\|_{\mathrm{op}} \geqslant \|\pi_*(f)\|_{\mathrm{op}}.$$

Proof. We start by proving the equivalences, and assume first that (ρ_n) converges to π as in (i). For any $f \in L^1(G)$ and $\varepsilon > 0$ the set

$$\mathscr{FO}(f, \|\pi_*(f)\|_{\mathrm{op}} - \varepsilon)$$

is an open neighbourhood of π, and so there exists some $N \in \mathbb{N}$ with

$$\rho_n \in \mathscr{FO}(f, \|\pi_*(f)\|_{\mathrm{op}} - \varepsilon)$$

for all $n \geqslant N$. This implies that

$$\liminf_{n\to\infty} \|\rho_{n,*}(f)\|_{\mathrm{op}} \geqslant \|\pi_*(f)\|_{\mathrm{op}} - \varepsilon.$$

Since $\varepsilon > 0$ and $f \in L^1(G)$ were arbitrary, condition (iii) follows.

Suppose now that (iii) holds, (n_k) is a strictly increasing sequence in $\mathbb{N}$, and define

$$\rho = \bigoplus_{k=1}^{\infty} \rho_{n_k}. \tag{4.23}$$

Then

$$\|\rho_*(f)\|_{\mathrm{op}} = \sup_{k\in\mathbb{N}} \|\rho_{n_k,*}(f)\|_{\mathrm{op}} \geqslant \liminf_{n\to\infty} \|\rho_{n,*}(f)\|_{\mathrm{op}} \geqslant \|\pi_*(f)\|_{\mathrm{op}}$$

for any $f \in L^1(G)$ (or $C_c(G)$). By the condition $(\pi \prec_{\mathrm{op}} \rho)$ of Theorem 4.28, this implies that $\pi \prec \rho$, as claimed in (ii).

So assume now that (ii) holds. If (ρ_n) does not converge to π, then there exists a neighbourhood of π such that for infinitely many $n \in \mathbb{N}$ the representation ρ_n does not belong to this neighbourhood. Without loss of generality we may assume that this neighbourhood of π is a principal Fell open set $\mathscr{FO}(f, \alpha)$ for some $f \in L^1(G)$ and $\alpha \in \mathbb{R}$. Hence there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ with $\|\rho_{n_k,*}(f)\|_{\mathrm{op}} \leqslant \alpha < \|\pi_*(f)\|_{\mathrm{op}}$. Defining ρ as in (4.23) gives

$$\|\rho_*(f)\| \leqslant \alpha < \|\pi_*(f)\|$$

and contradicts the assumed weak containment in (ii) by Theorem 4.28.

For the first claim in the corollary, we apply the above to the constant sequence defined by $\rho_n = \rho$ for all $n \in \mathbb{N}$ (see also Exercise 4.2(c)). □

We note that the Fell topology (by definition) does not distinguish between equivalent representations. More precisely, if $\pi, \rho \in \mathscr{U}(G)$ are equivalent representations, then $\pi \in \mathscr{O}$ if and only if $\rho \in \mathscr{O}$ for any Fell open set $\mathscr{O}$ in $\mathscr{U}(G)$. In particular, when restricting to irreducible representations the Fell topology can also be considered as a topology on $\widehat{G}$. We will also consider the Fell topology on other subsets of $\mathscr{U}(G)$.

Exercise 4.43. Show that a sequence (π_n) in $\mathscr{U}(G)$ converges to $\pi \in \mathscr{U}(G)$ if and only if π_n converges to $\pi|_{\langle v\rangle_\pi}$ as $n \to \infty$ for any unit vector $v \in \mathcal{H}_\pi$ (or any vector in a generating set $\{v_k \mid k \in \mathbb{N}\}$ satisfying $\mathcal{H}_\pi = \bigoplus_{k=1}^{\infty} \langle v_k\rangle_\pi$).

Exercise 4.44. Show that the operation $\oplus\colon \mathscr{U}(G) \times \mathscr{U}(G) \longrightarrow \mathscr{U}(G)$ is continuous with respect to the Fell topology. (Formally, we use for every $m, n \in \mathbb{N}$ some fixed isomorphisms $\mathcal{V}_m \oplus \mathcal{V}_n \cong \mathcal{V}_{m+n}$, $\mathcal{V}_n \oplus \mathcal{V}_\infty \cong \mathcal{V}_\infty \cong \mathcal{V}_\infty \oplus \mathcal{V}_n$, and $\mathcal{V}_\infty \oplus \mathcal{V}_\infty \cong \mathcal{V}_\infty$ to interpret $\oplus$ as an actual map from $\mathscr{U}(G) \times \mathscr{U}(G)$ to $\mathscr{U}(G)$.)

4.4.1 The Support of a Unitary Representation

Definition 4.45 (Support of a unitary representation). For a unitary representation ρ of G, we define the support of ρ as

$$\operatorname{supp}\rho = \{\pi \in \widehat{G} \mid \pi \prec \rho\}.$$

We note that by Corollary 4.42 we have $\operatorname{supp}\rho = \widehat{G} \cap \overline{\{\rho\}}$, where the closure is taken in the Fell topology, which shows in particular that the support is a Fell closed subset of $\widehat{G}$. We also note that Corollary 4.31 shows that the above definition of the support of a unitary representation agrees with the definition of the support in the case of an abelian group.

Exercise 4.46 (Weak containment and support). Let ρ_1, ρ_2 be unitary representations of G. Show that $\rho_1 \prec \rho_2$ if and only if $\operatorname{supp}\rho_1 \subseteq \operatorname{supp}\rho_2$.

With Definition 4.45 we can give a generalization of Corollary 4.36.

Exercise 4.47 (Support of restriction). Let $H < G$ be a closed subgroup, and let ρ be a unitary representation of G. Then

$$\overline{\bigcup_{\substack{\pi \in \widehat{G} \\ \pi \prec \rho}} \operatorname{supp}(\pi|_H)} = \operatorname{supp}(\rho|_H).$$

4.4.2 The Fell Topology Using Matrix Coefficients

We outline the alternative definition of the Fell topology using the compact-open topology on $\mathcal{P}^1(G)$, but will leave some steps as an exercise.

Definition 4.48 (Neighbourhoods in the Fell topology). For a compact set $Q \subseteq G$, $\varepsilon > 0$, and a continuous positive-definite function $\phi \in \mathcal{P}^1(G)$, we define the neighbourhood

$$CO(\phi, Q, \varepsilon) = \Big\{ F \in \mathcal{P}^1(G) \mid \|F - \phi\|_{Q,\infty} < \varepsilon \Big\}$$

in the compact-open topology. The *principal Fell open set* associated to Q, ε, and ϕ is defined by

$$\mathscr{F}\!\mathscr{O}_{\mathrm{diag}}(\phi, Q, \varepsilon) = \Big\{ \pi \in \mathscr{U}(G) \mid \text{there exists } \phi_\pi \in \mathcal{P}^1_\pi \text{ with } \phi_\pi \in CO(\phi, Q, \varepsilon) \Big\}.$$

Proposition 4.49 (Fell topology using matrix coefficients). *By varying $\phi \in \mathcal{P}^1(G)$, the compact subset $Q \subseteq G$, and $\varepsilon > 0$ we obtain another sub-basis of the Fell topology on $\mathscr{U}(G)$.*

In the proof of Proposition 4.49 we will make use of the phrase 'diag-Fell topology' to indicate the topology defined by the sub-basis in Definition 4.48. We will also need the elementary observations.

Lemma 4.50 (Inclusions). *Let $\phi_1, \phi_2 \in \mathcal{P}^1(G)$, Q_1, Q_2 compact subsets of G, and $\varepsilon_1, \varepsilon_2 > 0$. Suppose that*

$$CO(\phi_1, Q_1, \varepsilon_1) \subseteq CO(\phi_2, Q_2, \varepsilon_2).$$

Then

$$\mathscr{FO}_{\text{diag}}(\phi_1, Q_1, \varepsilon_1) \subseteq \mathscr{FO}_{\text{diag}}(\phi_2, Q_2, \varepsilon_2).$$

PROOF. The proof consists of checking the definitions. Indeed, suppose that $\pi \in \mathscr{FO}_{\text{diag}}(\phi_1, Q_1, \varepsilon_1)$. Then there exists a positive-definite function ϕ_π associated to π with $\phi_\pi \in CO(\phi_1, Q_1, \varepsilon_1)$. By our assumption, this implies that $\phi_\pi \in CO(\phi_2, Q_2, \varepsilon_2)$, and so $\pi \in \mathscr{FO}_{\text{diag}}(\phi_2, Q_2, \varepsilon_2)$. □

Lemma 4.51 (Sub-bases of neighbourhoods). *Let ρ be a unitary representation of G. By definition, every neighbourhood of ρ in the diag-Fell topology contains a finite intersection of sets of the form $\mathscr{FO}_{\text{diag}}(\phi, Q, \varepsilon)$, where $\phi \in \mathcal{P}^1(G)$, $Q \subseteq G$ is compact, and $\varepsilon > 0$. The same is true if we restrict ϕ to either of the following classes of functions:*

- *$\phi = \varphi_v^\rho$ for unit vectors $v \in \mathcal{H}_\rho$;*
- *$\phi = \varphi_w^\pi$ for unit vectors $w \in \mathcal{H}_\pi$ and irreducible unitary representation π weakly contained in ρ.*

PROOF. Let $\phi \in \mathcal{P}^1(G)$, $Q \subseteq G$ compact, and $\varepsilon > 0$ so $\rho \in \mathscr{FO}_{\text{diag}}(\phi, Q, \varepsilon)$. By definition of the principal Fell open set $\mathscr{FO}_{\text{diag}}(\phi, Q, \varepsilon)$, this means that there exist some positive-definite function $\phi_\rho \in \mathcal{P}_\rho^1$ associated to ρ with

$$\|\phi_\rho - \phi\|_{Q,\infty} < \varepsilon.$$

By definition of $\phi_\rho \in \mathcal{P}_\rho^1$ there exist vectors $v_1, \dots, v_n \in \mathcal{H}_\rho$ with the property that $\sum_{j=1}^n \|v_j\|^2 = 1$ and $\phi_\rho = \sum_{j=1}^n \varphi_{v_j}^\rho$. We suppose without loss of generality that $s_j = \|v_j\| > 0$ for $j = 1, \dots, n$. We set $\delta = \varepsilon - \|\phi_\rho - \phi\|_{Q,\infty} > 0$, define $\widetilde{v_j} = s_j^{-1} v_j \in \mathcal{H}_\rho$ for $j = 1, \dots, n$, and claim that

$$\mathscr{FO}_{\text{diag}}(\varphi_{\widetilde{v_1}}^\rho, Q, \delta) \cap \dots \cap \mathscr{FO}_{\text{diag}}(\varphi_{\widetilde{v_n}}^\rho, Q, \delta) \subseteq \mathscr{FO}_{\text{diag}}(\phi, Q, \varepsilon),$$

which will prove the first case in the lemma. To see the claim, suppose that π belongs to the intersection on the left-hand side. Then there exists, for every $j \in \{1, \dots, n\}$, a positive-definite function $\phi_j \in \mathcal{P}_\pi^1$ associated to π with $\|\phi_j - \varphi_{\widetilde{v_j}}^\rho\| < \delta$. We define $\phi_\pi = \sum_{j=1}^n s_j^2 \phi_j \in \mathcal{P}_\pi^1$, and note that $\varphi_{v_j}^\rho = s_j^2 \varphi_{\widetilde{v_j}}^\rho$ for $j = 1, \dots, n$. Together, we obtain

$$\|\phi_\pi - \phi_\rho\|_{Q,\infty} \leqslant \sum_{j=1}^n s_j^2 \|\phi_j - \varphi_{\widetilde{v_j}}^\rho\|_{Q,\infty} < \delta.$$

By the triangle inequality, this gives $\|\phi_\pi - \phi\|_{Q,\varepsilon} < \varepsilon$, and the claim follows from Lemma 4.50.

To see the second case of the lemma, it now suffices to consider a neighbourhood of the form $\mathscr{FO}_{\rm diag}(\varphi_v^\rho, Q, \varepsilon)$ for a unit vector $v \in \mathcal{H}_\rho$, a compact $Q \subseteq G$, and some $\varepsilon > 0$. We set $\delta = \frac{\varepsilon}{2}$ and apply Proposition 4.34 to find irreducible unitary representations $\pi_1, \dots, \pi_n \prec \rho$ and vectors $v_j \in \mathcal{H}_{\pi_j}$ for $j = 1, \dots, n$ so that $\sum_{j=1}^n \|v_j\|^2 = 1$ and $\|\varphi_v^\rho - \sum_{j=1}^n \varphi_{v_j}^{\pi_j}\|_{Q,\infty} < \frac{\varepsilon}{2}$. We again suppose that $s_j = \|v_j\| > 0$ and set $\widetilde{v_j} = s_j^{-1} v_j \in \mathcal{H}_{\pi_j}$ for $j = 1, \dots, n$, and use the same argument as above to show that

$$\mathscr{FO}_{\rm diag}(\varphi_{\widetilde{v_1}}^{\pi_1}, Q, \tfrac{\varepsilon}{2}) \cap \cdots \cap \mathscr{FO}_{\rm diag}(\varphi_{\widetilde{v_n}}^{\pi_n}, Q, \tfrac{\varepsilon}{2}) \subseteq \mathscr{FO}_{\rm diag}(\varphi_v^\rho, Q, \varepsilon).$$

This gives the second case in the lemma. □

Essential Exercise 4.52. Use Lemmas 4.50 and 4.51 to show that the diag-Fell topology is first countable.

Proof of Proposition 4.49. By Exercise 4.52 the diag-Fell topology is first countable. Recall that, by Lemma 4.41, this also holds for the Fell topology. Hence for both topologies the closed subsets can be characterized by limits of converging sequences (see Exercise 4.53). We claim that the characterization of convergence in terms of weak containment in Corollary 4.42(ii) also holds for the diag-Fell topology. Hence the Fell topology and the diag-Fell topology have the same convergent sequences with the same sets of possible limits. This implies that the Fell topology and the diag-Fell topology have the same closed sets, and so agree.

For the proof of the claim, we first assume that (ρ_n) is a sequence in $\mathscr{U}(G)$ converging to $\pi \in \mathscr{U}(G)$ in the diag-Fell topology. This implies quite directly (using the original definition of weak containment in Definition 4.1) that

$$\pi \prec \bigoplus_{k=1}^{\infty} \rho_{n_k} \tag{4.24}$$

for any increasing sequence $(n_k)_k$ in $\mathbb{N}$.

Finally, we suppose for the converse implication in the claim that (ρ_n) is a sequence in $\mathscr{U}(G)$ that does not converge in the diag-Fell topology. Hence there exists a neighbourhood U of π in the diag-Fell topology for which $\rho_{n_k} \notin U$ for all $k \in \mathbb{N}$ for some increasing sequence (n_k) in $\mathbb{N}$. By Lemma 4.51, we may assume that

$$U = \mathscr{FO}_{\rm diag}(\varphi_{w_1}^{\pi_1}, Q_1, \varepsilon_1) \cap \cdots \cap \mathscr{FO}_{\rm diag}(\varphi_{w_\ell}^{\pi_\ell}, Q_\ell, \varepsilon_\ell)$$

for some irreducible unitary representations $\pi_1, \dots, \pi_\ell \prec \pi$, some unit vectors $w_j \in \mathcal{H}_{\pi_j}$ for $j = 1, \dots, \ell$, compact subsets $Q_1, \dots, Q_\ell \subseteq G$, and numbers $\varepsilon_1, \dots, \varepsilon_\ell > 0$. Choosing a subsequence of $(n_k)_k$ (which we will still denote by $(n_k)_k$ for convenience), we may even suppose that

$$U = \mathscr{F}\mathscr{O}_{\mathrm{diag}}(\varphi_w^{\widetilde{\pi}}, Q, \varepsilon)$$

for some irreducible unitary representation $\widetilde{\pi} \prec \pi$, a unit vector $w \in \mathcal{H}_{\widetilde{\pi}}$, a compact set $Q \subseteq G$, and some $\varepsilon > 0$. We now show that π is not weakly contained in $\bigoplus_{k=1}^{\infty} \rho_{n_k}$, which will prove the claim above and hence also the proposition.

Suppose, for the purpose of a contradiction, that (4.24) holds. By the transitivity in Exercise 4.4 we also obtain $\widetilde{\pi} \prec \bigoplus_{k=1}^{\infty} \rho_{n_k}$. Therefore $\varphi_w^{\widetilde{\pi}}$ can be obtained as a limit of positive-definite functions associated to $\bigoplus_{k=1}^{\infty} \rho_{n_k}$. Since each matrix coefficient for $\bigoplus_{k=1}^{\infty} \rho_{n_k}$ is itself a converging sum of matrix coefficients of ρ_{n_k} for $k \in \mathbb{N}$, we deduce that $\varphi_w^{\widetilde{\pi}}$ can be approximated by finite sums of matrix coefficients of ρ_{n_k} for $k \in \mathbb{N}$. Since $\widetilde{\pi}$ is irreducible, we may apply Proposition 4.8 and obtain that $\varphi_w^{\widetilde{\pi}}$ is a limit (in the compact-open topology) of matrix coefficients of ρ_{n_k} for $k \in \mathbb{N}$. However, this is a contradiction of our construction of the subsequence $(n_k)_k$ since no positive-definite function associated to ρ_{n_k} for $k \in \mathbb{N}$ can approximate $\varphi_w^{\widetilde{\pi}}$ on Q with an error smaller than ε. □

Exercise 4.53. Let $\mathcal{T}$ be a topology on a set X satisfying the first countability axiom (that is, that every point in X has a countable basis of neighbourhoods). Show that $A \subseteq X$ is closed with respect to $\mathcal{T}$ if and only if it contains every limit point of every sequence of elements of A.

Exercise 4.54. Let $H < G$ be a closed subgroup, and let (ρ_n) be a sequence in $\mathscr{U}(G)$ converging to $\pi \in \mathscr{U}(G)$. Show that $(\rho_n|_H)$ converges to $\pi|_H$.

4.4.3 The Fell Topology as a Quotient Topology

As discussed in Section 1.5 (specifically, in Propositions 1.60 and 1.68), cyclic representations with a chosen generator are, up to unitary isomorphism, in a one-to-one correspondence with non-zero positive-definite functions. Moreover, pairs (π, v) where $\pi \in \widehat{G}$ is an irreducible representation and $v \in \mathcal{H}_\pi$ is a unit vector are in a one-to-one correspondence with extremal elements $\varphi_v^\pi \in \mathcal{E}^1(G)$, where we write $\mathcal{E}^1(G)$ for the extreme elements of $\mathcal{P}^1(G)$. This, and Corollary 4.32, gives another interpretation of the Fell topology on $\widehat{G}$. For $\phi \in \mathcal{E}^1(G)$ we write π^ϕ for the irreducible unitary representation from the GNS construction in Proposition 1.67, and write $[\pi^\phi] \in \widehat{G}$ for its equivalence class under unitary isomorphism.

Proposition 4.55 (Quotient topology of compact-open topology). *Using the compact-open topology on $\mathcal{E}^1(G)$, the map*

$$p\colon \mathcal{E}^1(G) \ni \phi \longmapsto [\pi^\phi] \in \widehat{G}$$

is both continuous and open with respect to the Fell topology on $\widehat{G}$. In particular, the Fell topology on $\widehat{G}$ is equal to the quotient topology of the compact-open topology on $\mathcal{E}^1(G)$ with respect to the map p.

PROOF. Suppose first that (ϕ_n) is a sequence in $\mathcal{E}^1(G)$ that converges in the compact-open topology to $\phi \in \mathcal{E}^1(G)$. Let $[\pi_n] = p(\phi_n) \in \widehat{G}$ and $[\pi] = p(\phi) \in \widehat{G}$ with generators $v_n \in \mathcal{H}_{\pi_n}$ and $v \in \mathcal{H}_\pi$ respectively, so that $\varphi^{\pi_n}_{v_n} = \phi_n$ for $n \in \mathbb{N}$ and $\varphi^{\pi}_{v} = \phi$. To prove that (π_n) converges to π in the Fell topology, we use the equivalent condition in Corollary 4.42(ii). So let (n_k) be an increasing sequence in $\mathbb{N}$. Since $\phi_{n_k} \to \phi$ as $k \to \infty$ it follows from Corollary 4.32 that $\pi \prec \rho = \bigoplus_{k=1}^{\infty} \pi_{n_k}$. As the subsequence (π_{n_k}) was arbitrary, it follows that (π_n) converges to π in the Fell topology. This gives continuity of the map p with respect to the Fell topology on $\widehat{G}$.

For the proof that p is an open map, we suppose that $\pi = p(\phi) \in \widehat{G}$ with $\phi \in \mathcal{E}^1(G)$, and that the sequence (π_n) in $\widehat{G}$ converges to π in the Fell topology. We claim that this implies the existence of unit vectors $v_n \in \mathcal{H}_{\pi_n}$ so that $\phi = \lim_{n\to\infty} \varphi^{\pi_n}_{v_n}$ in the compact-open topology. Assume the claim for now, and let $O \subseteq \mathcal{E}^1(G)$ be open. Applying the claim to $\pi = p(\phi)$ for some $\phi \in O$ and any sequence (π_n) converging to π in the Fell topology, we find some N so that $\varphi^{\pi_n}_{v_n} \in O$, and hence $\pi_n \in p(O)$, for $n \geqslant N$. As the Fell topology is first countable by Lemma 4.41, and the converging sequence with limit in $p(O)$ was arbitrary, this implies that $p(O)$ is open. Finally, note that continuity, openness, and surjectivity of p together show that a subset $U \subseteq \widehat{G}$ is open in the Fell topology if and only if $O = p^{-1}(U) \subseteq \mathcal{E}^1(G)$ is open in the compact-open topology, which makes the Fell topology the quotient topology on $\widehat{G}$.

So let $\pi = p(\phi)$ with $\phi \in \mathcal{E}^1(G)$ and $(\pi_n)_n$ converging to π, as in the claim. Moreover, let $Q \subseteq G$ be compact and let $\varepsilon > 0$. Suppose, for the purpose of a contradiction, that for infinitely many $n \in \mathbb{N}$ there is no unit vector $v_n \in \mathcal{H}_{\pi_n}$ with $\varphi^{\pi_n}_{v_n} \in CO(\phi, Q, \varepsilon)$. We choose an increasing sequence (n_k) of such integers. By our assumption that π_n converges to π, and by Corollary 4.42(ii), we have $\pi \prec \rho = \bigoplus_{k=1}^{\infty} \pi_{n_k}$. Since the matrix coefficients of ρ are uniformly converging sums of matrix coefficients of π_{n_k} for $k \in \mathbb{N}$, any positive-definite function associated to ρ can be approximated by finite sums of matrix coefficients of π_{n_k} for $k \in \mathbb{N}$. Applying the definition of $\pi \prec \rho$ to the generator $v \in \mathcal{H}_\pi$ with matrix coefficient ϕ and Proposition 4.8, it follows that ϕ can be approximated by matrix coefficients of unit vectors $v_{n_k} \in \mathcal{H}_{\pi_{n_k}}$ for some $k \in \mathbb{N}$. However, this contradicts our construction of the sequence (n_k) in $\mathbb{N}$. This contradiction shows that, for a fixed compact subset $Q \subseteq G$ and a fixed $\varepsilon > 0$, for all but finitely many $n \in \mathbb{N}$ there exists a unit vector v_n with $\varphi^{\pi_n}_{v_n} \in CO(\phi, Q, \varepsilon)$. Using the fact that G is σ-compact and taking $\varepsilon = \frac{1}{\ell}$ for $\ell \in \mathbb{N}$, this implies the claim and hence the proposition. □

4.5 Examples of the Fell Topology*

We will describe in this section the Fell topology on the unitary duals of the three metabelian groups discussed in detail in Chapter 3. Before doing so, we completely describe the abelian case.

4.5.1 Abelian Groups

Corollary 4.56 (Fell topology for abelian groups). *Let G be abelian. Then the Fell topology on $\widehat{G}$ agrees with the locally compact, σ-compact, metric topology of Corollary 2.5. Moreover, a sequence (ρ_n) in $\mathscr{U}(G)$ converges to $\pi \in \mathscr{U}(G)$ if and only if for any $t \in \operatorname{supp} \pi$ there exists $t_n \in \operatorname{supp} \rho_n$ for all $n \in \mathbb{N}$ such that $t_n \to t$ as $n \to \infty$.*

Proof. Recall that the topology on the Pontryagin dual in Corollary 2.5 can be defined as the compact-open topology for the characters viewed as elements of $C_b(G)$. Also note that all diagonal matrix coefficients for the unitary representation χ_t defined by $t \in \widehat{G}$ are positive multiples of χ_t and hence the only positive-definite function $\phi \in \mathcal{P}^1_{\chi_t}$ associated to χ_t is $\phi = \chi_t$ itself. Therefore

$$\mathscr{FO}_{\mathrm{diag}}(\phi, Q, \varepsilon) \cap \widehat{G} = \left\{ t \in \widehat{G} \mid \|\chi_t - \phi\|_{Q,\infty} < \varepsilon \right\}$$

for all $\phi \in \mathcal{P}^1(G)$, compact $Q \subseteq G$, and $\varepsilon > 0$. By Proposition 4.49, this shows that the Fell topology on the Pontryagin dual $\widehat{G}$ agrees with the compact-open topology.

Assume for the last claim, initially for any $t \in \operatorname{supp} \pi$, that there exists some choice of $t_n \in \operatorname{supp} \rho_n$ for all $n \in \mathbb{N}$ with $t_n \to t$ as $n \to \infty$. Let (n_k) be a strictly increasing sequence in $\mathbb{N}$. We then obtain from the assumption and the definition of the support of a unitary representation for an abelian group that

$$\operatorname{supp} \pi \subseteq \overline{\bigcup_{k=1}^{\infty} \operatorname{supp} \rho_{n_k}} = \operatorname{supp} \bigoplus_{k=1}^{\infty} \rho_{n_k}.$$

By Corollary 4.31, this implies that $\pi \prec \bigoplus_{k=1}^{\infty} \rho_{n_k}$. As the strictly increasing sequence (n_k) was arbitrary, Corollary 4.42 gives that $\rho_n \to \pi$ as $n \to \infty$.

Assume now that (ρ_n) is a sequence in $\mathscr{U}(G)$ converging to π. By Corollary 4.42 this shows that $\pi \prec \bigoplus_{n=1}^{\infty} \rho_{n_k}$ for any strictly increasing sequence (n_k) in $\mathbb{N}$. Hence by Corollary 4.31 and the definition of the support for unitary representations of abelian groups, we also obtain

$$\operatorname{supp} \pi \subseteq \overline{\bigcup_{k=1}^{\infty} \operatorname{supp} \rho_{n_k}}. \tag{4.25}$$

Now let $t \in \operatorname{supp} \pi$. It follows that for any neighbourhood O of t we have $O \cap \operatorname{supp} \rho_n \neq \emptyset$ for all but finitely many $n \in \mathbb{N}$, as otherwise we could construct a subsequence (n_k) that contradicts

$$t \in \overline{\bigcup_{k=1}^{\infty} \operatorname{supp} \rho_{n_k}}.$$

Now let $\widehat{G} = O_1 \supseteq O_2 \supseteq \cdots$ be a countable basis of the neighbourhoods of t, and choose recursively $1 = N_1 \leqslant N_2 \leqslant \cdots$ in $\mathbb{N}$ such that for every $\ell \in \mathbb{N}$ and $n \geqslant N_\ell$ the intersection $\operatorname{supp} \rho_n \cap O_\ell$ is non-empty. Finally choose for every $\ell \in \mathbb{N}$ and $n \in \mathbb{N}$ with $N_\ell \leqslant n < N_{\ell+1}$ the element $t_n \in \operatorname{supp} \rho_n$ such that t_n lies in $\operatorname{supp} \rho_n \cap O_\ell$. It follows that $t_n \to t$ as $n \to \infty$, which concludes the proof of the last statement in the corollary. □

4.5.2 The Isometry Group of the Plane

Let $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the group of isometries on $\mathbb{R}^2$. By Section 3.3.1, and using the same terminology, we have $\widehat{G} \cong \mathbb{Z} \sqcup (0, \infty)$, where the first set corresponds to all possible old representations χ_n for $n \in \mathbb{Z}$, and the second set to new representations π^r for all possible radii $r > 0$. We will show that on each of these sets separately, the Fell topology is the usual topology. More precisely, the Fell topology restricted to the old set $\mathbb{Z} \subseteq \widehat{G}$ is the discrete topology, and the Fell topology restricted to the Fell open new set $(0, \infty)$ is the standard metric topology. However, every old representation χ_n for $n \in \mathbb{Z}$ has a neighbourhood basis consisting of sets of the form $\{\chi_n\} \sqcup \{\pi^r \mid r \in (0, \delta)\}$ for $\delta > 0$. This is difficult to visualize, but Figure 4.1 is an attempt to do this. In particular, the Fell topology on $\widehat{G}$ is not Hausdorff but is T_1, as for every two points there is a neighbourhood of the first that does not contain the second.

Let us first discuss the Fell topology on the old set $\mathbb{Z} \subseteq \widehat{G}$. Here the old unitary representation is defined on $\mathcal{V}_1 = \mathbb{C}$ for every $n \in \mathbb{Z}$ by the character χ_n on $K \cong G/H$ and so there is only one choice of associated matrix coefficients to define a neighbourhood of χ_n (namely χ_n itself). If we choose the compact subset $Q = K$ (in Definition 4.48), it follows from the discreteness of $\widehat{K} \cong \mathbb{Z}$ that we can define a Fell neighbourhood of χ_n that does not contain any χ_m for $m \in \mathbb{Z} \smallsetminus \{n\}$.

To understand the Fell topology on all of $\widehat{G}$, we define the map

$$F\colon \begin{cases} \chi_n \longmapsto 0 & \text{for } n \in \mathbb{Z}, \\ \pi^r \longmapsto r & \text{for } r \in (0, \infty) \end{cases} \tag{4.26}$$

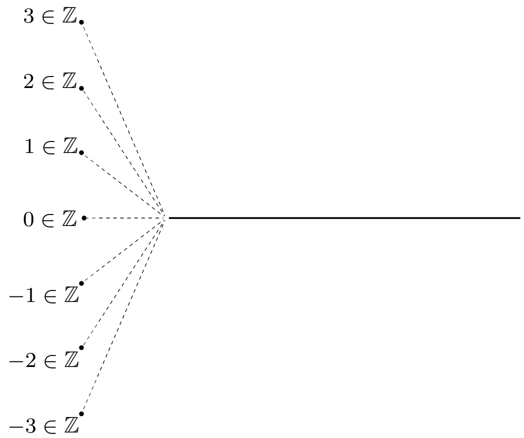

Fig. 4.1: The dotted lines from the elements $n \in \mathbb{Z}$ to the point 0 (which does not belong to $\widehat{G}$) indicates that these points are all limit points of π^r for $r \to 0$, while they themselves have no limit points.

and note that the last claim in Corollary 4.56 (applied to the restriction to $H \cong \mathbb{R}^2$) implies continuity of $F : \widehat{G} \to \mathbb{R}$ with respect to the Fell topology of $\widehat{G}$ (see Exercise 4.57). In particular, we can combine this with the above and deduce that for any $n \in \mathbb{Z}$ the set $\{\chi_n\} \sqcup \{\pi^r \mid r \in (0, \delta)\}$ is a neighbourhood of χ_n.

Exercise 4.57. Use Corollary 4.56 to prove continuity of the function F defined by (4.26).

The converse statement that for every Fell neighbourhood U of χ_n there exists some $\delta > 0$ with $\{\chi_n\} \sqcup (0, \delta) \subseteq U$ is easy to establish, as we may use the function $e_{-n} \in \mathcal{H}_r$ for sufficiently small $r > 0$ to approximate the matrix coefficient of $1 \in \mathbb{C}$ with respect to the representation χ_n of G. Informally, π^r degenerates for $r \searrow 0$ to the regular representation of K on $L^2(K)$ with H acting trivially, which should help in explaining the discussion above.

The description of a Fell neighbourhood of π^r for $r \in (0, \infty)$ is similar. By continuity of F, we have that

$$O = \{\pi^s \mid s \in (r - \delta, r + \delta)\}.$$

is a Fell open subset of $\widehat{G}$. On the other hand, it is easy to approximate the matrix coefficient of any $v_r \in \mathcal{H}_r$ by some $v_s \in \mathcal{H}_s$ if only s is sufficiently close to r. This shows that for any Fell neighbourhood U of π^r in $\widehat{G}$, there exists some $\eta > 0$ with

$$\{\pi^s \mid s \in (r - \eta, r + \eta)\} \subseteq U.$$

This concludes the description of the Fell topology on $\widehat{G}$ for $G = \mathrm{SO}_2(\mathbb{R}) \ltimes \mathbb{R}^2$.

Exercise 4.58. Describe the Fell topology on $\widehat{G_d}$, where G_d is as in Exercise 3.11.

4.5.3 The Affine Group in One Dimension

We now let $G = \mathbb{R}_{>0} \ltimes \mathbb{R}$ be the '$ax + b$'-group from Section 3.3.2. By Proposition 3.18, the unitary dual of G consists of the old characters on the multiplicative diagonal subgroup $A = \mathbb{R}_{>0}$ and two new irreducible representations π^+, π^- corresponding to the A-orbit $A{\bullet}1 = (0, \infty)$ and the A-orbit $A{\bullet}(-1) = (-\infty, 0)$ within the dual $\widehat{H} \cong \mathbb{R}$ of the additive subgroup $H \cong \mathbb{R}$.

By Corollary 4.56 (applied to the restrictions to H), we see that old representations can only converge to old representations, that π^+ (and, similarly, π^-) can only converge to itself and to old representations. In fact we have already shown in Section 4.3.4 that every old representation is weakly contained in π^+ (and similarly in π^-).

Finally, on the set $\widehat{A} \cong \mathbb{R}$ of old characters we have the standard metric topology. We summarize this in Figure 4.2.

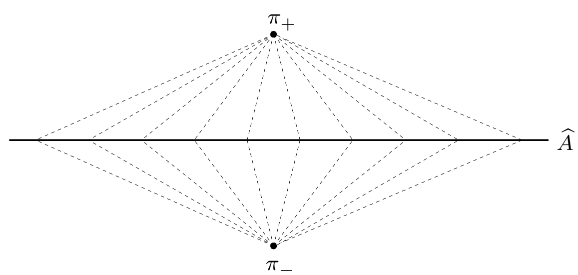

Fig. 4.2: The unitary dual of the '$ax + b$' group consists of the real line $\widehat{A} \cong \mathbb{R}$ and two 'nearly generic' points each of whose closures contain the real line but not the other of the two points.

4.5.4 The Heisenberg Group

We now fix $d \geqslant 1$ and let G be the $(2d+1)$-dimensional Heisenberg group as in Section 3.3.4. By the Stone–von Neumann theorem (Theorem 3.25), the unitary dual consists of the set $\widehat{\mathbb{R}^{2d}}$ of old representations arising from characters on $G/C(G) \cong \mathbb{R}^{2d}$ and one irreducible representation π^ξ for every non-trivial character on the centre defined by some $\xi \in \mathbb{R}^\times$.

By the last statement in Corollary 4.56 (applied to the restriction to the centre $C(G)$), the function $F\colon \widehat{G} \to \mathbb{R}$ defined by

$$F\colon \begin{cases} \chi \longmapsto 0 & \text{for } \chi \in \widehat{\mathbb{R}^{2d}}, \\ \pi^{\xi} \longmapsto \xi & \text{for } \xi \in \mathbb{R}^{\times} \end{cases}$$

is continuous. In particular, the set $\widehat{G/C(G)}$ of old characters is closed as a subset of $\widehat{G}$. Also, the Fell topology restricted to $\widehat{G/C(G)}$ becomes the standard metric topology on $\mathbb{R}^{2d}$.

Moreover, by the concrete description of π^{ξ} in (3.12) for $\xi \in \mathbb{R}^{\times}$ on the Hilbert space $L^2(\mathbb{R}^d)$ (independent of ξ) it is easy to see that if the sequence (ξ_n) in $\mathbb{R}^{\times}$ converges to $\xi \in \mathbb{R}^{\times}$, then the sequence (π^{ξ_n}) converges to π^{ξ} in the Fell topology. In the case of $\xi_n \to 0$ as $n \to \infty$ we also see that the definition of π^{ξ_n} degenerates to the multiplication representation of $\mathbb{R}^d$ on $L^2(\mathbb{R}^d)$. In particular, it is easy to find a sequence (v_n) in $L^2(\mathbb{R}^d)$ so that the matrix coefficient $\varphi_{v_n}^{\pi^{\xi_n}}$ converges to 1 in the compact-open topology as $n \to \infty$. This implies that π^{ξ_n} converges to the trivial character $\mathbb{1} \in \widehat{G/C(G)}$ as $n \to \infty$. For a non-trivial $\chi \in \widehat{G/C(G)}$ we can twist π^{ξ_n} as in Lemma 1.26 to obtain from Proposition 4.49 that $\pi^{\xi_n} \otimes \chi$ converges to $\chi \in \widehat{G/C(G)}$ as $n \to \infty$. By Theorem 3.25 we have that $\pi^{\xi_n} \otimes \chi$ is isomorphic to π^{ξ_n} (since $\chi(C(G)) = 1$), and so π^{ξ_n} converges to χ as $n \to \infty$ also. We again try to depict this (for the case $d = 1$) in Figure 4.3.

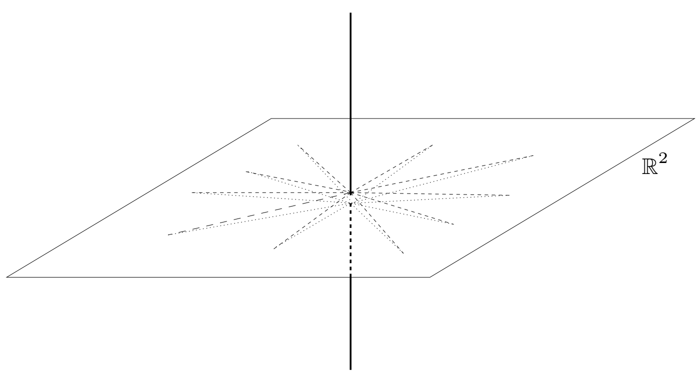

Fig. 4.3: The plane $\mathbb{R}^2$ consisting of old characters is closed, and has the standard topology. The vertical line parameterizes the new representations π^{ξ} with non-trivial central characters, and the dotted lines indicate that π^{ξ} converges as $\xi \to 0$ to all points in the plane simultaneously.

Exercise 4.59. Describe the Fell topology on the unitary dual of the group $G = \mathrm{Sol}$ from Exercise 3.23.

4.6 Summary and Outlook

As our examples clearly show, the notion of weak containment is an important generalization of the notion of containment in Section 1.2.7. We will use both weak containment and the related Fell topology sporadically in Chapter 5. However, weak containment will become crucial for the discussion of temperedness in Chapter 8.